has been questioned since it is not clear that the electron-phonon interaction can be calculated separately from correlation effects. For lack of anything better we will nevertheless assume that the effects of correlation and electron-phonon interactions can be estimated separately. Then the experimental data that $m^*/m=1.7$ for the bcc phase of sodium indicates that correlation alone increases the effective mass of the conduction electrons ($r_s=3.96$) about 40%. This result is in disagreement with the calculations of Quinn and Ferrell,[19] and Pines,[18] but is consistent with the results of DuBois[9] and Fletcher and Larson.[20] Figure 4 shows the theoretical calculations of DuBois as the solid line and the experimental point with approximate errors from the results on bcc sodium. The dashed line is a possible interpolation between the results.

The large value of the effective mass of the electrons in bcc sodium is quite surprising. Yet, it is not at variance with the theoretical calculations of DuBois, and Fletcher and Larson, and estimates of the effect of electron-phonon interactions. However, it would still be highly desirable to obtain further experimental measurements on the electronic specific heat of sodium. In particular, a measurement on a sample in which the transformation to the low-temperature phase has somehow been inhibited would be of great interest.

The large change in effective mass between the bcc and hcp phases is also surprising. The substantial contact of the Fermi surface to the Brillouin zone "A" face implies a fairly large energy gap at the zone face. This disagrees with the model of Cohen and Heine[29] for sodium. However, the contact explains why the hcp phase has the lower energy at absolute zero. The lowering of the energy levels in the vicinity of the contact lowers the total energy of the solid.

The author is indebted to Professor M. H. Cohen for a stimulating discussion and to Professor Ralph D. Myers for a critical and informative discussion.

[29] M. H. Cohen and V. Heine, *Advances in Physics*, edited by N. F. Mott (Taylor and Francis Ltd., London, 1958), Vol. 7, p. 395.

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Ultrasonic Attenuation in Superconductors*

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A general treatment of ultrasonic attenuation of both longitudinal and transverse waves in superconductors, valid for an arbitrary mean free path, is given on the basis of the Bardeen-Cooper-Schrieffer theory. The interaction between the ultrasonic waves and electrons is assumed to be given by a self-consistent electromagnetic field. Instead of the customary theory of the attenuation based on the Boltzmann equation, a different formulation is developed using the density-matrix formalism. The ratio of the attenuations in superconducting and normal metals for the longitudinal wave turns out to be approximately independent of the mean free path. The attenuation of the shear wave due to electromagnetic interaction is shown to be very small in the superconducting state.

1. INTRODUCTION

Ultrasonic attenuation in metals at low temperatures, predominantly electronic in origin, has been an object of active researches in recent years, yielding valuable information on the properties of normal metals as well as superconductors.[1,2] Bömmel and Mackinnon[3] first observed the rapid fall in the attenuation below the critical temperature. While it clearly reflected the decrease of the normal component, this steep drop could not be accounted for by the simple application of the two-fluid model and waited for the Bardeen-Cooper-Schrieffer (BCS) theory[4] for its satisfactory explanation. According to the latter theory the ratio, α_s/α_n, of the attenuations of a longitudinal sound wave in superconducting and in normal state varies with temperature as $2f(\epsilon_0)$, where f is the Fermi function and ϵ_0 is the temperature-dependent energy gap. The agreement between this formula and the experimental data, obtained for pure superconductors mostly in the frequency range around 50 Mc/sec, is reasonably good and is one of the strong supports of the BCS theory.[5] In fact this is one way of obtaining an empirical value of the

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[1] For general review of the subject, see R. W. Morse, *Progress in Cryogenics*, edited by K. Mendelssohn (Heywood & Company Ltd., London, 1959), Vol. I, p. 220.

[2] Concerning the ultrasonic attenuation in superconductors, see J. Bardeen and J. R. Schrieffer, *Progress in Low-Temperature Physics*, edited by C. J. Gorter, (North Holland Publishing Company, Amsterdam (to be published)].

[3] H. E. Bömmel, Phys. Rev. **96**, 220 (1954); L. Mackinnon, Phys. Rev. **98**, 1181, 1210 (1955).

[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[5] R. W. Morse and H. V. Bohm, Phys. Rev. **108**, 1094 (1957); R. W. Morse, H. V. Bohm, and J. D. Gavenda, Bull. Am. Phys. Soc. **3**, 44 and 203 (1958).

energy gap. The measurements of attenuation have been performed by Morse and Bohm[6] also for impure superconductors with $ql=1$, where q is the wave number of the sound and l the electron mean free path. The result indicates that the temperature variation of the ratio α_s/α_n depends slightly on the values of ql: for the smaller values of ql the attenuation decreases somewhat more rapidly than for $ql \gg 1$. Another case of interest is the attenuation of a shear wave. The result of detailed measurements by Morse and Bohm[6,1] taken in polycrystalline indium and tin showed an even more abrupt, apparently discontinuous drop in attenuation at the critical temperature. The change in the attenuation is more than 50% in the temperature range of 0.01°K below T_c, then followed by a gradual decrease which seems again to be described by the BCS formula.

So far there has been no theoretical treatment of the attenuations in superconductors with an arbitrary mean free path except one by Kresin[7] for the limiting case of $ql \ll 1$. It is the main purpose of the present work to construct a general theory of the attenuation of both longitudinal and shear waves on the basis of the BCS theory, valid for an arbitrary mean free path.

The first complete theory of the attenuation by the electron system in normal metals was proposed by Pippard[8] based on the idea of a distorted local Fermi surface. More recently, general treatments have been given by Blount[9] to discuss the attenuations for arbitrary frequency and band structures and by Cohen, Harrison, and Harrison[10] to analyze its magnetic dependence. Most of these theories are based on the Boltzmann equation for an electron distribution function, although the original treatment by Pippard did not make an explicit use of it. While this method is no doubt adequate for the case of normal metals, we cannot apply it to the case of superconductors, particularly in the present problem where an electromagnetic interaction is involved. Therefore we are led to take a different approach, closely related to the quantum mechanical derivation of conductivity from the density matrix formalism, first proposed by Kubo.[11] This kind of treatment has been used by Mattis and Bardeen[12] to derive the complex conductivity for a transverse electromagnetic field of a superconductor in the presence of scattering centers. Gorkov and Abrikosov[13] also calculated the conductivity by means of the Green's function formalism. We assume that the interaction between a long-wavelength sound wave and electrons in metals is mainly electromagnetic, so that it is necessary, first of all, to obtain the conductivities of the system for longitudinal as well as transverse fields, which we shall evaluate following the works mentioned above. In calculating the attenuation an extra complication arises from the fact that impurities, which we assume to be the only scattering mechanism, move with the lattice, thereby dragging electrons. This difficulty is dealt with by a canonical transformation, which has been used by Blount.[9]

In Sec. 2 we shall give an outline of the formulation of the problem, which is applied first to the case of normal metals in Sec. 3. The succeeding sections are devoted to the calculation of the attenuation in superconductors and to discussions of the results obtained. In the case of a longitudinal wave it becomes necessary to take into account the collective excitations in superconductors in order to guarantee the invariance of the theory against the canonical transformation mentioned above. The proof is presented in the Appendix.

2. FORMULATION

In the problem of ultrasonic attenuation by electrons it is convenient to consider three systems, namely, the impressed sound wave, the electrons, and the heat reservoir consisting of thermal phonons. We do not discuss the mechanism of transferring energy from the second to the last explicitly. The attenuation will be calculated from the amount of energy transferred to the electron system. For simplicity we suppose in this work that electrons are elastically scattered by impurities. Hence the unperturbed Hamiltonian for an electron may be written

$$H_0=-\frac{1}{2m}\nabla'^2+V_0(\mathbf{x}')+\sum_j V_{\mathrm{im}}(\mathbf{x}'-\mathbf{R}_j), \quad (2\text{-}1)$$

where $V_0(\mathbf{x}')$ is the periodic potential of the lattice and $V_{\mathrm{im}}(\mathbf{x}'-\mathbf{R}_j)$ is the potential due to the jth impurity. When the lattice is deformed by a sound wave, the perturbed potential is

$$V_0(\mathbf{x}'-\delta\mathbf{R})+V_1(\mathbf{x}',t)+\textstyle\sum_j V_{\mathrm{im}}(\mathbf{x}'-\delta\mathbf{R}-\mathbf{R}_j), \quad (2\text{-}2)$$

where $\delta\mathbf{R}(\mathbf{x}',t)$ is the smoothly varying function of $\mathbf{x}'$ such that at the lattice positions it is equal to the displacements of the ions. As pointed out by Blount,[9] $V_0(\mathbf{x}'-\delta\mathbf{R})-V_0(\mathbf{x}')$ may be large and the Bloch theorem no longer holds in the original coordinate system, so that it is necessary to make a transformation into the coordinate system, $\mathbf{x}$, fixed to the moving lattice[14];

$$\mathbf{x}=\mathbf{x}'-\delta\mathbf{R}(\mathbf{x}',t). \quad (2\text{-}3)$$

Another reason for making this transformation is that the scattering by impurities embedded in the lattice is

[6] R. W. Morse and H. V. Bohm, Bull. Am. Phys. Soc. 3, 225 (1958).
[7] B. Z. Kresin, J. Exptl. Theoret. Phys. U.S.S.R. 36, 1947 (1959), [translation: Soviet Phys.-JETP 36(9), 1385 (1959)].
[8] A. B. Pippard, Phil. Mag. 46, 1104 (1955).
[9] E. I. Blount, Phys. Rev. 114, 418 (1959).
[10] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. 117, 937 (1960).
[11] R. Kubo, Can. J. Phys. 34, 1274 (1956), J. Phys. Soc. Japan 12, 570 (1957).
[12] D. C. Mattis and J. Bardeen, Phys. Rev. 111, 412 (1958).
[13] A. A. Abrikosov and L. P. Gor'kov, J. Exptl. Theoret. Phys. U.S.S.R. 35, 1558 (1958) [translation: Soviet Phys.-JETP 35(8), 1090 (1959)].

