importance, as has been suggested by Castner and Känzig, and ought to influence the construction of the trial function, especially the nature of all the polarized orbitals in the vicinity of the hole. The proper way of dealing with the problem is presumably to set up a quite general wave function in which both the positions of the ions that determine the displaced equilibrium configuration and the linear combinations of ground and excited free-ion functions that determine the polarized orbitals are left free to be determined by an over-all variational calculation, instead of being specified from the beginning. Such a procedure would be very much more difficult since it would no longer be possible to isolate terms characteristic of a hole-free crystal, whose behavior can be calculated from the experimental properties of the macroscopic crystal. Furthermore, neither the excited orbitals for the K^+ and Cl^- ions nor multicenter matrix elements involving these orbitals are available. It will be seen that such a calculation lies outside the scope of the present work.

In conclusion, it has been possible to show that one can expect a valence-band hole in its ground state to be self-trapped; however, the details of the associated electronic and ion core configurations cannot be predicted without additional calculation.

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Frequency-Dependent Hall Effect in Normal and Superconducting Metals*†‡

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The Hall current flow occurring in a normal and a superconducting metal when both a static magnetic field (H_0) and an electromagnetic wave are applied on the metal is calculated. The entire frequency range of the electromagnetic wave is discussed although the emphasis is on the microwave range. The nonlocal, transverse Hall current in a normal metal is calculated by solving the Boltzmann equation. It is shown that the microwave Kerr rotation in a circular cylindrical cavity provides a good test for the nonlocal Hall current in a normal metal. The relation between a longitudinal and a transverse Hall current in a superconductor is briefly discussed. A detailed theory of the transverse Hall current in a superconductor based on the Bardeen-Cooper-Schrieffer model and including the effect of collective excitations is presented. The field H_0 is assumed constant in space and a general result for the Hall current in Q space is derived. When the electric field is constant in space ($Q\rightarrow 0$), it is shown that the Hall current is proportional to the microscopic analog of the fraction of normal electrons of a two-fluid model.

I. INTRODUCTION

THE Hall effect at audio frequencies is a well understood phenomenon in both metals and semiconductors. The experimental method used at these very low frequencies is a simple measurement of the Hall emf developed across the sample when a current flows in the sample and a static magnetic field is applied perpendicular to the current flow. The theory of this effect yields the simple and well-known result (we neglect any effect of band structure throughout),

$$j_{\mathrm{Hall}}=-R_0\sigma_0^2\mathbf{E}\times\mathbf{H}_0, \tag{1}$$

where

$$\sigma_0=ne^2\tau/m,\quad R_0=-(nec)^{-1}, \tag{2}$$

and where H_0 is the static magnetic field applied and n is the number of carriers per unit volume. We use the convention that R_0 is positive for electrons and negative for holes.

In general a Hall current or a Hall electric field will be produced by a microwave or an optical electric field with a static magnetic field perpendicular to the applied electric field. A simple measurement of a Hall emf is no longer feasible at these high frequencies. If a plane polarized electromagnetic wave is incident on a sample and if there is a static magnetic field present which is perpendicular to the incident electric field then both the

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reflected and the transmitted waves will have their planes of polarization rotated from the incident polarization and will be elliptically polarized. When we discuss the reflected wave it is customary to call it a Kerr (magneto) rotation whereas the transmitted wave is referred to as a Faraday rotation. Both the Kerr and the Faraday rotations yield information about Hall currents and have been experimentally measured in semiconductors, ferromagnetics and to a much smaller extent in normal metals.

In both semiconductors and ferromagnetics a local relation between current and electromagnetic fields is valid. In this case one may define complex indices of refraction for a right-handed polarized wave propagating through the sample (N_+) and for a left-handed circularly polarized wave (N_-). By a simple electromagnetic argument one may show that if the incident beam is plane polarized the reflected beam becomes elliptically polarized with the major axes rotated by the Kerr angle Φ_K and with ellipticity ϵ_K where,

$$\Phi_K = -\operatorname{Im}(N_+ - N_-)/(N^2-1), \tag{3}$$

$$\epsilon_K = -\operatorname{Re}(N_+ - N_-)/(N^2-1). \tag{4}$$

The transmitted beam has a Faraday rotation θ_F and an ellipticity ϵ_F given by,

$$\theta_F = (\omega d/2c)\operatorname{Re}(N_+ - N^-), \quad \text{for } \omega d(N_+ - N_-)/2c \ll 1. \tag{5}$$

$$\epsilon_F = (\omega d/2c)\operatorname{Im}(N_+ - N_-), \quad \text{for } \omega d(N_+ - N_-)/2c \ll 1. \tag{6}$$

Here d is the thickness of the sample.

The complex indices of refraction N_+ and N_- may be expressed in terms of the conductivity and polarizability tensors of the solid.[1] The conductivity ($\boldsymbol{\sigma}$) and polarizability ($\boldsymbol{\alpha}$) tensors are defined by

$$\mathbf{J} = \frac{1}{4\pi}\frac{\partial E}{\partial \mathbf{t}} + \boldsymbol{\alpha}\frac{\partial E}{\partial \mathbf{t}} + \boldsymbol{\sigma}\mathbf{E}. \tag{7}$$

Then if we let H_0 be along the z axes,

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_0 & -\alpha_1 & 0 \\ \alpha_1 & \alpha_0 & 0 \\ 0 & 0 & \alpha_0 \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_0 & -\sigma_1 & 0 \\ \sigma_1 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}, \tag{8}$$

and one finds for N_+ and N_-

$$\begin{aligned} N_+^2 &= (1+4\pi\alpha_0+4\pi\sigma_0/i\omega) - i(4\pi\alpha_1+4\pi\sigma_1/i\omega), \\ N_-^2 &= (1+4\pi\alpha_0+4\pi\sigma_0/i\omega) + i(4\pi\alpha_1+4\pi\sigma_1/i\omega). \end{aligned} \tag{9}$$

The conductivity and polarizability tensors are to be found from a microscopic model of the solid.

The Faraday rotation in semiconductors at microwave frequencies has been observed in several experiments.[2,3] The sample is inserted into a circular waveguide which has two degenerate TE_{11} modes. H_0 is applied along the axis of the waveguide and perpendicularly to the surface of the semiconductor. Using a simple free electron model, Rau and Caspari deduced a value for the Hall mobility from the measured rotation which was in approximate agreement with other dc experiments on the Hall mobility.

There has been a large amount of experimental work on the Faraday and Kerr effects in ferromagnetics, both in the microwave and optical frequency range.[4] In general these effects are several orders of magnitude larger than in normal metals; this is related to the large Hall effect in ferromagnetics. Experiments have shown that the Faraday, Kerr, and Hall effects are proportional to the net magnetization of the sample and not to the external magnetic field as is the case with the nonferromagnetic solids. Argyres has given a microscopic theory of the Faraday and Kerr effects in ferromagnetics at optical frequencies.[1]

The experimental data on normal metals in both the microwave and optical region are very meager up to the present time, mainly because the Faraday and Kerr rotations are several orders of magnitude smaller than in ferromagnetic metals. For example the Kerr rotation at optical frequencies is about a thousand times smaller in Ag than in magnetized iron.[5] The Kerr rotation at optical frequencies in nonferromagnetic metals was first observed by Majorana.[5] He was able to detect small rotations of the order of 0.01 minute per kilogauss by the use of a sensitive photoelectric detector. In the microwave region, the only published experiment at present is that of Cooke.[6] Cooke observed the Kerr rotation in a circular cylindrical cavity with two degenerate TE_{11} modes. The metal sample formed the end plate of the cavity and there was a static magnetic field perpendicular to the sample surface and along the axes of the cavity. Cooke was able to observe a rotation in several metals including bismuth, iron, and nickel, but no quantitative data on the rotation was given. He also observed that the angle of rotation increased with H_0. In Sec. III, we shall give a detailed theory of the Kerr rotation in a circular cylindrical cavity such as the one used by Cooke.

One can account for Majorana's results at optical frequencies by the following simple theory. At optical frequencies for good conductors we have $\omega\tau \gg 1$. Use the result of Sec. II for the Hall current when $\omega\tau \gg 1$,

$$j_{\text{Hall}} = R_0\sigma_0^2 E H_0/(i\omega\tau)^2. \tag{10}$$

Thus the conductivity and polarizability tensors are

$$\sigma_1 = -R_0\sigma_0^2 H_0/\omega^2\tau^2, \quad \alpha_1 = 0. \tag{11}$$

Let

$$N = n - ik, \tag{12}$$

[1] See, for example, Petros N. Argyres, Phys. Rev. **97**, 334 (1955).
[2] R. R. Rau and M. E. Caspari, Phys. Rev. **100**, 632 (1955).
[3] H. Suhl and G. L. Pearson, Phys. Rev. **92**, 858 (1953).
[4] See, for example, H. Konig, J. Optik **3**, 101 (1948); and C. Hogan, Revs. Modern Phys. **25**, 253 (1953).
[5] Q. Majorana, Nuovo cimento **2**, 1 (1944).
[6] S. P. Cooke, Phys. Rev. **74**, 701 (1948).

then from (3) we get

$$\phi_K = \frac{4\pi}{\omega} \operatorname{Im}\sigma_1 / \{(n-ik)[(n-ik)^2-1]\}. \tag{13}$$

We may get approximate values for n and k at optical frequencies by neglecting any polarization current[7]

$$j = \sigma_0 E/(1+i\omega\tau) + i\omega E/4\pi. \tag{14}$$

One then finds,

$$n^2-k^2 = 1-4\pi\sigma_0/\omega^2\tau, \quad 2nk = 4\pi\sigma_0/\omega^3\tau^2. \tag{15}$$

Thus from (13) and (15)

$$\phi_K = \frac{\sigma_0 R_0 H_0}{\omega\tau} \frac{1}{(4\pi\sigma_0/\omega^2\tau - 1)^{\frac{1}{2}}}. \tag{16}$$

The theory of Eq. (16) has been compared to the data of Majorana in Table I. A positive rotation Φ_K means a rotation from the x to the y axes with the reflected beam traveling along the $-z$ axis (xyz is a right-handed triad). We see that the simple theory is able to account for both the sign and the order of magnitude of the observed rotation and hence conclude that the Kerr rotation observed by Majorana at optical frequencies may be accounted for by a Hall current given by Eq. (11). A Kerr rotation at optical frequencies in normal metals has also been observed recently by Stern and Myers.[8]

We define a "transverse" Hall current to be a current whose divergence is zero and hence there is no associated charge density. For example a transverse Hall current flows in the experiment of Cooke. If the divergence of the Hall current is nonzero we shall call it a longitudinal Hall current and it will have an associated charge density. An example is the experiment of Spiewak at microwave frequencies where H_0 is parallel to the surface of the sample and the Hall current flows perpendicularly to the surface of the sample and has a nonzero divergence.[9]

An attempt to observe the Hall effect in superconductors at audio frequencies was made by Lewis.[10] The sample was a superconducting prolate spheroid with an audio frequency magnetic field applied. The emf between the equator and the pole was measured. A null result was found and Lewis concluded that the Hall coefficient R defined by,

$$\mathbf{E}_{\text{Hall}} = -R(\mathbf{J}\times\mathbf{H}_0), \tag{17}$$

was less than one fifth of its value in the normal state. We shall show that the result of the microscopic theory

TABLE I. Optical Kerr effect in normal metals; $\lambda\sim 5000$ A.