[14] The use of the transformation was first pointed out to the author by Professor J. Bardeen.

elastic not in the $\mathbf{x}'$ frame but in the $\mathbf{x}$ frame. The treatment here is essentially the same as one given by Blount, but for completeness we shall discuss it in some detail. Let $\psi'(\mathbf{x}',t)$ be a wave function satisfying

$$i\frac{\partial}{\partial t}\psi'(\mathbf{x}',t)=H_0\psi'(\mathbf{x}',t). \tag{2-4}$$

Since the transformation (2-3) is not in general volume conserving, we write for the wave function in the $\mathbf{x}$ frame,

$$\psi(\mathbf{x},t)=[1+(\nabla\delta\mathbf{R})]^{\frac{1}{2}}\psi'(\mathbf{x}'(\mathbf{x}),t), \tag{2-5}$$

so that to the first order in the displacement

$$\int\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)d\mathbf{x}=1, \tag{2-6}$$

if $\psi'(\mathbf{x}',t)$ is normalized. From (2-4) and (2-2) one can derive the wave equation for $\psi(\mathbf{x},t)$. Since we are interested in the linear response of the system, we consistently keep only the terms linear in the displacement. Using

$$\frac{d}{dt}\psi(\mathbf{x},t)=\frac{\partial\psi}{\partial t}-\frac{\partial\delta\mathbf{R}}{\partial t}\nabla\psi,\quad \nabla'\psi=\nabla\psi-\sum_j(\nabla\delta R_j)\nabla_j\psi,$$

we get, where as in what follows all the quantities refer to the new coordinate,

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t)=(H_0+H')\psi(\mathbf{x},t), \tag{2-7}$$

$$H_0=p^2/2m+V_0(\mathbf{x})+\textstyle\sum_j V_{\mathrm{im}}(\mathbf{x}-\mathbf{R}_j),$$

$$H'=-\frac{1}{4m}\sum_{ij}[p_j,[p_i,(\nabla_i\delta R_j)]_+]_+ -\tfrac{1}{2}\sum_j\left[p_j,\frac{\partial\delta R_j}{\partial t}\right]_+ +V_1(\mathbf{x}). \tag{2-8}$$

Note that this Hamiltonian is Hermitian. One can show that this transformation is equivalent to a unitary transformation, $U=\exp(iS)$ with $S=\frac{1}{2}\sum_j[p_j,\delta R_j]_+$. We shall consider H' as a perturbation and take the eigenfunctions of H_0 as our unperturbed states, which, if there is no scattering by impurities, we assume to be given by plane wave states. Since the wavelength of the impressed sound is much larger than the interatomic distance, we can consider $V_1(\mathbf{x})$ to be given by a long-range electromagnetic field produced by the sound wave, neglecting possible real metal effects.

Thus we arrive at a simple model: essentially a free electron gas with the background of positive charges capable of carrying the sound wave. The system is driven by the electromagnetic field determined by the Maxwell equations in a self-consistent manner and the scattering of electrons is by the fixed impurities. In addition there is a fictitious force described by the first part of H', which, as we shall see, can be physically interpreted as the dragging force due to the moving impurities.

The current density of electrons in the state ψ in the new frame can be obtained by calculating the time rate of change of the charge density $-e\psi^\dagger\psi$:

$$\partial\rho/\partial t=i\{(H\psi)^\dagger\psi-\psi^\dagger(H\psi)\}=\nabla\cdot\mathbf{j}, \tag{2-9}$$

where

$$\mathbf{j}(\mathbf{x},t)=\mathbf{j}_p(\mathbf{x},t)-\frac{e^2}{mc}\mathbf{A}\psi^\dagger\psi+e\frac{\partial\delta\mathbf{R}}{\partial t}\rho +\frac{ie}{2m}\sum_j[(\nabla_j\delta\mathbf{R})+(\nabla\delta R_j)] \times[(\nabla_j\psi)^\dagger\psi-\psi^\dagger(\nabla_j\psi)], \tag{2-10}$$

with ∇_j being the components of ∇, and

$$\mathbf{j}_p(\mathbf{x},t)=-\frac{i}{2m}\{(\nabla\psi)^\dagger\psi-\psi^\dagger(\nabla\psi)\}. \tag{2-11}$$

If there are no electromagnetic fields nor impurities, $\mathbf{j}_p(\mathbf{x},t)$ calculated to the first order in H' is equal to $-e\partial\delta\mathbf{R}/\partial t\cdot\rho$ which just cancels the third term in (2-10), as we shall explicitly show. Since we are keeping only terms linear in the displacement, we can drop the last term in (2-10), so that the expression is reduced to the customary one.

Let us describe the sound wave by a velocity field

$$\mathbf{u}(\mathbf{x},t)=\mathbf{u}\exp[i(\mathbf{q}\cdot\mathbf{x}-\omega t)]=-i\omega\delta\mathbf{R}(\mathbf{x},t),$$

with the wave vector $\mathbf{q}$ and the frequency ω. Rewriting (2-8) in terms of $\mathbf{u}$, we get

$$H=\frac{p^2}{2m}+\frac{e}{2mc}(\mathbf{A}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{A})-e\phi +\sum_j V_{\mathrm{im}}(\mathbf{x}-\mathbf{R}_j)+H_I, \tag{2-12}$$

where

$$H_I=\frac{1}{m\omega}[\mathbf{q}\cdot(\mathbf{p}+\tfrac{1}{2}\mathbf{q})][\mathbf{u}\cdot(\mathbf{p}+\tfrac{1}{2}\mathbf{q})]-\mathbf{u}\cdot(\mathbf{p}+\tfrac{1}{2}\mathbf{q}). \tag{2-13}$$

We shall assume that the impressed wave is either purely longitudinal or transverse. Since $\mathbf{A}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$, to be determined self-consistently, are already proportional to u, we can calculate the linear responses to the fields and to the fictitious force H_I separately. They are conveniently described by the conductivities defined by

$$\mathbf{j}(\mathbf{q},\omega)=\sigma(\mathbf{q},\omega)\mathbf{E}(\mathbf{q},\omega), \tag{2-14}$$

and

$$\mathbf{j}_I(\mathbf{q},\omega)=\sigma^I(\mathbf{q},\omega)(-m\mathbf{u}/e\tau). \tag{2-15}$$

where τ is the relaxation time to be determined later. This definition of σ^I is quite arbitrary at this stage but will turn out to be convenient. The total electronic cur-

rent will then be given by

$$\mathbf{j}_e=\mathbf{j}+\mathbf{j}_I=\sigma\mathbf{E}-\sigma^I(m\mathbf{u}/e\tau). \tag{2-16}$$

The attenuation constant α can be expressed in terms of σ and σ^I. We follow the derivation given by Cohen *et al.*,[10] with a slight generalization that σ^I may be different from σ. The electric field is given by the Maxwell equations,

$$\begin{aligned} E_{\parallel}&=\left(\frac{4\pi}{i\omega}\right)J_{\parallel},\\ E_{\perp}&=\left(\frac{4\pi i}{\omega}\right)\left(\frac{v_s}{c}\right)^2\left[1-\left(\frac{v_s}{c}\right)^2\right]^{-1}J_{\perp}, \end{aligned} \tag{2-17}$$

where $E_{\parallel}$, $J_{\parallel}$ and $E_{\perp}$, $J_{\perp}$ are the electric field and the total current components parallel and perpendicular to $\mathbf{q}$, respectively, and v_s is the sound velocity. The total current in the fixed frame is

$$\mathbf{J}=\mathbf{j}_e+Ne\mathbf{u}. \tag{2-18}$$

Combining (2-14) through (2-18), we can solve for $\mathbf{E}$:

$$\begin{aligned} E_{\parallel}&=(\sigma'+i\gamma)^{-1}(\sigma'^I-1)mu/e\tau,\\ E_{\perp}&=(\sigma'+i\beta)^{-1}(\sigma'^I-1)mu/e\tau, \end{aligned} \tag{2-19}$$

where we have introduced

$$\begin{aligned} &\sigma'=\sigma/\sigma_0, &&\sigma'^I=\sigma^I/\sigma_0,\\ &\beta=\omega c^2/(4\pi\sigma_0 v_s^2), &&\gamma=\beta(v_s/c)^2. \end{aligned} \tag{2-20}$$

$\sigma_0=Ne^2\tau/m$ is the dc conductivity. We shall neglect γ which is always very small. The power dissipated by the sound wave per unit volume is then given by

$$Q=\tfrac{1}{2}\,\mathrm{Re}[\mathbf{j}_e^*\mathbf{E}]. \tag{2-21}$$

Here we have neglected the correction related to the collision drag effect, which is important only at very high frequencies.[15] Substituting (2-19) into (2-21) and using the definition of the attenuation constant,

$$\alpha=Q/(\tfrac{1}{2}\rho_{\rm ion}|u|^2v_s), \tag{2-22}$$

where $\rho_{\rm ion}$ is the density of the ions, we obtain

$$\alpha=\frac{Nm}{\rho_{\rm ion}v_s\tau}\mathrm{Re}\left\{\frac{1-\sigma'^I}{\sigma'}\right\} \tag{2-23}$$

for the longitudinal wave and

$$\alpha=\frac{Nm}{\rho_{\rm ion}v_s\tau}\mathrm{Re}\left\{\frac{1-\sigma'^I}{\sigma'+i\beta}\right\} \tag{2-24}$$

for the transverse wave, respectively.

3. NORMAL METALS

The problem is now reduced to calculations of the appropriate conductivities defined by (2-14) and (2-15). We shall first treat the response of normal metals to the longitudinal wave by the method of density matrix. Following this example we can apply the method to the case of superconductors in a straightforward manner.

The average value of the charge density calculated by first order perturbation theory is compactly expressed by the formula,[16]

$$\bar{\rho}(\mathbf{x},t)=-i\int_{-\infty}^{t}dt'\int d\mathbf{x}'\langle[\rho(\mathbf{x},t),\rho(\mathbf{x}',t')]\rangle\phi(\mathbf{x}',t'), \tag{3-1}$$

where the angular bracket means $\langle a\rangle=\mathrm{Tr}(U_0a)$ with $U_0=\exp(-\beta\mathcal{H}_0)/\mathrm{Tr}[\exp(-\beta\mathcal{H}_0)]$. $\mathcal{H}_0$ is the unperturbed Hamiltonian of the entire system. One can also write down the response to the perturbation H_I in the same form,

$$\bar{\rho}_I(\mathbf{x},t)=-i\int_{-\infty}^{t}dt'\int d\mathbf{x}'\langle[\rho(\mathbf{x},t),\,H_I(\mathbf{x}',t')]\rangle. \tag{3-2}$$

One way of calculating these quantities is to apply the many-body Green's function formalism directly, as has been done by Abrikosov and Gorkov.[13] But the subsequent calculations are mathematically involved in the case of superconductors and become almost intractable at finite temperatures. Therefore, another method will be used here, which is similar to the one by Mattis and Bardeen.[12]

Let $\psi_n(\mathbf{x})$ be the eigensolutions of H_0 which include the impurity potentials:

$$H_0\psi_n=\epsilon_n\psi_n. \tag{3-3}$$

We use the second-quantized formalism and expand a field operator $\psi(\mathbf{x},t)$ in terms of ψ_n;

$$\begin{aligned} \psi(\mathbf{x},t)&=\sum_{n\sigma}\exp(i\mathcal{H}_0t)c_{n\sigma}\exp(-i\mathcal{H}_0t)\psi_{n\sigma}(\mathbf{x}),\\ \psi^\dagger(\mathbf{x},t)&=\sum_{n\sigma}\exp(i\mathcal{H}_0t)c_{n\sigma}^\dagger\exp(-i\mathcal{H}_0t)\psi_{n\sigma}(\mathbf{x}), \end{aligned} \tag{3-4}$$

where $c_{n\sigma}$ and $c_n{}^\dagger{}_\sigma$ are annihilation and creation operators. Then

$$\mathcal{H}_0=\sum_{n\sigma}\epsilon_n c_{n\sigma}^\dagger c_{n\sigma}. \tag{3-5}$$

The expression (3-1) can be written as

$$\bar{\rho}(\mathbf{x},t)=-\tfrac{1}{2}e^2\sum_{\omega}\int d\mathbf{y}\,\phi(\mathbf{x}\cdot\omega)e^{-i(\omega+i\eta)t}\sum_{i\neq f}U_0(E_i)\times\sum_{nn'\sigma}\left\{\frac{1}{E_i-E_f-\omega-i\eta}|\langle i|c_{n\sigma}^\dagger c_{n\sigma'}|f\rangle|^2\psi_n^*(\mathbf{x})\psi_{n'}(\mathbf{x})\psi_{n'}^*(\mathbf{y})\psi_n(\mathbf{y})+(\text{c.c. with }-\omega)\right\}, \tag{3-6}$$

[15] The condition for neglecting it is $\beta\ll1$, see reference 10, Sec. II-B.
[16] See for example, V. M. Galitskii and A. B. Migdal, J. Exptl. Theoret. Phys. U.S.S.R. 34, 139 (1958) [translation: Soviet Phys.-JETP 34(7), 96 (1958)]; P. C. Martin and J. Schwinger, Phys. Rev. 115, 1342 (1959).

where $|i\rangle$ and $|f\rangle$ are the eigenstates of $\mathcal{H}_0$ with the eigenvalues E_i and E_f, respectively, and an infinitesimal quantity η has been introduced for the adiabatic switching on of the interaction at $t=-\infty$. Since the scattering centers are randomly distributed in space and ψ_n's are a function of their positions, the only significant quantity is the average over this random distribution. In order to find the average it is convenient to separate out the product of ψ_n's from other factors in (3-6) in the following way. Carrying out the calculation of the matrix elements we get

$$\bar{\rho}(q,\omega)=-e^2\phi(q,\omega)\int d\mathbf{y}\, e^{-i\mathbf{q}(\mathbf{y}-\mathbf{x})}\int\int d\epsilon d\epsilon'[f(\epsilon)-f(\epsilon')]$$

$$\times\sum_{nn'}\left\{\frac{1}{\epsilon-\epsilon'-\omega-i\eta}\langle\psi_n{}^*(\mathbf{x})\psi_{n'}(\mathbf{x})\psi_{n'}{}^*(\mathbf{y})\psi_n(\mathbf{y})\delta(\epsilon_n-\epsilon)\delta(\epsilon_{n'}-\epsilon')\rangle_{\mathrm{av}}+(\text{c.c. with }-\omega)\right\},\quad(3\text{-}7)$$

where $\langle\ \rangle_{\mathrm{av}}$ means the average over the distribution of impurities and the energy is measured from the Fermi energy. We may rewrite this as

$$\bar{\rho}(q,\omega)=-e^2\phi(q,\omega)\int\int d\epsilon d\epsilon'\, M^{(+)}(\epsilon,\epsilon',\omega)F(q;\epsilon,\epsilon'),\quad(3\text{-}8)$$

where

$$M^{(\pm)}(\epsilon,\epsilon',\omega)=[f(\epsilon)-f(\epsilon')]\left\{\frac{1}{\epsilon-\epsilon'-\omega-i\eta}\pm\frac{1}{\epsilon-\epsilon'+\omega+i\eta}\right\},\quad(3\text{-}9)$$

and

$$F(q;\epsilon,\epsilon')=\int d\mathbf{y}\, e^{-i\mathbf{q}(\mathbf{y}-\mathbf{x})}\langle\Pi(\mathbf{x},\mathbf{y},\mathbf{y},\mathbf{x};\epsilon,\epsilon')\rangle_{\mathrm{av}},\quad(3\text{-}10)$$

$$\Pi(\mathbf{x}',\mathbf{y},\mathbf{y}',\mathbf{x};\epsilon,\epsilon')=\sum_{nn'}\psi_n{}^*(\mathbf{x}')\psi_n(\mathbf{y})\psi_{n'}{}^*(\mathbf{y}')\psi_{n'}(\mathbf{x})\delta(\epsilon_n-\epsilon)\delta(\epsilon_{n'}-\epsilon').\quad(3\text{-}11)$$

This is the correlation function of electron amplitudes in the presence of scattering centers. The similar quantity was discussed by Edwards[17] for the case of the dc conductivity, so that we shall just sketch the derivation of the results. In terms of the Green's function defined by

$$\begin{aligned}G_-(\mathbf{y},\mathbf{x},t)&=-i\eta(t)\textstyle\sum_n\psi_n(\mathbf{y})\psi_n{}^*(\mathbf{x})e^{i\epsilon_n t},\\ G_+(\mathbf{y},\mathbf{x},t)&=i\eta(t)\textstyle\sum_n\psi_n(\mathbf{x})\psi_n{}^*(\mathbf{y})e^{-i\epsilon_n t},\end{aligned}\quad(3\text{-}12)$$

where $\eta(t)=1$ for $t>0$ and $=0$ otherwise, one can write

$$\sum_n\psi_n(\mathbf{y})\psi_n{}^*(\mathbf{x})\delta(\epsilon-\epsilon_n)=\frac{i}{2\pi}\int_{-\infty}^{\infty}dt\{e^{-i\epsilon t}G_-(\mathbf{y},\mathbf{x},t)-e^{i\epsilon t}G_+(\mathbf{x},\mathbf{y},t)\}=\frac{i}{2\pi}\{G_-(\mathbf{y},\mathbf{x},\epsilon)-G_+(\mathbf{x},\mathbf{y},\epsilon)\}.\quad(3\text{-}13)$$

Hence

$$\langle\Pi(\mathbf{x},\mathbf{y}',\epsilon,\epsilon')\rangle_{\mathrm{av}}=-\frac{1}{(2\pi)^2}\langle[G_-(\mathbf{y},\mathbf{x},\epsilon)-G_+(\mathbf{x}',\mathbf{y},\epsilon)][G_-(\mathbf{x},\mathbf{y},\epsilon')-G_+(\mathbf{y},\mathbf{x},\epsilon')]\rangle_{\mathrm{av}}.\quad(3\text{-}14)$$

The average of the one-particle Green's function is approximately calculated by summing over the series of perturbations represented by diagrams shown in Fig. 1:

$$\langle G_\pm(\mathbf{p},\epsilon)\rangle_{\mathrm{av}}=(\epsilon-\epsilon_p\pm i/2\tau)^{-1},\quad(3\text{-}15)$$

where

$$\frac{1}{\tau}=\frac{np_0m}{(2\pi)^2}\int d\Omega|V_{\mathrm{im}}(\theta)|^2\quad(3\text{-}16)$$

is the relaxation time, n the average number density of impurities, and

$$\langle G_\pm(\mathbf{y},\mathbf{x},\epsilon)\rangle_{\mathrm{av}}=\int\frac{d\mathbf{p}}{(2\pi)^3}\langle G_\pm(\mathbf{p},\epsilon)\rangle_{\mathrm{av}}e^{\mp i\mathbf{p}(\mathbf{y}-\mathbf{x})},\quad(3\text{-}17)$$

with $\epsilon_p=p^2/2m-\epsilon_F$. In deriving (3-15) it has been assumed that the average separation between the scattering

[17] S. F. Edwards, Phil. Mag. **3**, 1020 (1958). See also P. R. Weiss and E. Abrahams, Phys. Rev. **111**, 722 (1958).