Metal	$(nec)^{-1}\times 10^{13}$ v-cm/amp-oersted (calculated)	$R_0\times 10^{13}$ (observed)	Φ_K (theory) min/kilogauss	Φ_K (Majorana expt) min/kilogauss
1. Ag	10.4	8.4	+0.0040	+0.0085
2. Au	10.5	7.2	+0.0046	+0.0095
3. Al	3.4	3.9		+0.0031
4. Pt		$\sim$2.0		+0.013
5. Bi	4.1	$\sim$1000		+0.0018

of Sec. IV is in qualitative accord with the null result at audio frequencies observed by Lewis.

No direct experiments have been reported as yet on the Hall current in superconductors in the microwave or the optical frequency range. However the Hall current does play a somewhat indirect role in the analysis of certain experiments on superconductors such as the magnetic field dependence of the surface impedance. For example in the experiment of Spiewak at microwave frequencies there is a longitudinal Hall electric field in the superconductor for that geometry where H_0 is perpendicular to the microwave electric field.[9] An analysis of the magnetic field dependence of the surface impedance for such an experiment includes effects due to a Hall current as well as magnetoresistance effects.[11] Both these effects give a contribution to the magnetic field dependence of the surface impedance which is quadratic in H_0 for small H_0. A more direct measure of the Hall current in a superconductor would be a measurement of a Faraday or a Kerr rotation. A measurement of a microwave Kerr rotation in a circular cylindrical cavity with the sample forming the end plate of the cavity and H_0 along the axis of the cavity (this is the geometry of Cooke,[6] as well as the geometry treated in Sec. II) cannot be considered for a bulk superconductor since the magnetic field inside the bulk superconductor will not be perpendicular to the surface and the sample would go into the intermediate state were such a field applied. However, if the sample were a thin superconducting film or in general, any small superconducting sample, the magnetic field H_0 could penetrate the sample and also be in the same direction as the axis of the cavity. We shall discuss the approximate dimensions required for this purpose in Sec. IV.

In Sec. II, we derive a relation for the nonlocal, transverse Hall current in the normal metal by solving the Boltzmann equation. In Sec. III, we give a detailed analysis of the microwave Kerr rotation in a circular cylindrical cavity with the normal metal forming the end plate of the cavity. The result for the Kerr rotation in the cavity shows that this experiment provides a good test for the validity of the nonlocal Hall current in normal metals.

In Sec. IV, we give a qualitative discussion of the

[7] See, for example, N. F. Mott and H. Jones, *Theory of the Properties of Metals and Alloys* (Dover Publications Inc., New York, 1936).

[8] E. A. Stern and R. D. Myers, Bull. Am. Phys. Soc. **3**, 416 (1958); E. A. Stern, Bull. Am. Phys. Soc. **5**, 150 (1960).

[9] M. Spiewak, Phys. Rev. **113**, 1479 (1959).

[10] H. W. Lewis, Phys. Rev. **92**, 1149 (1953); and Phys. Rev. **100**, 641 (1955).

[11] G. Dresselhaus and M. S. Dresselhaus, Phys. Rev. **118**, 77 (1960).

relation between a longitudinal and a transverse Hall current in a superconductor. We then give a detailed microscopic theory of the transverse Hall current in a superconductor based on the Bardeen-Cooper-Schrieffer model of the superconductor and including the effect of collective excitations by means of the generalized random phase approximation given by Anderson and by Rickayzen.[12,13] The final result for the Hall current is expressed in Q space. For small Q this gives the simple result that the Hall current is proportional to the microscopic analog of the fraction of normal electrons of a two-fluid model.

In Sec. IV, we also show that our result is in qualitative accord with the null result of Lewis at audio frequencies. We also discuss the application of our result for arbitrary Q to small superconducting samples such as thin films.

II. HALL CURRENT IN NORMAL METAL FROM THE BOLTZMANN EQUATION

In this section we treat the transverse case where the static magnetic field H_0 is perpendicular to the semi-infinite metal. In general we shall find a nonlocal relation between the Hall current and the electric field; this is similar to the nonlocal relation between current and field in the theory of the anomalous skin effect. The discussion of a nonlocal Hall current has already been treated by several authors. A quantum theory of the nonlocal Hall current (transverse case) was first given by Mattis.[14] His result was very similar to the equation to be derived in this section, but his derivation was somewhat incomplete due to neglect of certain terms in the Hamiltonian and the related problem of the choice of vector potential for H_0. In Sec. IV we present a quantum theory for the Hall current in both normal and superconducting metals which includes all the terms of the Hamiltonian. The result of Sec. IV for the normal metal is identical to the result to be obtained in this section by solving the Boltzmann equation. A frequency dependent Hall effect has also been discussed by Donovan.[15] The Hall current has also been included in discussions of the magnetic field dependence of ultrasonic attenuation in metals. For example, Kjeldaas and also Cohen, Harrison, and Harrison solved the Boltzmann equation and for the case of the sound wave parallel to H_0 they find a Hall conductivity,[16,17]

$$\sigma(q)=\tfrac{3}{4}R_0\sigma_0^2H_0\int_0^\pi \frac{d\theta\,\sin^3\theta}{[1-i(lq\cos\theta-\omega\tau)]^2}. \tag{1}$$

[12] P. W. Anderson, Phys. Rev. **114**, 1002 (1959).
[13] G. Rickayzen, Phys. Rev. **115**, 795 (1959).
[14] D. C. Mattis, thesis, University of Illinois, 1957 (unpublished).
[15] B. Donovan, Proc. Phys. Soc. (London) **A68**, 1026 (1955).
[16] T. Kjeldaas, Jr., Phys. Rev. **113**, 1473 (1959).
[17] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. **117**, 937 (1960).

To derive (1) one needs to assume

$$\omega_c\tau\ll 1, \tag{2}$$

where ω_c is the cyclotron frequency eH_0/mc.

The result to be derived in this section is identical to (1) although the derivation is carried out in a simple manner in real space by means of a general method due to Chambers.[18] Wtih the assumption that $\omega_c\tau\ll 1$, one finds for the Hall current in real space

$$\mathbf{J}_H(\mathbf{r},t)=\frac{3}{4\pi}\frac{R_0\sigma_0^2}{l^2}\int d^3R\frac{\mathbf{R}}{R^3}\times\mathbf{R}\cdot[\mathbf{E}(\mathbf{r}',\,t-R/v_0)\times\mathbf{H}_0]\exp(-R/l), \tag{3}$$

where

$$\mathbf{R}=\mathbf{r}-\mathbf{r}', \tag{4}$$

and v_0 is the Fermi velocity. An experimental test for the nonlocal Hall current given by (1) or equivalently by (3) is proposed in Sec. III on the microwave Kerr rotation.

Chambers has given a general solution to the Boltzmann equation which is a convenient starting point for the derivation of this section.[18] Heine has shown explicitly that Chambers solution satisfies the Boltzmann equation.[19]

The Chambers solution gives for the current

$$\mathbf{J}(\mathbf{r},t)=-\frac{2e^2}{h^3}\int d^3p\,\mathbf{v}\frac{\partial f_0}{\partial E}\int_c^t \mathbf{v}(\mathbf{r}',t')\cdot\mathbf{E}(\mathbf{r}',t')e^{-(t-t')/\tau}dt'. \tag{5}$$

The independent variables are $\mathbf{r}$, t, $\mathbf{p}$, and t'. The dependent variable $\mathbf{r}'$ is the position along its trajectory that an electron which has final momentum $\mathbf{p}$ at $(\mathbf{r},t)$ finds itself at time t'. Clearly the equation relating $\mathbf{r}'$ to the independent variables is determined by the equation of motion of the electron in the electric and magnetic fields present. The value of c depends on the boundary condition at the surface of the metal. For specular reflection $c=-\infty$ whereas for random scattering $c=-\infty$ except when the trajectory cuts the surface in which case c is the latest time prior to t that the trajectory cuts the surface. The Fermi function is denoted by f_0.

Consider a constant magnetic field H_0 perpendicular to the metal surface and an electric field $E(r,t)$ parallel to the surface. We keep terms in the current linear in E and in the product $\mathbf{E}\times\mathbf{H}_0$. The equation of motion of the electron gives

$$\mathbf{v}(\mathbf{r}',t')=\mathbf{p}/m+(e/c)\mathbf{v}\times\mathbf{H}_0, \tag{6}$$

where

$$\mathbf{v}=\mathbf{p}/m, \tag{7}$$

and $\mathbf{p}$ designates the momentum of the electron at $(\mathbf{r},t)$.

[18] R. G. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1952).
[19] V. Heine, Phys. Rev. **107**, 431 (1957).

From (5) and (6),

$$\mathbf{J}(\mathbf{r},t)=\frac{-2e^2}{h^3m^2}\int p^3 d\Lambda_p\,\hat{p}\frac{\partial f_0}{\partial E}\int_c^t\left[\frac{\mathbf{p}}{m}+\frac{e}{mc}\mathbf{v}\times\mathbf{H}_0(t-t')\right]\cdot\mathbf{E}(\mathbf{r}',t')e^{-(t-t')/\tau}dt',\quad(8)$$

where

$$\mathbf{p}\equiv p\hat{p}.\quad(9)$$

The integration over p is carried out to give

$$\mathbf{J}(\mathbf{r},t)=\frac{3ne^2v_0}{4\pi m}\int\hat{p}d\Lambda_p\int_c^t\left[\hat{p}+\frac{e}{mc}\hat{p}\times\mathbf{H}_0(t-t')\right]\cdot\mathbf{E}(\mathbf{r}',t')e^{-(t-t')/\tau}dt'.\quad(10)$$

All momenta appearing implicitly in (10) through the coordinate $\mathbf{r}'$ are to be evaluated at the Fermi surface. We now express the integration over momentum solid angle $d\Lambda_p$ in terms of the $\mathbf{R}$ solid angle $d\Lambda$. Since

$$\mathbf{R}=v_0\hat{p}(t-t')+\frac{ev_0}{2mc}(\hat{p}\times\mathbf{H}_0)(t-t')^2,\quad(11)$$

we have to first order in H_0 that

$$\hat{p}=\frac{1}{v_0}\left[\frac{\mathbf{R}}{t-t'}-\frac{e}{2mc}\mathbf{R}\times\mathbf{H}_0\right].\quad(12)$$

It is clear that (11) and hence (12) is valid only if the first term on the right-hand side of (11) is much larger than the second term. Since the important times in the integral are $t-t'\sim\tau$ then $\omega_c\tau\ll 1$. Thus (2) is a basic limitation on the validity of our derivation of the Hall current in the normal metal.