centers is large compared to the interatomic distance. Next, the interference of the two propagators can be found approximately by summing the series represented in Fig. 2, which results in an integral equation,

$$\langle \Pi^{(+-)}(\mathbf{p}_1,\epsilon;\mathbf{p}_2,\epsilon')\rangle_{\mathrm{av}}=\langle G_+(\mathbf{p}_1,\epsilon)\rangle_{\mathrm{av}}\langle G_-(\mathbf{p}_2,\epsilon')\rangle_{\mathrm{av}}\left[1+n\int\frac{d\mathbf{p}'}{(2\pi)^3}|V_{\mathrm{im}}(\mathbf{p}'-\mathbf{p})|^2\langle\Pi^{(+-)}(\mathbf{p}_1',\epsilon,\mathbf{p}_2'\epsilon')\rangle_{\mathrm{av}}\right] \qquad (3\text{-}18)$$

for the Fourier transform defined by

$$\int d\mathbf{y}e^{-i\mathbf{q}(\mathbf{y}-\mathbf{x})}\langle G_+(\mathbf{x},\mathbf{y},\epsilon)G_-(\mathbf{x},\mathbf{y},\epsilon')\rangle_{\mathrm{av}}=\int\frac{d\mathbf{p}}{(2\pi)^3}\langle\Pi^{(+-)}(\mathbf{p}_1,\epsilon,\mathbf{p}_2,\epsilon')\rangle_{\mathrm{av}}, \qquad (3\text{-}19)$$

where $\mathbf{p}_1=\mathbf{p}+\mathbf{q}/2$ and $\mathbf{p}_2=\mathbf{p}-\mathbf{q}/2$. The Eq. (3-18) can be readily solved; with the help of (3-15) we find, assuming that $\mathbf{q}$ is much smaller than the Fermi momentum p_0,

$$\langle\Pi^{(+-)}(\mathbf{p}_1,\epsilon,\mathbf{p}_2,\epsilon')\rangle_{\mathrm{av}}=\langle G_+(\mathbf{p}_1,\epsilon)\rangle_{\mathrm{av}}\langle G_-(\mathbf{p}_2,\epsilon')\rangle_{\mathrm{av}} \times\left[1+\frac{1}{2\alpha\tau}T(\epsilon-\epsilon',q)\right], \qquad (3\text{-}20)$$

where

$$T(x,q)=s(x,q)\left[1-\frac{1}{2\alpha\tau}s(x,q)\right]^{-1}, \qquad (3\text{-}21)$$

$$s(x,q)=-i\ln\frac{x/\alpha-1+i/\alpha\tau}{x/\alpha+1+i/\alpha\tau}, \qquad (3\text{-}22)$$

and $\alpha=qv_0$. In (3-20) we have assumed isotropic scat-

FIG. 1. Diagrams of impurity scattering included in $\langle G(\mathbf{p},\epsilon)\rangle_{\mathrm{av}}$.

tering although more general cases can be treated. The important point is that while for the case of the static conductivity or the one for a transverse field this interference effect vanishes unless the scattering is anisotropic, such that $(1/pp')\int d\Omega|V_{\mathrm{im}}(\mathbf{p}-\mathbf{p}')|^2(\mathbf{p}\cdot\mathbf{p}')\neq 0$, it is finite in the case of the response to the longitudinal wave of finite wavelength. Since $\langle G_\pm(\mathbf{p}_1,\epsilon)\rangle_{\mathrm{av}}\langle G_\pm(\mathbf{p}_2,\epsilon)\rangle_{\mathrm{av}}$ vanish upon integrating over p, we get

$$F(q,\epsilon,\epsilon')=\frac{N(0)}{4\pi\alpha}\{T(\epsilon'-\epsilon,q)+\mathrm{c.c.}\}, \qquad (3\text{-}23)$$

where $N(0)=mp_0/2\pi^2$ is the density of states at the Fermi surface.

Let us first calculate (3-8) for $\omega=0$. For this purpose it is easier to go back to (3-7). Since $f(\epsilon)-f(\epsilon')\approx(df/d\epsilon)(\epsilon-\epsilon')$ for important values of ϵ and ϵ',

$$\bar{\rho}(q,0)=-2e^2\phi\int d\mathbf{y}\,e^{-i\mathbf{q}(\mathbf{y}-\mathbf{x})}\int d\epsilon\frac{df}{d\epsilon}\sum_{nn'} \times\langle\psi_n^*(\mathbf{x})\psi_n(\mathbf{y})\psi_{n'}^*(\mathbf{y})\psi_{n'}(\mathbf{x})\rangle_{\mathrm{av}}\delta(\epsilon_n-\epsilon). \qquad (3\text{-}24)$$

Because of the completeness and of the relation

$$\sum_n\langle\psi_n^*(\mathbf{x})\psi_n(\mathbf{x})\rangle_{\mathrm{av}}\delta(\epsilon_n-\epsilon)\approx N(0), \qquad (3\text{-}25)$$

we get

$$\bar{\rho}(q,0)=-2e^2N(0)\phi(q,0). \qquad (3\text{-}26)$$

Thus

$$\bar{\rho}(q,\omega)=-2e^2N(0)\phi(q,\omega)\times\left\{1+\frac{1}{8\pi\alpha}\int\int d\epsilon d\epsilon'[f(\epsilon)-f(\epsilon')]\times\left[M^{(+)}(\epsilon,\epsilon',\omega)-\frac{2}{\epsilon-\epsilon'}\right]\times[T(\epsilon'-\epsilon,q)+\mathrm{c.c.}]\right\}. \qquad (3\text{-}27)$$

The integral can be done by contour integrations if one introduces the convergence factor $a^2/(\epsilon^2+\epsilon'^2+a^2)$ and then take the limit as $a\to\infty$.[12] Closing the contours such that the argument of $T(x,q)$ has no negative imaginary part, we finally get the expression for the longitudinal conductivity;

$$\sigma(q,\omega)=-i\frac{2N(0)e^2\omega}{q^2}\left\{1+\frac{i\omega}{2\alpha}T(\omega,q)\right\}. \qquad (3\text{-}28)$$

This is identical to the result which has been obtained by the semiclassical method based on the Boltzmann equation and which includes the diffusion current due to the local variation of the electron density.[18] It is interesting to note that the effect of the interference of the two propagators shown in Fig. 2 is closely related to the diffusion current: if we had not taken this into account we would have gotten, simply $S(\omega,q)$ instead of $T(\omega,q)$ in (3-28).

The response to the perturbation H_I can be calculated in the same manner from (3-2) and may be written

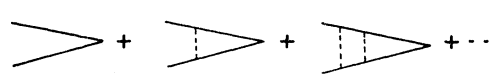

FIG. 2. Ladder-type diagrams included in the vertex corrections.

[18] J. L. Warren and R. A. Ferrell, Phys. Rev. **117**, 1253 (1960). See also reference 10.

in the form,

$$\bar{\rho}_I(q,\omega)=e\int\int d\epsilon d\epsilon'[M^{(+)}(\epsilon,\epsilon',\omega)F_1(q,\epsilon,\epsilon')+M^{(-)}(\epsilon,\epsilon',\omega)F_2(q,\epsilon,\epsilon')],\tag{3-29}$$

where

$$F_1(q,\epsilon,\epsilon')=\frac{1}{4m\omega}\int d\mathbf{y}\,e^{-i\mathbf{q}\cdot(\mathbf{y}-\mathbf{x})}\langle\lim_{\mathbf{y}'\to\mathbf{y}}[\mathbf{q}\cdot(\nabla_{y'}-\nabla_y)][\mathbf{u}\cdot(\nabla_{y'}-\nabla_y)]\Pi(\mathbf{x},\mathbf{y},\mathbf{y}',\mathbf{x};\,\epsilon,\epsilon')\rangle_{\mathrm{av}},\tag{3-30}$$

$$F_2(q,\epsilon,\epsilon')=\frac{i}{2}\int d\mathbf{y}\,e^{-i\mathbf{q}\cdot(\mathbf{y}-\mathbf{x})}\langle\lim_{\mathbf{y}'\to\mathbf{y}}[\mathbf{u}\cdot(\nabla_{y'}-\nabla_y)]\Pi(\mathbf{x},\mathbf{y},\mathbf{y}',\mathbf{x};\,\epsilon,\epsilon')\rangle_{\mathrm{av}}.\tag{3-31}$$

As before we first evaluate ρ_I putting $\omega=0$ in $M^{(\pm)}(\epsilon,\epsilon',\omega)$. The second term in (3-29) gives zero because of the symmetry, so that we get from the first term,

$$\frac{2eu}{m\omega q}\int d\epsilon\frac{df}{d\epsilon}\sum_n\langle\psi_n^*(\mathbf{x})(\mathbf{q}\cdot\nabla)^2\psi_n(\mathbf{x})\rangle_{\mathrm{av}}\delta(\epsilon-\epsilon_n)\approx-\frac{2eN(0)p_0^2q}{3m\omega}=-\frac{q}{\omega}Neu,\tag{3-32}$$

which cancels the "gauge current," Neu, in (2-10). One can calculate the correlation functions F_1 and F_2 by the same method; the only difference is to insert in (3-18) the verticies, $(\mathbf{q}\cdot\mathbf{p})(\mathbf{u}\cdot\mathbf{p})$ and $(\mathbf{u}\cdot\mathbf{p})$, respectively. The results are[19]

$$F_1(q,\epsilon,\epsilon')=F_1'(q,\epsilon,\epsilon')+\frac{\epsilon'-\epsilon}{\omega}F_2(q,\epsilon,\epsilon'),\tag{3-33}$$

$$F_1'(q,\epsilon,\epsilon')=-\frac{2qup_0^4}{(2\pi)^3m\omega v_0\alpha^2\tau}\left[\left\{1+\frac{i(\epsilon'-\epsilon)}{\omega}T(\epsilon'-\epsilon,\,q)\right\}+\mathrm{c.c.}\right],\tag{3-34}$$

$$F_2(q,\epsilon,\epsilon')=-\frac{up_0^3}{(2\pi)^3v_0\alpha^2}(\epsilon'-\epsilon)[T(\epsilon'-\epsilon,\,q)+\mathrm{c.c.}].\tag{3-35}$$

Substituting these into (3-29) one finds

$$\bar{\rho}_I(q,\omega)=-\frac{q}{\omega}Neu-i\left(\frac{mu}{e\tau}\right)\frac{2e^2N(0)}{q}\left[1+\frac{i\omega}{2\alpha}T(\omega,q)\right],\tag{3-36}$$

the terms involving F_2 having cancelled each other. When $\tau\to\infty$, it reduces to $-(q/\omega)Neu$. Hence

$$\sigma^I(q,\omega)=\sigma(q,\omega).\tag{3-37}$$

The total electronic current is, therefore,

$$\mathbf{j}_e(q,\omega)=\sigma(q,\omega)\left(\mathbf{E}-\frac{m\mathbf{u}}{e\tau}\right).\tag{3-38}$$

[See Eq. (2-40) of reference 10]. One can see that the inclusion of H_I takes care of the fact that electrons relax onto the displaced Fermi distribution after colliding with moving impurities.

If we keep terms of the lowest order in $\omega/\alpha=v_s/v_0$, we get Pippard's result for the attenuation constant;

$$\alpha_n=\frac{mNv_0}{\rho_{\mathrm{ion}}v_sl}\left[\frac{1}{3}\frac{(ql)^2\tan^{-1}(ql)}{ql-\tan^{-1}(ql)}-1\right],\tag{3-39}$$

where $l=\tau v_0$ is the mean free path.

[19] In performing the integral over p it is sufficient to consider only the contributions from the poles of $\langle G\rangle_{\mathrm{av}}$. This may be justified by subtracting the corresponding quantity with $\tau\to\infty$, which can be evaluated by the ordinary method.

4. CONDUCTIVITIES OF SUPERCONDUCTORS

In place of a real superconductor we shall take the simplified model of the BCS theory, supplemented by a suitable consideration of the collective excitations when it is necessary. As the perturbation Hamiltonian we assume that the same expression (2-12) is still adequate. Actually this is an approximation. In the BCS model we have an attractive interaction between electrons via phonons, responsible for the superconducting transition. This interaction would also be modified by the deformation due to the impressed wave, so that there should appear an additional term in (2-12), which would bring about the local variation of the energy gap. We assume that this effect is small. Then the response of a superconductor may be obtained by simply extending the method developed in the previous section.

The pairing of the states in the presence of scattering is known to be $(n\uparrow,-n\downarrow)$ if we define $\psi_{-n}(\mathbf{x})\equiv\psi_n^*(\mathbf{x})$. Let us introduce the quasi-particle operators,

$$\begin{aligned}\gamma_{n0}&=u_nc_{n\uparrow}-v_nc_{-n\downarrow}^\dagger, & \gamma_{n1}&=u_nc_{-n\downarrow}+v_nc_{n\uparrow}^\dagger,\\ u_n&=\tfrac{1}{2}(1+\epsilon_n/E_n)^{\frac{1}{2}}, & v_n&=\tfrac{1}{2}(1-\epsilon_n/E_n)^{\frac{1}{2}},\end{aligned}\tag{4-1}$$

where $E_n=(\epsilon_n+\epsilon_0^2)^{\frac{1}{2}}$. The statistical operator for a superconductor is given by[20]

$$U_0=C^{-1}\sum_{j=0}^{\infty}\sum_{m_1<\cdots<m_j}\lambda_{m_1}\cdots\lambda_{m_j}\gamma_{m_1}^{\dagger}\cdots\gamma_{m_j}^{\dagger}|0\rangle \times\langle 0|\gamma_{m_1}\cdots\gamma_{m_j} \quad (4\text{-}2)$$

where

$$\lambda_m=e^{-\beta E^m},\quad C=\prod_m(1+\lambda_m). \quad (4\text{-}3)$$

$|0\rangle$ is the BCS ground state, $\gamma_m|0\rangle=0$, and the m's here include the indices 1 and 0.

(a) Longitudinal Wave

One can evaluate the first order response in just the same way as for normal metals:

$$\rho(q,\omega)=-e^2\phi(q,\omega)\int\int d\epsilon d\epsilon'\left\{L^{(-)}(\epsilon,\epsilon',\omega)-2\frac{f(\epsilon)-f(\epsilon')}{\epsilon-\epsilon'}\right\}F(q,\epsilon,\epsilon')-2e^2N(0)\phi(q,\omega), \quad (4\text{-}4)$$

where $L^{(-)}$ is the matrix element involving the coherence factors:

$$L^{(\mp)}(\epsilon,\epsilon',\omega)=\frac{1}{2}\left(1+\frac{\epsilon\epsilon'\mp\epsilon_0^2}{EE'}\right)[f(\mathrm{E})-f(\mathrm{E}')]\left(\frac{1}{E-E'-\omega-i\eta}+\frac{1}{E-E'+\omega+i\eta}\right)$$
$$-\frac{1}{2}\left(1-\frac{\epsilon\epsilon'\mp\epsilon_0^2}{EE'}\right)[1-f(\mathrm{E})-f(\mathrm{E}')]\left(\frac{1}{E+E'-\omega-i\eta}+\frac{1}{E+E'+\omega+i\eta}\right). \quad (4\text{-}5)$$

Making use of the symmetry in ϵ and ϵ' of the above expression, we get the following expression for the conductivity:

$$\sigma(q,\omega)=-\frac{2N(0)e^2\omega}{q^2}\left\{i+\frac{1}{4qv_0}I(q,\omega)\right\}, \quad (4\text{-}6)$$

where

$$I(q,\omega)=\int_{\epsilon_0-\omega}^{\epsilon_0}dE[1-2f(E+\omega)]\{[g_1(E)+1]T(\epsilon_1-\epsilon_2,q)+[g_1(E)-1]T(\epsilon_1+\epsilon_2,q)\}$$
$$-\int_{\epsilon_0}^{\infty}dE[1-2f(E+\omega)]\{[g_1(E)+1]T(\epsilon_2-\epsilon_1,q)+[g_1(E)-1]T(-\epsilon_1-\epsilon_2,q)\}$$
$$+\int_{\epsilon_0}^{\infty}dE[1-2f(E)]\{[g_1(E)+1]T(\epsilon_2-\epsilon_1,q)+[g_1(E)-1]T(\epsilon_1+\epsilon_2,q)\}. \quad (4\text{-}7)$$

We defined

$$\epsilon_1=(E^2-\epsilon_0^2)^{\frac{1}{2}},\quad \epsilon_2=[(E+\omega)^2-\epsilon_0^2]^{\frac{1}{2}},\quad g_1(E)=(E^2+E\omega-\epsilon_0^2)/\epsilon_1\epsilon_2, \quad (4\text{-}8)$$

where for a negative argument we take $(-x)^{\frac{1}{2}}=ix^{\frac{1}{2}}$. In the limit $\epsilon_0\rightarrow 0$ this expression reduces to the value given in (3-28).

In evaluating σ^I for the longitudinal wave we encounter the same kind of difficulty as in the problem of the gauge invariance of the BCS theory. This is naturally to be expected from the expression (2-13), which involves a term identical in form to a longitudinal vector potential. Since what we did to get H_I is merely a coordinate transformation, the theory should be in the absence of impurities invariant against this transformation. To ensure this invariance it is necessary to consider not only the quasi-particle excitations but also the collective excitations, just as in the case of the gauge invariance. In the Appendix we shall give a proof with the aid of the random phase approximation generalized to finite temperatures that the theory is indeed invariant against the canonical transformation when there are no scattering centers.

In terms of the correlation functions (3-33)–(3-35), the response to H_I is equal to

$$\rho_I(q,\omega)=-e\int\int d\epsilon d\epsilon'\{L'^{(-)}(\epsilon,\epsilon',\omega)F_1(q,\epsilon,\epsilon')+L_0'(\epsilon,\epsilon',\omega)F_2(q,\epsilon,\epsilon')\}-\frac{q}{\omega}Neu, \quad (4\text{-}9)$$

where $L'=L-2\{[f(\epsilon)-f(\epsilon')]/(\epsilon-\epsilon')\}$ and L_0 is a new type of matrix element given by

$$L_0(\epsilon,\epsilon',\omega)=\frac{1}{2}\frac{E'\epsilon+E\epsilon'}{EE'}[f(E)-f(E')]\left(\frac{1}{E-E'-\omega-i\eta}-\frac{1}{E-E'+\omega+i\eta}\right)$$
$$-\frac{1}{2}\frac{E'\epsilon-E\epsilon'}{EE'}[1-f(E)-f(E')]\left(\frac{1}{E+E'-\omega-i\eta}-\frac{1}{E+E'+\omega+i\eta}\right). \quad (4\text{-}10)$$

[20] J. G. Valatin, Nuovo cimento 7, 843 (1958).

The appearance of L_0 for F_2 in contrast to $L^{(-)}$ for F_1 may be understood from the fact that the first and second term of H_I contains an even and odd power of momentum operators, respectively. With the help of (3-33), the above expression may be rewritten as

$$\rho_I(q,\omega)=-e\int\int d\epsilon d\epsilon'\left\{L'^{(-)}(\epsilon,\epsilon',\omega)\frac{\epsilon'-\epsilon}{\omega}-L_0'(\epsilon,\epsilon',\omega)\right\}F_2(q,\epsilon,\epsilon')$$
$$-e\int\int d\epsilon d\epsilon'\, L'^{(-)}(\epsilon,\epsilon',\omega)F_1'(q,\epsilon,\epsilon')-\frac{q}{\omega}Neu. \quad (4\text{-}11)$$

As was remarked above, in the absence of impurities the first term would vanish if we had used the correct matrix elements instead of the one given by the quasi-particle excitations. We assume that this is the case even in the presence of impurities. For a more rigorous treatment one has to set up the theory of the collective excitations in superconductors in the presence of scattering, which is beyond the scope of the present work. However, it is thought that the error which might result from this unsubstantiated assumption would be of the same order of magnitude as the direct contribution of the collective excitations to the absorption of the longitudinal sound wave, which was estimated by Rickayzen to be negligible at least for relatively low frequencies.[21]