From (11), neglecting terms of order H_0^2,

$$|\mathbf{R}|=R=v_0(t-t').\quad(13)$$

Thus,

$$\mathbf{J}(\mathbf{r},t)=\frac{-3\sigma_0}{4\pi l v_0^2}\int d\Lambda_p\left(\frac{\mathbf{R}}{t-t'}-\frac{e}{2mc}\mathbf{R}\times\mathbf{H}_0\right)\times\int_{c'}^0 dR\left(\frac{\mathbf{R}}{t-t'}+\frac{e}{2mc}\mathbf{R}\times\mathbf{H}_0\right)\cdot\mathbf{E}(\mathbf{r}',t')e^{-R/l}.\quad(14)$$

Since,

$$d\Lambda_p=d\Lambda+\sim\omega_c\tau d\Lambda,\quad(15)$$

$$\mathbf{J}(\mathbf{r},t)=\frac{-3\sigma_0}{4\pi l v_0^2}\bigg\{v_0^2\int\int_{c'}^0 d\Lambda_p dR\,\frac{\mathbf{R}}{R}\frac{\mathbf{R}}{R}\cdot\mathbf{E}(\mathbf{r}',t')e^{-R/l}-\frac{ev_0}{2mc}\int\int_{c'}^0\frac{\mathbf{R}}{R}\mathbf{R}\cdot[\mathbf{E}(\mathbf{r}',t')\times\mathbf{H}_0]e^{-R/l}dRd\Lambda-\frac{ev_0}{2mc}\int\int_{c'}^0 dRd\Lambda\frac{\mathbf{R}}{R}\mathbf{E}(\mathbf{r}',t')\cdot(\mathbf{R}\times\mathbf{H}_0)e^{-R/l}\bigg\}.\quad(16)$$

Define the z axis as along H_0. Denote the polar angles of R by (θ,φ) and of $v_0(t-t')\hat{p}$ by (θ_p,φ_p). Then from (11)

$$\theta=\theta_p,\quad(17)$$

$$\varphi=\varphi_p-\omega_c R/2v_0.\quad(18)$$

The first term of (16) may be written in terms of θ, φ as

$$v_0^2\int_0^\pi d\theta\sin\theta\int_{-\omega_c R/2v_0}^{2\pi-\omega_c R/2v_0}d\varphi\int_{c'}^0 dR\frac{\mathbf{R}}{R}\frac{\mathbf{R}}{R}\cdot\mathbf{E}(\mathbf{r}',t')e^{-R/l}.\quad(19)$$

Make the change of variable in (19)

$$\varphi'=\varphi+\omega_c R/2v_0,\quad(20)$$

and use that

$$\mathbf{R}(\theta,\varphi'-\omega_c R/2v_0,R)=\mathbf{R}(\theta,\varphi',R)-\omega_c R(\mathbf{H}_0\times\mathbf{R})/2v_0H_0.\quad(21)$$

Then the total current in (16) reduces to

$$\mathbf{J}(\mathbf{r},t)=\frac{3\sigma_0}{4\pi l}\int\int_0^{c'}\frac{\mathbf{R}}{R^4}\mathbf{R}\cdot\mathbf{E}(\mathbf{r}',t-R/v_0)e^{-R/l}d^3R+\frac{3R_0\sigma_0^2}{4\pi l^2}\int\int_{c'}^0 d\Lambda dR\frac{\mathbf{R}}{R}\mathbf{R}\cdot[\mathbf{E}(\mathbf{r}',t-R/v_0)\times\mathbf{H}_0]e^{-R/l},\quad(22)$$

where c' is the value of R corresponding to c.

The first term gives the well-known Chambers formula for the anomalous skin effect; the second term gives the nonlocal transverse Hall current in a normal metal.

III. MICROWAVE KERR ROTATION IN CIRCULAR CYLINDRICAL CAVITY

In this section we give a detailed theory for the microwave Kerr rotation in a circular cylindrical cavity which has two degenerate modes, such as the cavity used by Cooke.[6] The Kerr rotation is expressed in terms of the microscopic, wave number dependent conductivity tensor of the sample by the use of an electrodynamic perturbation theorem derived by Redfield.[20] The result is explicitly applied to the case of a normal metal whose conductivity tensor has been derived in Sec. II. It is shown that a measurement of the microwave Kerr rotation in the cavity as a function of the mean free path gives a good experimental test for the nonlocal Hall current in a way which is very similar to the verification of the anomalous skin effect by measurements on surface impedance as a function of mean free path. Although the cavity geometry described in this section cannot be used to measure a transverse Hall current in a bulk superconductor it seems that it could be used to measure a transverse Hall current in small superconducting samples such as very thin films.

[20] A. G. Redfield, J. Appl. Phys. 25, 1021 (1954).

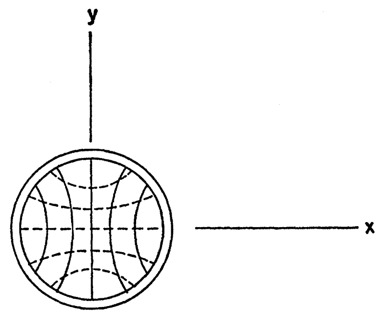

FIG. 1. The two degenerate TE_{11} modes in a cylindrical cavity of circular cross section.

The cavity is a cylindrical one of circular cross section which has two degenerate TE_{11} modes. The two degenerate modes are sketched in Fig. 1. The sample forms the end plate of the cavity and a static magnetic field H_0 is applied along the cavity axis and perpendicular to the metal surface. Clearly this geometry causes a transverse Hall current flow in the sample. The Kerr and Faraday rotations in a generalized microwave gyrator have been discussed by Redfield.[20] The circular cylindrical cavity described in this section is a special type of microwave gyrator. We define mode 1 as the mode excited by a coupling loop (A) entering the cavity when $H_0=0$. Another coupling loop (B) enters the cavity and can be rotated by 90° around the cavity axis so that it is coupled with either mode 1 or mode 2. Let I_{b1} denote the current in the coupling loop B when it is in position 1 and coupled completely with mode 1 and let I_{b2} denote the current in loop B when it is in position 2 and completely coupled with mode 2. I_{b2} is zero unless the dc magnetic field couples the two modes and its magnitude is proportional to the rotation of the microwaves in the cavity. If the gyrator is a cavity near resonance so that most of the loss occurs inside the cavity then by use of reasonable assumptions about the cavity coupling Redfield shows that[20]

$$\Theta \equiv \frac{I_{b2}}{I_{b1}} = \frac{-Q}{\omega\epsilon} \int (\mathbf{E}_1\sigma^a\mathbf{E}_2 - i\omega\mu_0\mathbf{H}_1\chi^a\mathbf{H}_2), \qquad (1)$$

where Q is the Q of the cavity, ϵ is the energy stored in the cavity for fields $\mathbf{E}_1$, $\mathbf{H}_1$ in the cavity, $\mathbf{E}_1$ and $\mathbf{H}_1$ are the unperturbed ($H_0=0$) fields in the cavity at resonance, and $\mathbf{E}_2$ and $\mathbf{H}_2$ differ from $\mathbf{E}_1$ and $\mathbf{H}_1$ by a 90° rotation. Also σ^a is the antisymmetric part of the conductivity tensor and χ^a the antisymmetric part of the magnetic susceptibility tensor. Local conductivity and susceptibility tensors have been assumed in (1). Also (1) is valid only for small rotations $\Theta \ll 1$.

Since we wish to consider nonlocal conductivity tensors we give a slight generalization of Eq. (1). We start with the electrodynamic perturbation theorem derived by Redfield[20]

$$a_{ij} - a_{ji} = -\int \mathbf{E}_j 2\sigma^a \mathbf{E}_i dv, \qquad (2)$$

where a_{ij} is the ij component of the admittance matrix of the microwave gyrator and $\mathbf{E}_j$ is the field in the gyrator when $V_j=1$ and $V_i=0$ whereas $\mathbf{E}_i$ is the field in the gyrator when $V_i=1$, $V_j=0$. For our purpose i and j refer to the coupling loops A and B. We follow the conventions of Montgomery *et al.*[21] in the definition of the admittance matrix. Using the derivation of Redfield, one may generalize (2) to give

$$a_{ij} - a_{ji} = -\int \mathbf{E}_j(\mathbf{q})\sigma^a(\mathbf{q})\mathbf{E}_i(\mathbf{q})d\mathbf{q}, \qquad (3)$$

where $\sigma^a(\mathbf{q})$ denotes the $\mathbf{q}$ Fourier component of the antisymmetric part of the conductivity tensor defined as

$$\mathbf{j}(\mathbf{q}) = \sigma(\mathbf{q})\mathbf{E}(\mathbf{q}). \qquad (4)$$

Using (3) one may show the generalization of (1) is

$$\Theta = \frac{-Q}{\omega\epsilon} \int \mathbf{E}_1(\mathbf{q})\sigma^a(\mathbf{q})\mathbf{E}_2(\mathbf{q})d\mathbf{q}, \qquad (5)$$

where for our purpose we have put $\chi^a(q)=0$.

We apply (5) to the TE_{11} mode of the cavity. Let the cavity axis and H_0 be along the z axis, then we may separate out the z dependence in the electric fields

$$\begin{aligned} E_{1x} &= E_{1x}(x,y)E(z), \quad E_{2x} = E_{2x}(x,y)E(z), \\ E_{1y} &= E_{1y}(x,y)E(z), \quad E_{2y} = E_{2y}(x,y)E(z). \end{aligned} \qquad (6)$$

Also $\sigma^a(q)$ will be a function of q_z only so that (5) reduces to

$$\Theta = \frac{Q}{\omega\epsilon} \int [E(q_z)]^2 \sigma^a(q_z) dq_z \times \int\int dxdy [E_{1x}(x,y)E_{2y}(-x,-y) - E_{2x}(x,y)E_{1y}(-x,-y)], \qquad (7)$$

where $\sigma^a(q_z)$ now denotes a number and not a tensor; that is

$$\sigma^a(q) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma^a(q_z).$$

We assume the boundary condition of specular reflection at the surface of the metal. Then the semi-infinite metal slab may be replaced by an infinite metal medium if we take[22]

$$E(z) = E(-z), \quad E(q_z) = E(-q_z). \qquad (8)$$

Also if the magnetic field is perpendicular to the metal surface it must satisfy

$$H_0(z) = H_0(-z). \qquad (9)$$

[21] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, *Principles of Microwave Circuits* (McGraw-Hill Book Company, Inc., New York, 1948).
[22] D. C. Mattis and G. Dresselhaus, Phys. Rev. **111**, 403 (1957).

The metal surface is at $z=0$. If the magnetic field is parallel to the metal surface one must take

$$H_0(z)=-H_0(-z). \tag{10}$$

Then using a method outlined by Serber and also used by Mattis and Dresselhaus[22] one may add a current sheet at $z=0$ in Maxwell's equations and show that

$$E(q_z)=-\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{E'(0)}{q_z^2+K(q_z)}, \tag{11}$$

where $E'(0)$ denotes dE/dz at $z=0$ and $K(q_z)$ is defined by

$$\mathbf{j}(q_z)=\frac{c^2}{4\pi i\omega}K(q_z)\mathbf{E}(q_z). \tag{12}$$

If we place our coordinate system at the center of the cavity then for the TE_{111} mode we have

$$E_{1y}(x,y)=E_{1y}(-x,-y),\quad E_{2y}(x,y)=E_{2y}(-x,-y), \tag{13}$$

so that,

$$\Theta=\frac{4Q[E'(0)]^2}{\omega\epsilon\pi}\int_0^\infty dq_z\frac{\sigma^a(q_z)}{[q_z^2+K(q_z)]^2}\times\int\int dxdy[E_{1x}(x,y)E_{2y}(x,y)-E_{2x}(x,y)E_{1y}(x,y)]. \tag{14}$$

For the TE_{111} mode,

$$\begin{aligned}E_{1x}(r,\theta)&=\cos^2\theta J_1(k,r)/k_1r+\sin^2\theta J_1'(k_1r),\\E_{1y}(r,\theta)&=\sin\theta\cos\theta J_1(k_1r)/k_1r-\sin\theta\cos\theta J_1'(k_1r),\end{aligned} \tag{15}$$

and for mode 2,

$$\begin{aligned}E_{2x}(r,\theta)&=\cos\theta\sin\theta J_1(k_1r)/k_1r-\cos\theta\sin\theta J_1'(k_1r),\\E_{2y}(r,\theta)&=\sin^2\theta J_1(k_1r)/k_1r+\cos^2\theta J_1'(k_1r),\end{aligned} \tag{16}$$

where $J_1(x)$ is the Bessel function of first order and $J_1'(x)$ is the first derivative, and

$$E(z)=-(\omega/k_1)e^{-i\omega t}\sin k_3z, \tag{17}$$

where we have introduced polar coordinates (r,θ). Also

$$k_1=1.841/a,\quad k_3=\pi/C, \tag{18}$$

where a is the radius and C the length of the cavity. The electric field lines for the two degenerate modes are shown in Fig. 1. The eigenfrequency ω is given by

$$\omega^2/c^2=k_1^2+k_3^2. \tag{19}$$

To evaluate (14) we use

$$E_{1x}(r,\theta)E_{2y}(r,\theta)-E_{1y}(r,\theta)E_{2x}(r,\theta)=J_1(k_1r)J_1'(k_1r)/k_1r. \tag{20}$$