Carrying out the ϵ' integral in the remaining expression, we finally get for σ^I,

$$\sigma^I=i\frac{e^2N(0)\omega}{q}\left(\frac{2}{\omega}J_1+\frac{1}{2\alpha}J_2\right), \quad (4\text{-}12)$$

where

$$J_1=\int_{\epsilon_0-\omega}^{\epsilon_0}dE[1-2f(E+\omega)]g_1(E)+2\int_{\epsilon_0}^{\infty}dE[f(E+\omega)-f(E)]g_1(E), \quad (4\text{-}13)$$

and

$$J_2=\int_{\epsilon_0-\omega}^{\epsilon_0}dE[1-2f(E+\omega)]\left\{\left(\frac{E+\omega}{\epsilon_2}+\frac{E}{\epsilon_1}\right)T(\epsilon_1-\epsilon_2,\,q)+\left(\frac{E+\omega}{\epsilon_2}-\frac{E}{\epsilon_1}\right)T(\epsilon_1+\epsilon_2,\,q)\right\}$$
$$+\int_{\epsilon_0}^{\infty}dE[1-2f(E+\omega)]\left\{\left(\frac{E+\omega}{\epsilon_2}+\frac{E}{\epsilon_1}\right)T(\epsilon_2-\epsilon_1,\,q)+\left(\frac{E+\omega}{\epsilon_2}-\frac{E}{\epsilon_1}\right)T(-\epsilon_1-\epsilon_2,\,q)\right\}$$
$$-\int_{\epsilon_0}^{\infty}dE[1-2f(E)]\left\{\left(\frac{E+\omega}{\epsilon_2}+\frac{E}{\epsilon_1}\right)T(\epsilon_2-\epsilon_1,\,q)-\left(\frac{E+\omega}{\epsilon_2}-\frac{E}{\epsilon_1}\right)T(\epsilon_1+\epsilon_2,\,q)\right\}. \quad (4\text{-}14)$$

In deriving (4-12)–(4-14) we have made use of (3-34) and the relations

$$\begin{aligned}[g_1(E)+1](\epsilon_2-\epsilon_1)&=\omega\left(\frac{E}{\epsilon_1}+\frac{E+\omega}{\epsilon_2}\right),\\ [g_1(E)-1](\epsilon_2+\epsilon_1)&=\omega\left(\frac{E}{\epsilon_1}-\frac{E+\omega}{\epsilon_2}\right).\end{aligned} \quad (4\text{-}15)$$

The above expression reduces to the correct normal value in the limit of $\epsilon_0\to 0$.

(b) Transverse Wave

In the present method the conductivity for the transverse field can be calculated from the expression

$$j_\alpha(q,\omega)=\frac{e^2}{m^2c}A_\alpha(q,\omega)\int\int d\epsilon d\epsilon'\, L'^{(+)}(\epsilon,\epsilon',\omega)F_3(q,\epsilon,\epsilon'), \quad (4\text{-}16)$$

$$F_3(q,\epsilon,\epsilon')=\int d\mathbf{y}\, e^{-i\mathbf{q}(\mathbf{y}-\mathbf{x})}\langle\lim_{\substack{x'\to x\\ y'\to y}}(\nabla_{x_\alpha}-\nabla_{x_\alpha'})(\nabla_{y_\alpha}-\nabla_{y_\alpha'})\Pi(x',y,y',x;\,\epsilon,\epsilon')\rangle_{\mathrm{av}}, \quad (4\text{-}17)$$

[21] G. Rickayzen, Phys. Rev. **115**, 795 (1959).

for the component parallel to $\mathbf{A}$ ($\mathbf{A}\cdot\mathbf{q}=0$) which we take in the direction of the x_α axis. As remarked before, if the scattering is isotropic the interference, or vertex correction vanishes in this case. Then,

$$F_3(q,\epsilon,\epsilon')=\frac{1}{(2\pi)^2}\int\frac{d\mathbf{p}}{(2\pi)^3}p_\alpha{}^2[\langle G_-(\mathbf{p}_2,\epsilon)\rangle_{\mathrm{av}}\langle G_+(\mathbf{p}_1,\epsilon')\rangle_{\mathrm{av}}+\langle G_+(\mathbf{p}_1,\epsilon)\rangle_{\mathrm{av}}\langle G_-(\mathbf{q}_2,\epsilon')\rangle_{\mathrm{av}}]$$
$$=\frac{1}{(2\pi)^2}\frac{3\pi mN}{4\alpha}[S(\epsilon-\epsilon',q)+\text{c.c.}], \tag{4-18}$$

where

$$S(x,q)=-\frac{2}{\alpha\tau}(1-ix\tau)+s(x,q)\left\{1+\frac{1}{\alpha^2\tau^2}(1-ix\tau)^2\right\}, \tag{4-19}$$

with $S(x,g)$ defined in (3-23). The final result is

$$\sigma_{(t)}(q,\omega)=-(3Ne^2/8m\alpha\omega)I_{(t)}(q,\omega), \tag{4-20}$$

where

$$I_{(t)}(q,\omega)=\int_{\epsilon_0-\omega}^{\epsilon_0}dE[1-2f(E+\omega)]\{[g_2(E)+1]S(\epsilon_1-\epsilon_2,q)+[g_2(E)-1]S(\epsilon_1+\epsilon_2,q)\}$$
$$-\int_{\epsilon_0}^{\infty}dE[1-2f(E+\omega)]\{[g_2(E)+1]S(\epsilon_2-\epsilon_1,q)+[g_2(E)-1]S(-\epsilon_2-\epsilon_1,q)\}$$
$$+\int_{\epsilon_0}^{\infty}dE[1-2f(E)]\{[g_2(E)+1]S(\epsilon_2-\epsilon_1,q)+[g_2(E)-1]S(\epsilon_2+\epsilon_1,q)\}, \tag{4-21}$$

with $g_2(E)=(E^2+E\omega+\epsilon_0{}^2)/\epsilon_1\epsilon_2$. This result can also be gotten simply by taking the Fourier transform of the Eqs. (3-3)–(3-5) of Mattis-Bardeen's article. The above expression reduces to the correct value for normal state in the limit of $\epsilon_0\rightarrow 0$:

$$\sigma_{N(t)}(q,\omega)=(3e^2N/4m\alpha)S(\omega,q). \tag{4-22}$$

Let us proceed to the evaluation of σ^I. The second term of H_I is identical in form with the electromagnetic interaction, so that this part, together with the "gauge current," Neu, gives the following contribution to the response current;

$$j_b{}^I(q,\omega)=-\frac{3\pi eNu}{4\alpha}\int\int\frac{d\epsilon d\epsilon'}{(2\pi)^2}L'^{(+)}(\epsilon,\epsilon',\omega)[S(\epsilon-\epsilon',q)+\text{c.c.}]. \tag{4-23}$$

From the first term of H_I we find the contribution

$$j_a{}^I(q,\omega)=i\frac{3\pi eNu}{4\omega\alpha}\int\int\frac{d\epsilon d\epsilon'}{(2\pi)^2}L_0'(\epsilon,\epsilon',\omega)\left\{\frac{1}{\tau}[S(\epsilon-\epsilon',q)-\text{c.c.}]-i(\epsilon-\epsilon')[S(\epsilon-\epsilon',q)+\text{c.c.}]\right\}. \tag{4-24}$$

Because of the identity

$$(\epsilon-\epsilon')L_0(\epsilon,\epsilon',\omega)=\omega L^{(+)}(\epsilon,\epsilon',\omega), \tag{4-25}$$

the proof of which is given at the end of the Appendix, the second term of (4-24) exactly cancels $j_b{}^I$, so that we are left with the term which vanishes for $\tau\rightarrow\infty$, as it should be. $L_0'(\epsilon,\epsilon',\omega)$ may be written under the integrals as

$$L_0=[1-2f(E)]\left\{\frac{\epsilon'+\epsilon-\epsilon\omega/E}{E'^2-(E-\omega-i\eta)^2}-\frac{\epsilon'+\epsilon+\epsilon\omega/E}{E'^2-(E+\omega+i\eta)^2}\right\}. \tag{4-26}$$

Thus we finally obtain

$$\sigma_{(t)}{}^I(q,\omega)=(3Ne^2/8\omega\alpha m)J_{(t)}(q,\omega), \tag{4-27}$$

where

$$J_{(t)}(q,\omega)=\int_{\epsilon_0-\omega}^{\epsilon_0}dE[1-2f(E+\omega)]\left\{\left(\frac{E+\omega}{\epsilon_2}+\frac{E}{\epsilon_1}\right)S(\epsilon_1-\epsilon_2,q)+\left(\frac{E+\omega}{\epsilon_2}-\frac{E}{\epsilon_1}\right)S(\epsilon_1+\epsilon_2,q)\right\}$$
$$+\int_{\epsilon_0}^{\infty}dE[1-2f(E+\omega)]\left\{\left(\frac{E+\omega}{\epsilon_2}+\frac{E}{\epsilon_1}\right)S(\epsilon_2-\epsilon_1,q)+\left(\frac{E+\omega}{\epsilon_2}-\frac{E}{\epsilon_1}\right)S(-\epsilon_1-\epsilon_2,q)\right\}$$
$$-\int_{\epsilon_0}^{\infty}dE[1-2f(E)]\left\{\left(\frac{E+\omega}{\epsilon_2}+\frac{E}{\epsilon_1}\right)S(\epsilon_2-\epsilon_1,q)-\left(\frac{E+\omega}{\epsilon_2}-\frac{E}{\epsilon_1}\right)S(\epsilon_1+\epsilon_2,q)\right\}. \tag{4-28}$$

In the limit of $\epsilon_0\rightarrow 0$, one can show that $\sigma_{(t)}{}^I\rightarrow\sigma_{N(t)}$.

5. ATTENUATIONS IN SUPERCONDUCTORS

Since the final integration in these expressions obtained for σ and σ^I cannot be carried out analytically, further discussions have to specialize in various limiting cases. We first note that the frequencies of ultrasonic waves available are much smaller than the gap frequency except at T very close to the critical temperature. (The gap frequency is, say for tin at $T=0.8T_c$, 1.2×10^{12} cps whereas the highest frequency so far attained is of the order of 10^{10} cps.) Unlike the problem of the surface impedance, we have a definite dispersion relation in our problem, namely, $\omega=qv_s$ which, together with the fact that $v_s/v_0\sim10^{-2}\ll1$, simplifies our analysis considerably.

(a) Longitudinal Wave: $\omega\ll\varepsilon_0(T)$

Let us first look at the imaginary part of σ given by (4-6) and (4-7). In order to estimate the first integral of (4-7) we put the arguments of the T functions equal to ω. This is actually not permissible unless $qv_0\gg(2\epsilon_0\omega)^{\frac{1}{2}}$, but presently we are interested in the order of magnitude. Then, this term can be expressed in terms of elliptic integrals and its contribution to σ turns out to be approximately equal to

$$-i\frac{\pi e^2N(0)\omega^3}{q^2\alpha\epsilon_0}[1-2f(\epsilon_0)].$$

Therefore, compared to the first term in (4-6), which is the Thomas-Fermi screening factor, it is at least smaller by v_s/v_0 and is negligible. The remaining two integrals are approximated by

$$4\int_{\epsilon_0}^{\infty}dE[f(E+\omega)-f(E)]T(\epsilon_2-\epsilon_1,\,q)$$

since $g_1(E)$ approaches 1 very rapidly as E departs from ϵ_0; an increase by a few ω is enough to make g_1-1 negligibly small. Noting that $\epsilon_2-\epsilon_1\leq(2\epsilon_0\omega)^{\frac{1}{2}}/\alpha$ and approaches ω as E increases, we may replace $T(\epsilon_2-\epsilon_1)$ again by $T(\omega)$. Thus, the real part of σ is given by

$$\sigma_1'\equiv\sigma_1/\sigma_0\approx3\left(\frac{v_s}{v_0}\right)^2\frac{1}{ql}T(\omega,q)f(\epsilon_0)$$
$$\approx6\left(\frac{v_s}{v_0}\right)^2\frac{1}{ql}\frac{\tan^{-1}(ql)}{1-\tan^{-1}(ql)/ql}f(\epsilon_0). \quad (5\text{-}1)$$

The imaginary part is equal to

$$\sigma_2'\equiv\sigma_2/\sigma_0=-(3/ql)(v_s/v_0). \quad (5\text{-}2)$$

In evaluating σ^I we take advantage of the fact that $\sigma_1'^I$ and $\sigma_2'^I$ appear in the expression for the attenuation constant (2-23) each multiplied by σ_1' and σ_2', respectively. Hence $\sigma_2'^I$ is more important. The similar approximations as above yield

$$\sigma_2'^I\approx-\frac{6}{ql}\left(\frac{v_s}{v_0}\right)f(\epsilon_0), \quad (5\text{-}3)$$

and $\sigma_1'^I$ is proportional to $(v_s/v_0)^2$. Substituting (5-1)–(5-3) into (2-22) we get

$$\alpha_s\simeq\frac{mNv_0}{\rho_{\text{ion}}v_sl}\left[\frac{1}{3}\frac{(ql)^2\tan^{-1}(ql)}{ql-\tan^{-1}(ql)}-1\right]2f(\epsilon_0). \quad (5\text{-}4)$$

Therefore, the ratio is,

$$\alpha_s/\alpha_n=2f(\epsilon_0), \quad (5\text{-}5)$$

just as in a pure superconductor.

(b) Longitudinal Wave: $\omega>2\varepsilon_0(T)$; $\beta\varepsilon_0(T)$, $\beta\omega\ll1$; $l=\infty$

In view of the recent progress in generating high frequency sound waves, it may be of interest to study the attenuation in the region of temperature just below T_C, where ω exceeds the gap frequency. For simplicity we put $l=\infty$. Further, since in this case $qv_0/\epsilon_0\gg1$ we may set $T(x,q)\approx\pi$ as in the extreme anomalous limit. Because $\omega>2\epsilon_0$, a real part appears in the first integral of (4-7), which corresponds to the direct excitation of quasi-particles across the gap. It is approximated by

$$I_1=-\pi\beta\epsilon_0\omega\left\{\left(1+\frac{\omega}{2\epsilon_0}\right)E(k)-\frac{\omega}{\epsilon_0}\left(1+\frac{\omega}{2\epsilon_0}\right)^{-1}K(k)\right\}, \quad (5\text{-}6)$$

where E and K are the complete elliptic integrals and $k=|(\omega/\epsilon_0)-2|/|(\omega/\epsilon_0)+2|$. We have expanded $f(E)$ in terms of βE. The remaining real part of (4-7) may be evaluated by the following approximations:

$$-4\pi\int_{\epsilon_0}^{\infty}dE[f(E)-f(E+\omega)]g_1(E)$$
$$\approx-4\pi\int_{\epsilon_0}^{\infty}dE\,E(E^2-\epsilon_0^2)^{\frac{1}{2}}[f(E)-f(E+\omega)]$$
$$\approx-4\pi\omega\int_0^{\infty}dy\exp[(y^2+\beta^2\epsilon_0^2)^{\frac{1}{2}}]$$
$$\times\{\exp[(y^2+\beta^2\epsilon_0^2)^{\frac{1}{2}}]+1\}^{-2}=-2\pi\omega\rho_n(T)/\rho, \quad (5\text{-}7)$$

where ρ_n is the density of the normal component, $\rho_n(T)/\rho\equiv1-\Lambda(0)/\Lambda(T)$.[4] We emphasize that this expression is valid only when $\omega\gg\epsilon_0(T)$ and $\beta\epsilon_0(T)\gg1$. Thus the ratio to the normal value, $\sigma_{1n}=\pi N(0)e^2\omega^2/q^3v_0$

is equal to

$$\frac{\sigma_1}{\sigma_{1n}}=\tfrac{1}{2}\beta\epsilon_0\left\{\left(1+\frac{\omega}{2\epsilon_0}\right)E(k)-\frac{\omega}{\epsilon_0}\left(1+\frac{\omega}{2\epsilon_0}\right)^{-1}K(k)\right\}+\rho_n(T)/\rho, \quad (5\text{-}8)$$

which is α_s/α_n in this case. The first term represents the extra absorption due to breaking up of the ground pairs. An estimate of (5-8) for $\omega/\epsilon_0(0)=1/10$ seems to give a peak just below T_C, but its height exceeds 1 by only a few percent. For higher frequencies this effect would become more noticeable.

(c) Transverse Wave: $\omega<2\epsilon_0(T)\ll qv_0$

If these conditions are satisfied, one can take the extreme anomalous limit, replacing S factors by a constant $S(\omega,q)$ in (4-21). Then we have

$$\sigma_{(t)}=\frac{3Ne^2}{4\alpha m\omega}(I_a+I_b)S(\omega,q), \quad (5\text{-}9)$$

where

$$I_a=\int_{\epsilon_0-\omega}^{\epsilon_0}dE[1-2f(E+\omega)]g_2(E),$$
$$I_b=2\int_{\epsilon_0}^{\infty}dE[f(E)-f(E+\omega)]g_2(E). \quad (5\text{-}10)$$

Similarly from (4-27) and (4-28) we get

$$\sigma_{(t)}{}^{I}=\frac{3Ne^2}{4\alpha m}S(\omega,q)=\sigma_{N(t)}. \quad (5\text{-}11)$$

Using the expression for the ratio

$$\frac{\alpha_s}{\alpha_n}=\left|\frac{\sigma_N}{\sigma_{(t)}}\right|^2\times\frac{\sigma_{1(t)}/\sigma_N-(\sigma_N\sigma_0)^{-1}(\sigma_{1(t)}\sigma_{1(t)}{}^{I}+\sigma_{2(t)}\sigma_{2(t)}{}^{I})}{1-\sigma_N/\sigma_0} \quad (5\text{-}12)$$

and neglecting v_s/v_0, we get

$$\frac{\alpha_s}{\alpha_n}\approx\left|\frac{\sigma_N}{\sigma_{(t)}}\right|^2\frac{I_b}{\omega}\approx\left|\frac{\sigma_1{}^{\mathrm{ex}}}{\sigma_N}\right|\Big/\left|\frac{\sigma_2{}^{\mathrm{ex}}}{\sigma_N}\right|^2, \quad (5\text{-}13)$$

where σ^{ex} is the value of a pure superconductor at the extreme anomalous limit. This ratio is clearly much smaller than unity.

In the opposite case of $qv_0\ll\epsilon_0$ the mean free path enters into the conductivities in a very complicated manner, so that we have not been able to see whether the discontinuous drop becomes small as $ql\to 0$.

(d) Transverse Wave: $qv_0\ll\epsilon_0(T)$; $l=\infty$

The conductivity σ in this case has been given by Khalatnikov and Abrikosov[22]:

$$\frac{\sigma_1}{\sigma_N}=1,\quad \frac{\sigma_2}{\sigma_N}=\frac{4\alpha}{3\pi\omega}\frac{\Lambda}{\Lambda_T}, \quad (5\text{-}14)$$

which leads to

$$\frac{\alpha_s}{\alpha_n}=\left[1+\left(\frac{4}{3\pi}\frac{v_0}{v_s}\frac{\Lambda}{\Lambda_T}\right)^2\right]^{-1}. \quad (5\text{-}15)$$

Because Λ/Λ_T rapidly approaches unity below T_C, α_s/α_n again drops to a negligible value.

6. CONCLUDING REMARKS

Let us first discuss the attenuation of longitudinal waves. Most of the measurements have been performed in the frequency range 30–100 Mc/sec, so that our condition $\omega\ll\epsilon_0$ of Sec. 5(a) is amply satisfied. Our result is then independent of ql and fails to explain the observed fact that α_s/α_n decreases slightly more steeply with decreasing temperature, the smaller the value of ql.[1,5] The error due to the approximation ($g_1\approx 1$) seems to be too small to affect this conclusion. Two other assumptions have been made in our calculation: the effect of deformation on the attractive electron-electron interaction is ignored and the use is made of the result obtained from the theory of collective excitation in the absence of scattering. In these respects further improvement of the theory would be necessary. It seems, however, that speaking only of the theoretical side, there are other factors that need to be taken into account before expecting any better agreement, such as the anisotropic energy gap[23] and the scattering of quasi-particle excitations by phonons.