The energy density of one mode in the cavity is,

$$\epsilon=\frac{\pi\omega^2}{16k_1^2k_3}\int_0^a rdr\{[J_1'(k_1r)]^2+J_1(k_1r)/(k_1r)^2\}. \tag{21}$$

Define

$$M=\int_0^{1.841}dx\,x[(J_1'(x))^2+J_1^2(x)/x^2], \tag{22}$$

so that

$$\epsilon=\pi\omega^2M/16k_1^4k_3. \tag{23}$$

Also define

$$N=\int_0^{1.841}dx\,J_1(x)J_1'(x), \tag{24}$$

then (14) becomes

$$\Theta=\left(\frac{8N}{M}\right)\left[\frac{Q}{\pi}\frac{k_3^2c^2}{\omega^3}\right]\times\left[\frac{16\omega^2}{c^3}\int_0^\infty dq_z\frac{\sigma^2(q_z)}{[q_z^2+K(q_z)]^2}\right]\left[\frac{k_3c}{\omega}\right], \tag{25}$$

and for a TE_{111} mode $N=0.14$ and $M=0.45$. The result (25) may readily be interpreted in terms of a simple physical picture. The Kerr rotation after *one* reflection from the sample for a plane wave of infinite extent in the xy plane may be shown to be equal to the third factor in Eq. (25). The mean time of damping of the electric field of the wave is $2Q/\omega$. If v_g is the group velocity of the wave in the z direction the wave makes

$$Qv_gk_3/\omega\pi \tag{26}$$

reflections before being damped out. Also

$$v_g=k_3c^2/\omega. \tag{27}$$

Thus the second factor in (25) corresponds to the mean number of reflections made by the wave in the cavity before it is damped out.

The last factor k_3c/ω arises because the value of $E'(0)$ at the metal surface is smaller by this factor in a waveguide than for a plane wave of infinite extent in the xy plane, for a given electric field strength far away from the metal surface. Since the primary current is proportional to $E'(0)$, the Kerr rotation in a single reflection is reduced in the guide from its value for a plane wave of infinite extent in the xy plane by the factor k_3c/ω. The first factor is a number of order unity which depends on the mode under consideration and hence may be referred to as a structure factor. The result of (25) may readily be generalized to a TE_{11n} mode.

The preceding discussion makes clear an important advantage of any experiment on the Kerr rotation in a microwave resonant cavity, namely that the observed rotation Θ is of the order of Q times the Kerr rotation for a single reflection. Since we shall show that the rotation for a single reflection is very small and since Q values of order 10^4 are available this is an important advantage of this type of experiment.

The real part of Θ corresponds to the angle of rotation whereas the imaginary part of Θ corresponds to the ellipticity of the elliptically polarized wave.

We now apply (25) to the nonlocal current in a

normal metal. The main features of the result are most clearly brought out by considering two limiting cases.

(*a*) The microwave region at room temperature has $\omega\tau \ll 1$ and $l \ll \lambda$ where λ is the penetration depth (or skin depth). Local relations for both the Ohmic current and the Hall current are valid. Using the result of Sec. II for small q gives for the Hall conductivity,

$$\sigma_{\mathrm{Hall}}(q_z) = \sigma^a(q_z) = R_0\sigma_0^2 H_0, \tag{28}$$

and

$$K(q) = 4\pi i\omega\sigma_0/c^2. \tag{29}$$

Then (25) gives

$$\Theta = \left(\frac{8N}{M}\right)\left(\frac{Qk_3^2c^2}{\pi\omega^2}\right)\left(\frac{-\delta\omega(1+i)\sigma_0 H_0 R_0}{2c}\right)\left(\frac{k_3 c}{\omega}\right), \tag{30}$$

where δ is the classical skin depth

$$\delta = c/(2\pi\omega\sigma_0)^{\frac{1}{2}}. \tag{31}$$

The order of magnitude of the rotation is one minute per kilogauss for a Q of 10^4.

(*b*) The extreme anomalous limit is valid when

$$qv_0/\pi\omega \gg 1 \quad \text{and} \quad l \gg \lambda. \tag{32}$$

Then from Sec. II the Hall current for large q is

$$\sigma_{\mathrm{Hall}}(q_z) = \sigma^a(q_z) = \frac{3\sigma_0}{l}\frac{R_0\sigma_0}{l}\frac{H_0}{q_z^2}, \tag{33}$$

and also

$$K(q_z) = \frac{4\pi i\omega}{c^2}\frac{3\pi\sigma_0}{4q_z l}. \tag{34}$$

Both (33) and (34) are independent of mean free path. From (25)

$$\Theta_\infty = \left(\frac{8N}{M}\right)\left(\frac{Qk_3^2c^2}{\pi\omega^2}\right)\left(-\frac{128}{9\sqrt{3}}\frac{\omega}{c}R_0H_0(\sqrt{3}+i)\right.$$
$$\left.\times\frac{\sigma_0^2 v_0\lambda_L^2(0)}{l^2c^2}\left(\frac{4v_0\lambda_L^2(0)}{3\pi\omega}\right)^{\frac{2}{3}}\right)\left(\frac{k_3c}{\omega}\right). \tag{35}$$

Like the surface impedance in the extreme anomalous limit the Kerr rotation Θ_∞ is independent of mean free path. Here $\lambda_L^2(0)$ is defined by

$$\lambda_L^2(0) = mc^2/ne^2 4\pi. \tag{36}$$

It is convenient to examine the Kerr rotation as a function of the parameter α defined as

$$\alpha = 3l^2/2\delta^2. \tag{37}$$

Then if we plot the ratio Θ/Θ_∞ as a function of $\alpha^{1/6}$ it will approach unity for α large compared to one whereas for small α the rotation increases linearly with $\alpha^{1/6}$ as given by Eq. (30). Denoting the real part of Θ by Θ^R and the imaginary part by Θ^I, we have

$$\frac{\Theta^R}{\Theta_\infty^R} = \frac{9\pi}{64}\left(\frac{27}{8}\right)^{1/6}\pi^{2/3}\alpha,^{1/6} \quad \text{for } \alpha^{1/6} \ll 1, \tag{38}$$

and

$$\frac{\Theta^I}{\Theta_\infty^I} = \sqrt{3}\,\Theta^R/\Theta_\infty^R \quad \text{for } \alpha^{1/6} \ll 1. \tag{39}$$

Both Θ^R/Θ_∞^R and Θ^I/Θ_∞^I are plotted versus $\alpha^{1/6}$ in Fig. 2. The dotted parts of Fig. 2 have been interpolated. For comparison we show in Fig. 3 a plot of the inverse of the surface resistance and the surface reactance in the microwave region as derived by Reuter and Sondheimer, plotted as a function of $\alpha^{1/6}$ along the abscissa.[23] The constant A is independent of α,

$$A = 6^{\frac{1}{2}}(\pi\omega/ec^2)^{\frac{2}{3}}(mv_0/3n)^{\frac{1}{3}}. \tag{40}$$

Also specular reflection has been assumed. The similar behavior of the Kerr rotation and of the surface impedance are evident from a comparison of Figs. 2 and 3. It is clear that the saturation of the Kerr rotation and of the surface impedance for large $\alpha^{1/6}$ is simply due to the currents being limited by the skin depth rather than the mean free path.

IV. THEORY OF HALL CURRENT IN SUPERCONDUCTORS

There are several important differences between a transverse and a longitudinal Hall current in a superconductor. In the longitudinal case H_0 is applied parallel to the surface of the superconductor so that a bulk sample can be used. In this geometry the Q wavevector of the incident microwave field is parallel to the Hall current. In the transverse case when H_0 is applied perpendicular to the surface of the superconductor a bulk specimen cannot be used. This is because in any specimen whose thickness is large compared to the penetration depth the magnetic field lines will be very

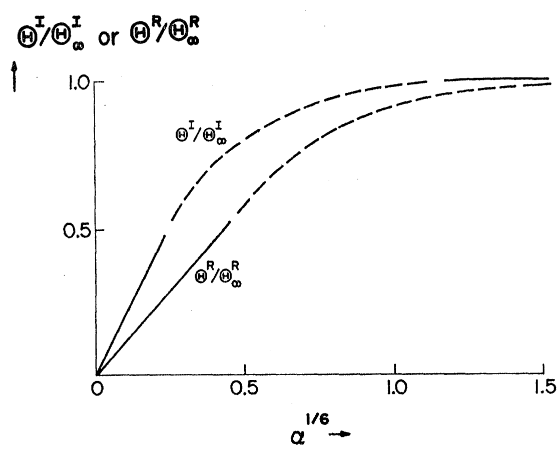

FIG. 2. The rotation ratios Θ^R/Θ_∞^R and Θ^I/Θ_∞^I are plotted versus the parameter $\alpha^{1/6}$ ($\alpha = 3l^2/2\delta^2$).

[23] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948).

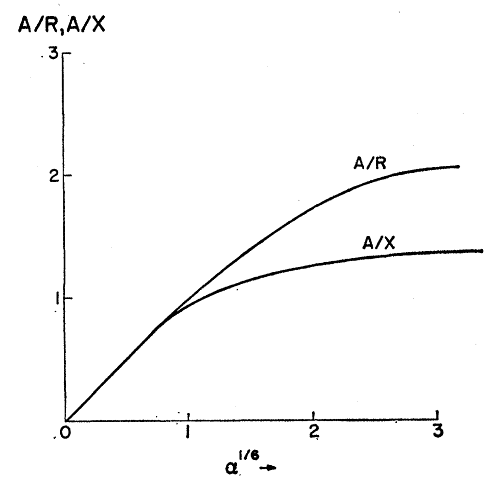

FIG. 3. Surface impedance in the microwave region versus the parameter $\alpha^{1/6}$.

nearly parallel to the surface. Consider for example a thin superconducting disk of radius r_0 and thickness d, with an applied field H_0 normal to the plane of the disk. If one assumes the current in the disk is given by the London equation,

$$\mathbf{j}=(-1/c\Lambda_T)\mathbf{A}, \tag{1}$$

where $\nabla\cdot\mathbf{A}=0$ and $\mathbf{A}_\perp$ to surface$=0$, then one may readily show that the magnetic field inside the disk will be approximately in the z direction (direction of H_0) and approximately equal to the applied field H_0 if

$$dr_0/\lambda_L^2(T)\lesssim 1, \tag{2}$$

where $\lambda_L(T)$ is the London penetration depth. Since typical values of $\lambda_L(T)$ are of order 10^{-5} cm and if $d=10^{-7}$ cm as in the thin films of Ginsburg and Tinkham[24] then the maximum value of r_0 is $r_0=10^{-3}$ cm. Such small superconducting disks have been used in experiments of Androes and Knight on the Knight shift.[25] For thin films with $d=10^{-7}$ cm the important values of wave vector q are $q\sim d^{-1}$ or $q\xi_0\sim\xi_0/d\gg 1$. In this region a nonlocal theory is known to be valid and the London equation (1) gives a current which is much too large. A more accurate criterion than (2) would allow the radius r_0 to be much larger, of the order of $r_0\sim 10^{-1}$ cm.