An attenuation measurement at a microwave frequency would be of considerable interest, because it may be possible to observe the absorption due to the excitations across the gap [Sec. 5(b)] as well as to study the effect of the collective excitations.

In the case of a transverse wave our analysis has shown that the attenuation due to the electromagnetic interaction drops abruptly to a negligible value at T_C owing to the appearance of the large value of σ_2. This confirms the qualitative explanation that the discontinuous drop is due to the strong screening of the transverse field by the Meissner effect.[1,2] As pointed out by Morse,[1] the subsequent gradual decrease of α_s observed seems to indicate two distinct processes for

[22] I. M. Khalatnikov and A. A. Abrikosov, *Advances in Physics*, edited by N. F. Mott (Taylor and Francis, Ltd., London, 1959), Vol. 8, p. 45.

[23] R. W. Morse, T. Olsen, and J. D. Gavenda, Phys. Rev. Letters **3**, 15 (1959); **4**, 193 (Erratum) (1959). P. O. Bezuglyi, A. A. Galkin, and A. P. Karolyuk, J. Exptl. Theoret. Phys. U.S.S.R. **36**, 1951 (1959) [translation: Soviet Phys.-JETP **36(9)**, 1388 (1959)].

attenuating the transverse wave, one electromagnetic, discussed here, and the other due to some real metal effects which may be described by a deformation potential. If one assumes a phenomenological potential, as in the work by Blount,[9] this latter effect may be calculated by a quantum mechanical treatment similar to the present one.

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APPENDIX

For the purpose of proving the invariance of the theory in the absence of scattering centers against the canonical transformation leading to (2-13), it is more convenient to resort to a formalism slightly different from the one used in the text, which was developed by Anderson[24] and Rickayzen[21] on the basis of the set of linear equations of motion. We shall use the form given by Rickayzen: for the detail of the derivations of the basic equations and for the notation used, the reference should be made to his article. Since only the case of zero temperature was treated in the latter article we first generalize the theory of random phase approximation to finite temperature. The procedure is simple; one linearizes the equations of motion for the pairs of quasi-particle operators by replacing $n_{k\sigma}=c_{k\sigma}^+c_{k\sigma}$, $b_k=c_{-k\downarrow}c_{k\uparrow}$ and b_k^+ by their average values at finite temperatures instead of their expectation values in the BCS ground state. One expects this approximation would become worse as temperature increases. The resulting equations of motion are as follows.

$$
\begin{aligned}
[H,\gamma_{k+q0}^+\gamma_{k1}^+]&=(E_k+E_{k+q})\gamma_{k+q0}^+\gamma_{k1}^+ +(1-f_k-f_{k+q})\{V_D(q)m(k,q)\rho(q)+\tfrac{1}{2}n(k,q)B_k(q)-\tfrac{1}{2}l(k,q)A_k(q)\},\\
[H,\gamma_{k+q1}\gamma_{k0}]&=-(E_k+E_{k+q})\gamma_{k+q1}\gamma_{k0}-(1-f_k-f_{k+q})\{V_D(q)m(k,q)\rho(q)+\tfrac{1}{2}n(k,q)B_k(q)+\tfrac{1}{2}l(k,q)A_k(q)\},\\
[H,\gamma_{k+q0}^+\gamma_{k0}]&=(E_{k+q}-E_k)\gamma_{k+q0}^+\gamma_{k0}+(f_{k+q}-f_k)\{-V_D(q)n(k,q)\rho(q)+\tfrac{1}{2}m(k,q)B_k(q)+\tfrac{1}{2}p(k,q)A_k(q)\},\\
[H,\gamma_{k1}^+\gamma_{k+q1}]&=-(E_{k+q}-E_k)\gamma_{k1}^+\gamma_{k+q1}-(f_{k+q}-f_k)\{-V_D(q)n(k,q)\rho(q)+\tfrac{1}{2}m(k,q)B_k(q)-\tfrac{1}{2}p(k,q)A_k(q)\}.
\end{aligned}
\tag{A-1}
$$

Here $V_D(q)$ is the direct interaction between electrons, hence predominantly the Coulomb interaction. When we calculate the effect of the perturbation H_I, we can drop terms with the direct interactions, since we shall use the screened potential.

The second term of H_I, (2-13) is identical in form to a longitudinal vector potential coupling, so that invoking the gauge invariance already proved we may replace it by an equivalent scalar potential $e\phi'=m\omega uq^{-1}$. Hence, effectively we have

$$
H_I=\textstyle\sum_k h(k,q)\{\rho_k{}^q+\bar{\rho}_k{}^q\}^*-e\phi'\rho(q),
\tag{A-2}
$$

where

$$
h(k,q)=-\frac{1}{m\omega}\left[\mathbf{q}\cdot\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)\right]\left[\mathbf{u}\cdot\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)\right]=-\frac{mu}{q\omega}(\epsilon_{k+q}-\epsilon_k)^2.
$$

Adding this to the Hamiltonian we can derive the equations of motion for the pairs of quasi-particle operators with the driving terms and from them the equations determining the collective variables $\rho(q)$, $A_k(q)$, and $B_k(q)$ to the first order in H_I. The driving terms in the equation for $A_k(q)$ can be transformed as follows:

$$
\begin{aligned}
-m(k,q)[h(k,q)-\phi']S_{(+)}&=-\frac{2mu}{\omega q}(\epsilon_{k+q}-\epsilon_k)p(k,q)+\frac{2mu}{q}l(k,q)\epsilon_0S_{(-)},\\
n(k,q)[h(k,q)-\phi']S_{(+)}'&=-\frac{2mu}{\omega q}(\epsilon_{k+q}-\epsilon_k)l(k,q)-\frac{2mu}{q}p(k,q)\epsilon_0S_{(-)}',
\end{aligned}
\tag{A-3}
$$

where

$$
\begin{aligned}
S_{(\pm)}&=(E_{k+q}+E_k+\omega+i\eta)^{-1}\pm(E_{k+q}+E_k-\omega-i\eta)^{-1},\\
S_{(\pm)}'&=(E_{k+q}-E_k+\omega+i\eta)^{-1}\pm(E_{k+q}-E_k-\omega-i\eta)^{-1}.
\end{aligned}
\tag{A-4}
$$

We have used the following identities:

$$
\begin{aligned}
m(k,q)(E_{k+q}+E_k)&=-p(k,q)(\epsilon_{k+q}-\epsilon_k)+2\epsilon_0l(k,q),\\
n(k,q)(E_{k+q}-E_k)&=l(k,q)(\epsilon_{k+q}-\epsilon_k)+2\epsilon_0p(k,q).
\end{aligned}
\tag{A-5}
$$

[24] P. W. Anderson, Phys. Rev. **112**, 1900 (1959).

In this way we can reduce the equations for the collective variables:

$$\rho(q)=\sum_k\Bigg\{(1-f_k-f_{k+q})\Bigg[-\frac{2mu}{\omega q}(\epsilon_{k+q}-\epsilon_k)p(k,q)+\tfrac{1}{2}\bigg(A_k(q)+\frac{4mu\epsilon_0}{q}\bigg)l(k,q)S_{(-)}\Bigg]m(k,q)$$
$$+(f_{k+q}-f_k)\Bigg[-\frac{2mu}{\omega q}(\epsilon_{k+q}-\epsilon_k)l(k,q)-\tfrac{1}{2}\bigg(A_k(q)+\frac{4mu\epsilon_0}{q}\bigg)p(k,q)S_{(-)}'\Bigg]n(k,q)\Bigg\}, \quad \text{(A-6)}$$

$$A_K(q)=-\sum_k V(\mathbf{K},\mathbf{k})\Bigg\{(1-f_k-f_{k+q})\Bigg[-\frac{2mu}{q}m(k,q)+\tfrac{1}{2}\bigg(A_k(q)+\frac{4mu\epsilon_0}{q}\bigg)l(k,q)S_{(+)}\Bigg]l(k,q)$$
$$+(f_{k+q}-f_k)\Bigg[\frac{2mu}{q}n(k,q)-\tfrac{1}{2}\bigg(A_k(q)+\frac{4mu\epsilon_0}{q}\bigg)p(k,q)S_{(+)}'\Bigg]p(k,q)\Bigg\}, \quad \text{(A-7)}$$

where we have omitted $B_K(q)$, which is negligible because of the symmetry around the Fermi surface. In the last equation the driving term for $A_k(q)$ can be shown to be equal to $-4mu\epsilon_0q^{-1}$. Therefore, if we take $A_k(q)$ to be $-4mu\epsilon_0q^{-1}$, we are left with

$$\rho(q)=-\frac{2mu}{\omega q}\sum_k\{(1-f_k-f_{k+q})m(k,q)p(k,q)+(f_{k+q}-f_k)n(k,q)l(k,q)\}(\epsilon_{k+q}-\epsilon_k)$$
$$=-q\omega^{-1}Nu. \quad \text{(A-8)}$$

This completes our proof of the invariance.

For the transverse case, $\mathbf{u}\cdot\mathbf{q}=0$, it can be readily seen that the collective excitations are not important unless $V(\mathbf{K},\mathbf{k})$ is anisotropic. The invariance is maintained even if we limit our consideration to the quasi-particle excitations. The proof rests on the relation (4-26) which may be shown to be true if one notes that in the present notation

$$L'(\epsilon,\epsilon',\omega)=p(k,q)m(k,q)(1-f_k-f_{k+q})S_{(-)}-l(k,q)n(k,q)(f_k-f_{k+q})S_{(-)}', \quad \text{(A-9)}$$

and the identities

$$\begin{aligned} m(k,q)(\epsilon_{k+q}-\epsilon_k)&=-p(k,q)(E_{k+q}+E_k),\\ n(k,q)(\epsilon_{k+q}-\epsilon_k)&=l(k,q)(E_{k+q}-E_k). \end{aligned} \quad \text{(A-10)}$$