Another important difference between the transverse and longitudinal Hall current is the charge density associated with the longitudinal current. Let H_0 be along the z axis and define a wave-number-dependent conductivity tensor by

$$j_y(Q)=\sigma_{yx}(Q)E_x(Q)+\sigma_{yy}(Q)E_y(Q). \tag{3}$$

Let the incident microwave electric field be along x, then for the transverse case as in the experiment of Cooke we may take $E_y(Q)=0$. For the longitudinal case we get from the continuity equation for current and $\nabla\cdot\mathbf{E}=4\pi\rho$

$$E_y(Q)=-4\pi j_y(Q)/i\omega. \tag{4}$$

In general one would expect that the driving term would be similar in both the transverse and longitudinal current except for effects arising when diffusion takes place in the longitudinal case. The detailed microscopic theory of this section will be carried out explicitly only for the transverse current; however since we qualitatively expect the driving term $\sigma_{yx}(Q)$ in the longitudinal case to be the same as in the transverse case we may then apply the results of the microscopic theory in a qualitative manner to the Lewis experiment and also compare this theory with a two fluid model theory postulated by Dresselhaus and Dresselhaus for the longitudinal case.[11]

The detailed microscopic theory for the transverse case makes the following basic assumptions. We let H_0 be along z and the incident microwave field along x so that the Hall current will be along y only and it will be transverse,

$$\partial j_y/\partial y=0. \tag{5}$$

Further it is assumed that H_0 is uniform throughout the sample; such an assumption would be valid for small superconducting specimens as discussed earlier. A perturbation theory is used to include the effect of H_0 and the microwave field so that the Hall current will be proportional to H_0. Clearly such an assumption fails in large magnetic fields; when applied to small samples perturbation theory will be valid when

$$\omega_c d/v_0\ll 1, \tag{6}$$

where d is the small dimension of the sample and ω_c is the cyclotron frequency eH_0/mc.

We assume the BCS model[26] for the superconductor and also include the effects of collective excitations by the generalized random phase approximation given by Anderson and by Rickayzen.[12,13] We shall show explicitly that the collective coordinates are zero in the transverse Hall current for an appropriate choice of gauge if the two-body interaction $V(\mathbf{k},\mathbf{k}')$ is independent of the angle between $\mathbf{k}$ and $\mathbf{k}'$. This shows that in the transverse case, for the appropriate choice of gauge, the inclusion of collective coordinates is not mandatory; one can obtain the same result by the use of perturbation theory assuming the BCS ground state (without collective effects being included) to be the eigenstate when the electromagnetic fields are zero. The formalism used in this section includes the collective excitations explicitly and hence could be used to treat the longitudinal Hall current.

The notation used in this section corresponds closely to that of reference 13 and is briefly summarized here.

[24] D. M. Ginsburg and M. Tinkham, Phys. Rev. **118**, 990 (1960).
[25] G. M. Androes and W. D. Knight, Phys. Rev. Letters **2**, 386 (1959).
[26] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957); hereafter called BCS.

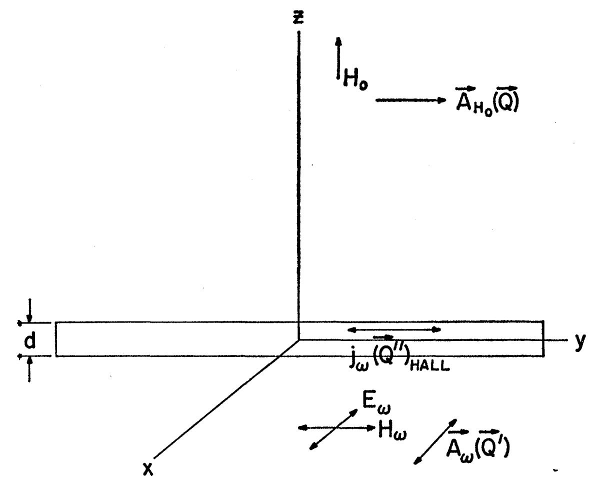

FIG. 4. Geometry for transverse Hall current in superconductor.

The potential $V(\mathbf{k},\mathbf{k}')$ denotes the interaction responsible for the superconducting transition. The operator which creates an electron in the state of momentum $\mathbf{k}$ and spin $\boldsymbol{\sigma}$ is denoted as $c_{k,\sigma}^*$. It is more convenient to use the quasi-particle operators introduced by Bogoliubov and by Valatin,

$$\gamma_{k0}=u_kc_{k\uparrow}-v_kc_{-k\downarrow}^*,\quad \gamma_{k1}=u_kc_{-k\downarrow}+v_kc_{k\uparrow}^*, \tag{7}$$

where,

$$u_k^2=\tfrac{1}{2}(1+\epsilon_k/E_k),\quad v_k^2=\tfrac{1}{2}(1-\epsilon_k/E_k). \tag{8}$$

The energy of an electron in the normal state is $\epsilon_{k'}$ measured from the Fermi surface. The energy gap is I_k and

$$E_k=(\epsilon_k^2+I_k^2)^{\frac{1}{2}}. \tag{9}$$

The collective variables in the superconductor are defined as[13]

$$\rho(\mathbf{Q})=\sum\nolimits_{\mathbf{k}} m(k,Q)(\gamma_{k+Q0}^*\gamma_{k1}^*+\gamma_{k+Q1}\gamma_{k0}) +n(k,Q)(\gamma_{k1}^*\gamma_{k+Q1}+\gamma_{k+Q0}^*\gamma_{k0}), \tag{10}$$

$$B_k(\mathbf{Q})=\sum\nolimits_{\mathbf{k}} V(K,k)[n(k,Q)(\gamma_{k+Q0}^*\gamma_{k1}^*+\gamma_{k+Q1}\gamma_{k0}) -m(k,Q)(\gamma_{k+Q0}^*\gamma_{k0}+\gamma_{k1}^*\gamma_{k+Q1})], \tag{11}$$

$$A_k(\mathbf{Q})=\sum\nolimits_{\mathbf{k}} V(K,k)[l(k,Q)(\gamma_{k+Q0}^*\gamma_{k1}^*-\gamma_{k+Q1}\gamma_{k0}) +p(k,Q)(\gamma_{k+Q0}^*\gamma_{k0}-\gamma_{k1}^*\gamma_{k+Q1})], \tag{12}$$

where,

$$\begin{aligned} l(k,Q)&=u_ku_{k+Q}+v_kv_{k+Q}, & m(k,Q)&=u_kv_{k+Q}+v_ku_{k+Q},\\ n(k,Q)&=u_ku_{k+Q}-v_kv_{k+Q}, & p(k,Q)&=u_kv_{k+Q}-v_ku_{k+Q}. \end{aligned} \tag{13}$$

The microwave electromagnetic field is described by the vector potential $\mathbf{A}(\mathbf{r},t)$ and

$$\mathbf{A}(\mathbf{r},t)=\sum\nolimits_\omega e^{i(\omega-is)t}\mathbf{A}_\omega(\mathbf{r}). \tag{14}$$

Here s^{-1} will be identified with the phenomenological relaxation time as in the work of Mattis and Dresselhaus.[22] We further express $\mathbf{A}_\omega(\mathbf{r})$ as

$$\mathbf{A}_\omega(\mathbf{r})=\sum\nolimits_{\mathbf{Q}'}\mathbf{a}_\omega(-\mathbf{Q}')\exp(-i\mathbf{Q}'\cdot\mathbf{r}). \tag{15}$$

Since the microwave electric field is in the x direction, $a_\omega(-Q')$ is along x and for the transverse case $\mathbf{Q}'$ is along the z axis. A diagram of the geometry is shown in Fig. 4. Clearly it suffices to consider one value of Q' in (15) so that we drop the sum over Q'. Also for this gauge,

$$\nabla\cdot\mathbf{A}(\mathbf{r},t)=0. \tag{16}$$

The static field H_0 will be represented by the vector potential

$$\mathbf{A}_{H0}(\mathbf{r})=\mathbf{a}_{H_0}(-\mathbf{Q})\exp(-i\mathbf{Q}\cdot\mathbf{r}) +\mathbf{a}_{H_0}(\mathbf{Q})\exp(i\mathbf{Q}\cdot\mathbf{r}), \tag{17}$$

where we arbitrarily choose $\mathbf{a}_{H_0}(-\mathbf{Q})$ to be in the y direction and $\mathbf{Q}$ to be along x. We then take the limit that $\mathbf{Q}\rightarrow 0$ in the final result since H_0 is assumed not to vary along the x direction. Since H_0 is real,

$$a_{H_0}^*(Q)=a_{H_0}(-Q). \tag{18}$$

This choice of magnetic field representation has several important advantages. We shall show explicitly that for this choice all collective coordinates are zero; such a result would not be valid in a more general choice. For example with the choice of magnetic field

$$\mathbf{A}_{H0}(\mathbf{r})=a_{H_0}(Q_y)\exp(iQ_yy)\hat{x}+\text{c.c.}, \tag{19}$$

neither the charge density nor the other collective coordinates would be zero and thus would greatly complicate the formalism. (For example, the charge density would go to zero only in the limit $Q_y\rightarrow 0$. Another advantage of (17) is that a current proportional to $a_\omega(-Q')a_{H_0}(-Q)e^{i(\omega-is)t}$ flows in the y direction only. For the choice of magnetic field (19) there would be an added current flow in the z direction which is proportional to $a_\omega(-Q')a_{H_0}(-Q)e^{i(\omega-is)t}$ and which arises from the Lorentz force exerted by the microwave magnetic field on the static London current which flows along the direction of $\mathbf{a}_{H_0}(-\mathbf{Q})$. However, for a small superconducting sample (i.e., d small in Fig. 4) any Hall current flow in the z direction would not be important physically due to the presence of the boundary and hence can be neglected.

For our choice of gauge,

$$\begin{gathered}\mathbf{Q}\cdot\mathbf{a}_{H_0}(-\mathbf{Q})=\mathbf{Q}'\cdot\mathbf{a}_\omega(-\mathbf{Q}')=0,\\ \mathbf{Q}'\cdot\mathbf{a}_{H_0}(-\mathbf{Q})=\mathbf{a}_{H_0}(-\mathbf{Q})\cdot\mathbf{a}_\omega(-\mathbf{Q}')=0,\end{gathered} \tag{20}$$

and

$$\mathbf{Q}\cdot\mathbf{a}_\omega(-\mathbf{Q}')=Qa_\omega(-Q')\neq 0. \tag{21}$$

Also define

$$\mathbf{Q}''=\mathbf{Q}+\mathbf{Q}'. \tag{22}$$

The field free Hamiltonian of the system is[13]

$$H_0=\sum_{k\sigma}\epsilon_kc_{k\sigma}^*c_{k\sigma}+\sum_{kk'q\sigma\sigma'}V_D(\mathbf{k},\mathbf{k}') \times c_{k'\sigma'}^*c_{-k'+q\sigma}^*c_{-k+q\sigma}c_{k\sigma'}, \tag{23}$$

where $V_D(\mathbf{k},\mathbf{k}')$ includes the unscreened Coulomb and electron phonon interactions. For our choice of gauge

the perturbation H_1 is

$$H_1=H_{A(H_0)}+H_{A\omega}, \tag{24}$$

where

$$H_{A(H_0)}=-2\alpha a_{H_0}(-Q)\sum_k k_y[l(k,-Q)\times(\gamma_{k-Q0}{}^*\gamma_{k0}-\gamma_{k1}{}^*\gamma_{k-Q1})-p(k,-Q)(\gamma_{k-Q0}{}^*\gamma_{k1}{}^*-\gamma_{k-Q1}\gamma_{k0})]+\text{H.c.}, \tag{25}$$

$$H_{A\omega}=-2\alpha a_\omega(-Q')e^{it(\omega-is)}\sum_k k_x[l(k,-Q')\times(\gamma_{k-Q'0}{}^*\gamma_{k0}-\gamma_{k1}{}^*\gamma_{k-Q'1})-p(k,-Q')\times(\gamma_{k-Q'0}{}^*\gamma_{k1}{}^*-\gamma_{k-Q'1}\gamma_{k0})]+\text{H.c.}, \tag{26}$$

and $\alpha=e\hbar/2mc$. For our choice of gauge the term $\mathbf{a}_{H_0}(-\mathbf{Q})\cdot\mathbf{a}_\omega(-\mathbf{Q}')$ is zero. Also the paramagnetic current operator is

$$\mathbf{j}_\omega(\mathbf{Q}'')=(e\hbar/2m)\sum_k(2\mathbf{k}+\mathbf{Q}'')[l(k,Q'')\times(\gamma_{k+Q''0}{}^*\gamma_{k0}-\gamma_{k1}{}^*\gamma_{k+Q''1})-p(k,Q'')\times(\gamma_{k+Q''0}{}^*\gamma_{k1}{}^*-\gamma_{k+Q''1}\gamma_{k0})]+\text{H.c.} \tag{27}$$

It is proved in Appendix A that the diamagnetic part of the Hall current is zero for our gauge so that Eq. (27) gives the total Hall current.

The method we use to evaluate the Hall current is similar to a method used by Rickayzen in his treatment of the dielectric constant of a superconductor.[13] The random phase approximation is used to derive equations of motion for the $\gamma\gamma$ operators and including the effect of the driving terms H_1 in the Hamiltonian. Only terms which will give a contribution to the current linear in the product $a_{H_0}(-Q)a_\omega(-Q')$ are retained. Since we assume only the Fourier components Q and Q' are present [namely $a_{H_0}(-Q)$ and $a_\omega(-Q')$] then the only component of current proportional to the product $a_{H_0}(-Q)a_\omega(-Q')$ excited is $\mathbf{j}(\mathbf{Q}'')$. Thus we need only consider equations of motion for γ products of the form

$$\gamma_{k+Q''0}{}^*\gamma_{k1}{}^*,\quad \gamma_{k+Q''1}\gamma_{k0},\quad \gamma_{k+Q''0}{}^*\gamma_{k0},\quad \gamma_{k1}{}^*\gamma_{k+Q''1}. \tag{28}$$

The equations of motion with no driving forces are given in the random phase approximation by[27]

$$[H_0,\gamma_{k+Q''0}{}^*\gamma_{k1}{}^*]=\nu_k(Q'')\gamma_{k+Q''0}{}^*\gamma_{k1}{}^*+(1-f_k-f_{k+Q''})\times\{V_D(Q'')m(k,Q'')\rho(Q'')+\tfrac{1}{2}n(k,Q'')B_k(Q'')-\tfrac{1}{2}l(k,Q'')A_k(Q'')\}. \tag{29}$$

$$[H_0,\gamma_{k+Q''1}\gamma_{k0}]=-\nu_k(Q'')\gamma_{k+Q''1}\gamma_{k0}-(1-f_k-f_{k+Q''})\times\{V_D(Q'')m(k,Q'')\rho(Q'')+\tfrac{1}{2}n(k,Q'')B_k(Q'')+\tfrac{1}{2}l(k,Q'')A_k(Q'')\}. \tag{30}$$

$$[H_0,\gamma_{k+Q''0}{}^*\gamma_{k0}]=\bar{E}_k(Q'')\gamma_{k+Q''0}{}^*\gamma_{k0}+(f_{k+Q''}-f_k)\times\{-V_D(Q'')\rho(Q'')n(k,Q'')+\tfrac{1}{2}m(k,Q'')B_k(Q'')+\tfrac{1}{2}p(k,Q'')A_k(Q'')\}. \tag{31}$$

$$[H_0,\gamma_{k1}{}^*\gamma_{k+Q''1}]=-\bar{E}_k(Q'')\gamma_{k1}{}^*\gamma_{k+Q''1}-(f_{k+Q''}-f_k)\times\{-V_D(Q'')\rho(Q'')n(k,Q'')+\tfrac{1}{2}m(k,Q'')B_k(Q'')-\tfrac{1}{2}p(k,Q'')A_k(Q'')\}. \tag{32}$$

These reduce to the equations given by Rickayzen when $T=0$.[13] In this notation,

$$\nu_k(Q'')=E_k+E_{k+Q''},\quad \bar{E}_k(Q'')=E_{k+Q''}-E_k, \tag{33}$$

$$f_k=f(E_k), \tag{34}$$

where f is the Fermi function.

When we include the driving terms H_1 the equations of motion are

$$\begin{aligned}[H,\gamma_{k+Q''0}{}^*\gamma_{k1}{}^*]={}&\nu_k(Q'')\gamma_{k+Q''0}{}^*\gamma_{k1}{}^*+(1-f_k-f_{k+Q''})\{V_D(Q'')m(k,Q'')\rho(Q'')+\tfrac{1}{2}n(k,Q'')B_k(Q'')\\&-\tfrac{1}{2}l(k,Q'')A_k(Q'')\}-\alpha a_{H_0}(-Q)\{2k_yl(k+Q',Q)\gamma_{k+Q'0}{}^*\gamma_{k1}{}^*+2k_yl(k,Q)\gamma_{k+Q1}{}^*\gamma_{k+Q+Q'0}{}^*\\&-2k_yp(k+Q',Q)\gamma_{k+Q'1}\gamma_{k1}{}^*+2k_yp(k,Q)\gamma_{k+Q'+Q0}{}^*\gamma_{k+Q0}\}-\alpha\exp(i\tilde{\omega}t)a_\omega(-Q')\\&\times\{(2k_x+2Q)l(k+Q,Q')\gamma_{k+Q0}{}^*\gamma_{k1}{}^*+2k_xl(k,Q')\gamma_{k+Q'1}{}^*\gamma_{k+Q+Q'0}{}^*\\&-(2k_x+2Q)p(k+Q,Q')\gamma_{k+Q1}\gamma_{k1}{}^*+2k_xp(k,Q')\gamma_{k+Q+Q'0}{}^*\gamma_{k+Q'0}\}.\end{aligned} \tag{35}$$

$$\begin{aligned}[H,\gamma_{k+Q''1}\gamma_{k0}]={}&-\nu_k(Q'')\gamma_{k+Q''1}\gamma_{k0}-(1-f_k-f_{k+Q''})\{V_D(Q'')m(k,Q'')\rho(Q'')+\tfrac{1}{2}n(k,Q'')B_k(Q'')\\&+\tfrac{1}{2}l(k,Q'')A_k(Q'')\}-\alpha a_{H_0}(-Q)\{-2k_yl(k,Q)\gamma_{k+Q+Q'1}\gamma_{k+Q0}-2k_yl(k+Q',Q)\gamma_{k0}\gamma_{k+Q'1}\\&-2k_yp(k,Q)\gamma_{k+Q+Q'1}\gamma_{k+Q1}{}^*+2k_yp(k+Q',Q)\gamma_{k+Q'0}{}^*\gamma_{k0}\}-\alpha a_\omega(-Q')\exp(it\tilde{\omega})\\&\times\{-2k_xl(k,Q')\gamma_{k+Q+Q'1}\gamma_{k+Q'0}-(2k_x+2Q)l(k+Q,Q')\gamma_{k0}\gamma_{k+Q1}\\&-2k_xp(k,Q')\gamma_{k+Q+Q'1}\gamma_{k+Q'1}{}^*+(2k_x+2Q)p(k+Q,Q')\gamma_{k+Q0}{}^*\gamma_{k0}\}.\end{aligned} \tag{36}$$

$$\begin{aligned}[H,\gamma_{k+Q''0}{}^*\gamma_{k0}]={}&\bar{E}_k(Q'')\gamma_{k+Q''0}{}^*\gamma_{k0}+(f_{k+Q''}-f_k)\{-V_D(Q'')\rho(Q'')n(k,Q'')+\tfrac{1}{2}m(k,Q'')B_k(Q'')\\&+\tfrac{1}{2}p(k,Q'')A_k(Q'')\}-\alpha a_{H_0}(-Q)\{2k_yl(k+Q',Q)\gamma_{k+Q'0}{}^*\gamma_{k0}-2k_yl(k,Q)\gamma_{k+Q'+Q0}{}^*\gamma_{k+Q0}\\&-2k_yp(k+Q',Q)\gamma_{k+Q'1}\gamma_{k0}-2k_yp(k,Q)\gamma_{k+Q'+Q0}{}^*\gamma_{k+Q1}{}^*\}-\alpha a_\omega(-Q')\exp(it\tilde{\omega})\\&\times\{(2k_x+2Q)l(k+Q,Q')\gamma_{k+Q0}{}^*\gamma_{k0}-2k_xl(k,Q')\gamma_{k+Q'+Q0}{}^*\gamma_{k+Q'0}\\&-(2k_x+2Q)p(k+Q,Q')\gamma_{k+Q1}\gamma_{k0}-2k_xp(k,Q')\gamma_{k+Q+Q'0}{}^*\gamma_{k+Q'1}{}^*\}.\end{aligned} \tag{37}$$

[27] T. Tsuneto, Phys. Rev. **121**, 402 (1960).

$$\begin{aligned}[H,\gamma_{k1}{}^*\gamma_{k+Q''1}]=&-\tilde{E}_k(Q'')\gamma_{k1}{}^*\gamma_{k+Q''1}-(f_{k+Q''}-f_k)\{-V_D(Q'')\rho(Q'')n(k,Q'')+\tfrac{1}{2}m(k,Q'')B_k(Q'')\\&-\tfrac{1}{2}p(k,Q'')A_k(Q'')\}-\alpha a_{H0}(-Q)\{-2k_yl(k,Q)\gamma_{k+Q1}{}^*\gamma_{k+Q''1}+2k_yl(k+Q',Q)\gamma_{k1}{}^*\gamma_{k+Q'1}\\&+2k_yp(k+Q',Q)\gamma_{k1}{}^*\gamma_{k+Q'0}{}^*+2k_yp(k,Q)\gamma_{k+Q0}\gamma_{k+Q+Q'1}\}-\alpha a_\omega(-Q')\exp(it\tilde{\omega})\\&\times\{-2k_xl(k,Q')\gamma_{k+Q'1}{}^*\gamma_{k+Q''1}+(2k_x+2Q)l(k+Q,Q')\gamma_{k1}{}^*\gamma_{k+Q1}\\&+(2k_x+2Q)p(k+Q,Q')\gamma_{k1}{}^*\gamma_{k+Q0}{}^*+2k_yp(k,Q')\gamma_{k+Q'0}\gamma_{k+Q+Q'1}\},\end{aligned}\tag{38}$$

where $\tilde{\omega}=\omega-is$.

The Eqs. (35) through (38) are equations involving operators. If we let Ψ be the wave functional of the system in the presence of H_1 and Ψ_0 be the wave functional when $H_1=0$, then we need to find the expectation value

$$\langle\Psi|\mathbf{j}_\omega(\mathbf{Q}'')|\Psi\rangle.\tag{39}$$

Thus we form expectation values in Eqs. (35)–(38) with the wave functional Ψ. The equations from now on will always refer to such expectation values and not to operators.

We also use the simple theorem

$$\langle\Psi|H,\gamma_{k+Q''0}{}^*\gamma_{k1}{}^*|\Psi\rangle=\tilde{\omega}\langle\Psi|\gamma_{k+Q''0}{}^*\gamma_{k1}{}^*|\Psi\rangle,\tag{40}$$

which is valid as long as H may be written as the sum of a static and time dependent Hamiltonian and if Ψ is only needed to first order perturbation theory with the time dependent part of the Hamiltonian being the perturbation. The result (40) is clearly applicable to our case since the only time dependent part of H is $H_{A\omega}$.

In Eqs. (35)–(40), we need expectation values of products which differ by momentum Q (i.e., $\gamma_{k+Q0}{}^*\gamma_{k1}{}^*$) to first order in $a_{H0}(-Q)$ and expectation values of products which differ by momentum Q' (i.e., $\gamma_{k+Q'0}{}^*\gamma_{k1}{}^*$) to first order in $a_\omega(-Q)$. The total equation of motion for $\gamma_{k+Q}\gamma_k$ quantities to first order in a_{H0} is

$$\begin{aligned}0=&\nu_k(Q)\gamma_{k+Q0}{}^*\gamma_{k1}{}^*+(1-f_k-f_{k+Q})\{V_D(Q)m(k,Q)\rho(Q)+\tfrac{1}{2}n(k,Q)B_k(Q)-\tfrac{1}{2}l(k,Q)A_k(Q)\}\\&+\alpha a_{H0}(-Q)2k_y(1-f_k-f_{k+Q})p(k,Q),\\0=&-\nu_k(Q)\gamma_{k+Q1}\gamma_{k0}-(1-f_k-f_{k+Q})\{V_D(Q)m(k,Q)\rho(Q)+\tfrac{1}{2}n(k,Q)B_k(Q)+\tfrac{1}{2}l(k,Q)A_k(Q)\}\\&+\alpha a_{H0}(-Q)2k_y(1-f_k-f_{k+Q})p(k,Q),\\0=&\tilde{E}_k(Q)\gamma_{k+Q0}{}^*\gamma_{k0}+(f_{k+Q}-f_k)\{-V_D(Q)n(k,Q)\rho(Q)+\tfrac{1}{2}m(k,Q)B_k(Q)+\tfrac{1}{2}p(k,Q)A_k(Q)\}\\&-\alpha a_{H0}(-Q)2k_y(f_k-f_{k+Q})l(k,Q),\\0=&-\tilde{E}_k(Q)\gamma_{k1}{}^*\gamma_{k+Q1}-(f_{k+Q}-f_k)\{-V_D(Q)n(k,Q)\rho(Q)+\tfrac{1}{2}m(k,Q)B_k(Q)-\tfrac{1}{2}p(k,Q)A_k(Q)\}\\&-\alpha a_{H0}(-Q)2k_y(f_k-f_{k+Q})l(k,Q).\end{aligned}\tag{41}$$

The commutator of γ pairs with H has been set equal to zero since $a_{H0}(-Q)$ is static. When $V(\mathbf{K},\mathbf{k})$ is independent of angle a self-consistent solution of (41) is

$$\rho(Q)=B_k(Q)=A_k(Q)=0,\tag{42}$$

and

$$\begin{aligned}\gamma_{k+Q0}{}^*\gamma_{k1}{}^*&=-2\alpha a_{H0}(-Q)k_y(1-f_k-f_{k+Q})p(k,Q)\nu_k{}^{-1}(Q),\\\gamma_{k+Q1}\gamma_{k0}&=2\alpha a_{H0}(-Q)k_y(1-f_k-f_{k+Q})p(k,Q)\nu_k{}^{-1}(Q),\\\gamma_{k+Q0}{}^*\gamma_{k0}&=2\alpha a_{H0}(-Q)k_y(f_k-f_{k+Q})l(k,Q)\tilde{E}_k(Q)^{-1},\\\gamma_{k1}{}^*\gamma_{k+Q1}&=-2\alpha a_{H0}(-Q)k_y(f_k-f_{k+Q})l(k,Q)\tilde{E}_k(Q)^{-1}.\end{aligned}\tag{43}$$

In the same manner one finds the expectation values of γ products which differ by Q' to be

$$\rho(Q')=A_k(Q')=B_k(Q')=0,\tag{44}$$

and,

$$\begin{aligned}\gamma_{k+Q'0}{}^*\gamma_{k1}{}^*&=2k_xa_\omega(-Q')\exp(it\tilde{\omega})p(k,Q')(1-f_k-f_{k+Q'})[\tilde{\omega}-\nu_k(Q')]^{-1},\\\gamma_{k+Q'1}\gamma_{k0}&=2k_xa_\omega(-Q')\exp(it\tilde{\omega})p(k,Q')(1-f_k-f_{k+Q'})[\tilde{\omega}+\nu_k(Q')]^{-1},\\\gamma_{k+Q'0}{}^*\gamma_{k0}&=-2k_xa_\omega(-Q')\exp(it\tilde{\omega})l(k,Q')(f_k-f_{k+Q'})[\tilde{\omega}-\tilde{E}_k(Q')]^{-1},\\\gamma_{k1}{}^*\gamma_{k+Q'1}&=-2k_xa_\omega(-Q')\exp(it\tilde{\omega})l(k,Q')(f_k-f_{k+Q'})[\tilde{\omega}+\tilde{E}_k(Q')]^{-1}.\end{aligned}\tag{45}$$

To derive (45) we have replaced the commutator by $\tilde{\omega}\gamma\gamma$ because there is an $\exp(it\tilde{\omega})$ time dependence in $H_{A\omega}$.

The final equations of motion are derived from (35)–(38) using the expectation values given by (43) and (45).

We also use (40) to replace the commutators in (35)–(38) by $\tilde{\omega}\gamma\gamma$. Thus one finds

$$\begin{aligned}\tilde{\omega}\gamma_{k+Q''0}{}^*\gamma_{k1}{}^* = {} & \nu_k(Q'')\gamma_{k+Q''0}{}^*\gamma_{k1}{}^* + (1-f_k-f_{k+Q''})\{V_D(Q'')m(k,Q'')\rho(Q'')+\tfrac{1}{2}n(k,Q'')B_k(Q'')-\tfrac{1}{2}l(k,Q'')A_k(Q'')\}\\ & -4\alpha^2 a_{H0}(-Q)a_\omega(-Q')\exp(it\tilde{\omega})k_y\{l(k+Q',Q)k_x p(k,Q')(1-f_k-f_{k+Q'})[\tilde{\omega}-\nu_k(Q')]^{-1}\\ & -l(k,Q)(k_x+Q)p(k+Q,Q')(1-f_{k+Q}-f_{k+Q''})[\tilde{\omega}-\nu_{k+Q}(Q')]^{-1}-k_x p(k+Q',Q)l(k,Q')\\ & \times(f_k-f_{k+Q'})[\tilde{\omega}+\tilde{E}_k(Q')]^{-1}-(k_x+Q)p(k,Q)l(k+Q,Q')(f_{k+Q}-f_{k+Q''})[\tilde{\omega}-\tilde{E}_{k+Q}(Q')]^{-1}\\ & -(k_x+Q)l(k+Q,Q')p(k,Q)(1-f_k-f_{k+Q})\nu_k{}^{-1}(Q)+k_x l(k,Q')p(k+Q',Q)(1-f_{k+Q'}-f_{k+Q''})\\ & \times\nu_{k+Q'}{}^{-1}(Q)-(k_x+Q)p(k+Q,Q')l(k,Q)(f_k-f_{k+Q})\tilde{E}_k(Q)^{-1}\\ & +k_x p(k,Q')l(k+Q',Q)(f_{k+Q'}-f_{k+Q''})\tilde{E}_{k+Q'}{}^{-1}(Q)\}. \qquad (46)\end{aligned}$$

$$\begin{aligned}\tilde{\omega}\gamma_{k+Q''1}\gamma_{k0} = {} & -\nu_k(Q'')\gamma_{k+Q''1}\gamma_{k0}-(1-f_k-f_{k+Q''})\{V_D(Q'')m(k,Q'')\rho(Q'')+\tfrac{1}{2}n(k,Q'')B_k(Q'')\\ & +\tfrac{1}{2}l(k,Q'')A_k(Q'')\}-4\alpha^2 a_{H0}(-Q)a_\omega(-Q')\exp(it\tilde{\omega})k_y\{-l(k,Q)(k_x+Q)p(k+Q,Q')\\ & \times(1-f_{k+Q}-f_{k+Q''})[\tilde{\omega}+\nu_{k+Q}(Q')]^{-1}+l(k+Q',Q)k_x p(k,Q')(1-f_k-f_{k+Q'})[\tilde{\omega}+\nu_k(Q')]^{-1}\\ & -p(k,Q)(k_x+Q)l(k+Q,Q')(f_{k+Q}-f_{k+Q''})[\tilde{\omega}+\tilde{E}_{k+Q}(Q')]^{-1}-p(k+Q',Q)k_x l(k,Q')(f_k-f_{k+Q'})\\ & \times[\tilde{\omega}-\tilde{E}_k(Q')]^{-1}-k_x l(k,Q')p(k+Q',Q)(1-f_{k+Q}-f_{k+Q''})\nu_{k+Q'}{}^{-1}(Q)\\ & +(k_x+Q)l(k+Q,Q')p(k,Q)(1-f_k-f_{k+Q})\nu_k{}^{-1}(Q)-k_x p(k,Q')l(k+Q',Q)\\ & \times(f_{k+Q'}-f_{k+Q''})\tilde{E}_{k+Q'}(Q)^{-1}+(k_x+Q)p(k+Q,Q')l(k,Q)(f_k-f_{k+Q})\tilde{E}_k(Q)^{-1}\}. \qquad (47)\end{aligned}$$

$$\begin{aligned}\tilde{\omega}\gamma_{k+Q''0}{}^*\gamma_{k0} = {} & \tilde{E}_k(Q'')\gamma_{k+Q''0}{}^*\gamma_{k0}+(f_{k+Q''}-f_k)\{-V_D(Q'')\rho(Q'')n(k,Q'')+\tfrac{1}{2}m(k,Q'')B_k(Q'')+\tfrac{1}{2}p(k,Q'')V_k(Q'')\}\\ & -4\alpha^2 a_{H0}(-Q)a_\omega(-Q')\exp(it\tilde{\omega})k_y\{-l(k+Q',Q)k_x l(k,Q')(f_k-f_{k+Q'})[\tilde{\omega}-\tilde{E}_k(Q')]^{-1}\\ & +l(k,Q)l(k+Q,Q')(f_{k+Q}-f_{k+Q''})(k_x+Q)[\tilde{\omega}-\tilde{E}_{k+Q}(Q')]^{-1}-p(k+Q',Q)p(k,Q')k_x(1-f_k-f_{k+Q'})\\ & \times[\tilde{\omega}+\nu_k(Q')]^{-1}-p(k,Q)(k_x+Q)p(k+Q,Q')(1-f_{k+Q}-f_{k+Q''})[\tilde{\omega}-\nu_{k+Q}(Q')]^{-1}\\ & +(k_x+Q)l(k+Q,Q')l(k,Q)(f_k-f_{k+Q})\tilde{E}_k{}^{-1}(Q)-l(k,Q')k_x l(k+Q',Q)(f_{k+Q'}-f_{k+Q''})\tilde{E}_{k+Q'}{}^{-1}(Q)\\ & -(k_x+Q)p(k+Q,Q')p(k,Q)(1-f_k-f_{k+Q})\nu_k{}^{-1}(Q)+p(k,Q')k_x p(k+Q',Q)\\ & \times(1-f_{k+Q'}-f_{k+Q''})\nu_{k+Q'}{}^{-1}(Q)\}. \qquad (48)\end{aligned}$$

$$\begin{aligned}\tilde{\omega}\gamma_{k1}{}^*\gamma_{k+Q''1} = {} & -\tilde{E}_k(Q'')\gamma_{k1}{}^*\gamma_{k+Q''1}-(f_{k+Q''}-f_k)\{-V_D(Q'')\rho(Q'')n(k,Q'')+\tfrac{1}{2}n(k,Q'')B_k(Q'')\\ & -\tfrac{1}{2}p(k,Q'')A_k(Q'')\}-4\alpha^2 a_{H0}(-Q)a_\omega(-Q')\exp(i\tilde{\omega}t)k_y\{l(k,Q)(k_x+Q)l(k+Q,Q')(f_{k+Q}-f_{k+Q''})\\ & \times[\tilde{\omega}+\tilde{E}_{k+Q}(Q')]^{-1}-l(k+Q',Q)k_x l(k,Q')(f_k-f_{k+Q'})[\tilde{\omega}+\tilde{E}_k(Q')]^{-1}-p(k+Q',Q)p(k,Q')\\ & \times(1-f_k-f_{k+Q'})k_x[\tilde{\omega}-\nu_k(Q')]^{-1}-p(k,Q)(k_x+Q)p(k+Q,Q')(1-f_{k+Q}-f_{k+Q''})\\ & \times[\tilde{\omega}+\nu_k(Q')]^{-1}+k_x l(k,Q')l(k+Q',Q)(f_{k+Q'}-f_{k+Q''})\tilde{E}_{k+Q'}{}^{-1}(Q)-(k_x+Q)l(k+Q,Q')l(k,Q)\\ & \times(f_k-f_{k+Q})\tilde{E}_k(Q)^{-1}+(k_x+Q)p(k+Q,Q')p(k,Q)(1-f_k-f_{k+Q})\nu_k{}^{-1}(Q)\\ & -p(k,Q')p(k+Q',Q)k_x(1-f_{k+Q'}-f_{k+Q''})\nu_{k+Q'}{}^{-1}(Q)\}. \qquad (49)\end{aligned}$$

Equations (46)–(49) are the basic equations of motion for the Hall current with driving terms proportional to $a_{H0}(-Q)a_\omega(-Q)$.

A self-consistent solution of (46)–(49) together with the equations defining the collective variables is gotten simply for the case that $V(\mathbf{k},\mathbf{k}')$ is independent of angle which we assume throughout.

$$\rho_k(Q'')=B_k(Q'')=A_k(Q'')=0. \qquad (50)$$

The solution (50) is easily seen to be self-consistent since the driving terms in the equations of the collective variables are proportional to an odd function of k_y and hence yield zero upon symmation over $\mathbf{k}$.

Using (46)–(50), the current may be simplified to give

$$j_\omega(Q'')_z=j_\omega(Q'')_x=0, \qquad (51)$$

$$j_\omega(Q'')_y=-8\alpha^2(e\hbar/m)a_{H0}(-Q)a_\omega(-Q')\exp(i\tilde{\omega}t)\sum_k k_y{}^2 k_x M(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q''}). \qquad (52)$$

where

$$M(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q''})$$
$$=\frac{-p(k,Q'')l(k+Q',Q)p(k,Q')(1-f_k-f_{k+Q'})}{[\tilde{\omega}-\nu_k(Q'')][\tilde{\omega}-\nu_k(Q')]}+\frac{p(k,Q'')p(k+Q',Q)l(k,Q')(f_k-f_{k+Q'})}{[\tilde{\omega}+\bar{E}_k(Q')][\tilde{\omega}-\nu_k(Q'')]}$$
$$-\frac{p(k,Q'')l(k,Q')p(k+Q',Q)(1-f_{k+Q'}-f_{k+Q''})}{[\tilde{\omega}-\nu_k(Q'')]\nu_{k+Q'}(Q)}-\frac{p(k,Q'')p(k,Q')l(k+Q',Q)(f_{k+Q'}-f_{k+Q''})}{[\tilde{\omega}-\nu_k(Q'')]\bar{E}_{k+Q'}(Q)}$$
$$+\frac{l(k,Q'')l(k,Q')l(k+Q',Q)(f_k-f_{k+Q'})}{[\tilde{\omega}+\bar{E}_k(Q'')][\tilde{\omega}+\bar{E}_k(Q')]}+\frac{l(k,Q'')p(k+Q',Q)p(k,Q')(1-f_k-f_{k+Q'})}{[\tilde{\omega}+\bar{E}_k(Q'')][\tilde{\omega}-\nu_k(Q')]}$$
$$-\frac{l(k,Q'')l(k,Q')l(k+Q',Q)(f_{k+Q'}-f_{k+Q''})}{[\tilde{\omega}+\bar{E}_k(Q'')]\bar{E}_{k+Q'}(Q)}+\frac{l(k,Q'')p(k,Q')p(k+Q',Q)(1-f_{k+Q'}-f_{k+Q''})}{[\tilde{\omega}+\bar{E}_k(Q'')]\nu_{k+Q'}(Q)}-(\tilde{\omega}\to-\tilde{\omega}). \quad (53)$$

Also,

$$M(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q''})=M(\epsilon_k,\epsilon_{k+Q''},\epsilon_{k+Q'}). \quad (54)$$

The case of physical interest is the limit $Q\to 0$ since this gives a magnetic field which is uniform in the x direction. Thus using a series expansion in powers of Q, we find

$$M(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q''})=M(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q'})+\frac{\hbar^2}{m}k_xQ\left[\frac{\partial M}{\partial\epsilon_{k+Q''}}(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q''})\right]_{Q=0}+\cdots. \quad (55)$$

The first term in (55) gives zero upon summation over $\mathbf{k}$. Using (54), one can show that

$$\left[\frac{\partial M}{\partial\epsilon_{k+Q''}}(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q''})\right]_{Q=0}=\frac{1}{2}\frac{\partial M}{\partial\epsilon_{k+Q'}}(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q'}). \quad (56)$$

The partial derivative on the right of (56) means ϵ_k is to be kept constant. Thus the current is

$$j_y(Q')=-4\alpha^2\frac{\hbar^3}{m^2}Qa_{H_0}(-Q)a_\omega(-Q')\exp(i\tilde{\omega}t)\times\sum_{\mathbf{k}}k_x^2k_y^2\frac{\partial M}{\partial\epsilon_{k+Q'}}(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q'}), \quad (57)$$

where

$$M(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q'})=\frac{p^2(k,Q')(1-f_k-f_{k+Q'})}{[\tilde{\omega}-\nu_k(Q')]^2}+\frac{p^2(k,Q')}{\tilde{\omega}-\nu_k(Q')}\frac{\partial f(E')}{\partial E'}+\frac{l^2(k,Q')(f_k-f_{k+Q'})}{[\tilde{\omega}+\bar{E}_k(Q')]^2}+\frac{l^2(k,Q')}{\tilde{\omega}+\bar{E}_k(Q')}\frac{\partial f(E')}{\partial E'}-(\tilde{\omega}\to-\tilde{\omega}). \quad (58)$$

Since $iQa_{H_0}(-Q)=-H_0$, the current is proportional to H_0. The current may be put into a more convenient form by an integration by parts over the polar angle θ between $\mathbf{k}$ and $\mathbf{Q}'$. This gives

$$(\hbar^2Q/2m)\sum_{\mathbf{k}}k_x^2k_y^2\frac{\partial M}{\partial\epsilon_{k+Q'}}(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q'})=Q/2Q'\sum_{\mathbf{k}}k_y^2k_zM(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q'}), \quad (59)$$

so that,

$$j_y(Q')=-4\alpha^2\frac{\hbar e}{m}iH_0a_\omega(-Q')\exp(i\tilde{\omega}t)[Q']^{-1}\times\sum_{\mathbf{k}}k_y^2k_zM(\epsilon_k,\epsilon_{k+Q'},\epsilon_{k+Q'}). \quad (60)$$

As a special case of (60) we first consider the normal state and prove that the quantum theory for the transverse Hall current given by (60) is identical to the result from the Boltzmann equation given in Sec. II. The matrix element for the normal state is

$$M(\epsilon,\epsilon',\epsilon')=\frac{f(\epsilon')-f(\epsilon)}{(\tilde{\omega}+\epsilon-\epsilon')^2}+\frac{\partial f(\epsilon')}{\partial\epsilon'}\frac{1}{\tilde{\omega}+\epsilon-\epsilon'}-(\tilde{\omega}\to-\tilde{\omega}), \quad (61)$$

where

$$\epsilon\equiv\epsilon_k,\quad \epsilon'\equiv\epsilon_{k+Q'}. \quad (62)$$

Thus,

$$[Q']^{-1}\sum_k k_y^2k_zM(\epsilon,\epsilon',\epsilon')$$
$$=\frac{\pi}{Q'(2\pi)^3}\int_0^\pi d\theta\,\sin^3\theta\cos\theta\int_0^\infty dk\,k^5[f(\epsilon')-f(\epsilon)]\times\left[\frac{1}{(\tilde{\omega}+\epsilon-\epsilon')^2}-\frac{1}{(\tilde{\omega}-\epsilon+\epsilon')^2}\right]$$
$$+\frac{\pi}{Q'(2\pi)^3}\int_0^\pi d\theta\,\sin^3\theta\cos\theta\times\int_0^\infty dk\,k^5\frac{\partial f(\epsilon')}{\partial\epsilon'}\left[\frac{1}{\tilde{\omega}+\epsilon-\epsilon'}+\frac{1}{\tilde{\omega}-\epsilon+\epsilon'}\right]. \quad (63)$$

The integration yields

$$[Q']^{-1}\sum_{\mathbf{k}} k_y^2 k_z M(\epsilon,\epsilon',\epsilon') = \frac{1}{(2\pi)^2}\frac{m\tilde{\omega}}{\hbar^2}k_F^3\int_0^\pi \frac{\sin^3\theta d\theta}{[\tilde{\omega}+\hbar v_0 Q'\cos\theta]^2}, \quad (64)$$

so that

$$j_y(Q') = \tfrac{3}{4}R_0\sigma_0^2 H_0\left[-\frac{i}{c}a_\omega(-Q')\exp(i\tilde{\omega}t)\tilde{\omega}\right] \times\int_0^\pi \frac{d\theta \sin^3\theta}{[1-i(lQ'\cos\theta-\omega\tau)]^2}, \quad (65)$$

where we have taken the phenomenological relaxation time s^{-1} to be equal to τ. Since

$$E(-Q') = -\frac{i}{c}a_\omega(-Q')\exp(i\tilde{\omega}t)\tilde{\omega}, \quad (66)$$

the result (66) gives the same Hall conductivity as derived by the Boltzmann equation treatment of Sec. II. When expressed in real space (65) gives the same result as Eq. (3) of Sec. II.

We now return to the case of the superconductor given by (60). The general result may be written as a double integral over energy ϵ and over angle θ as

$$j_y(Q') = -\frac{\alpha^2 e}{2\pi^2\hbar Q'}iH_0 a_\omega(-Q')\exp(i\tilde{\omega}t) \times\int_0^\pi d\theta\sin^3\theta\cos\theta\int_{-\infty}^\infty d\epsilon\, k^3 M(\epsilon,\epsilon',\epsilon'), \quad (67)$$

where in general,

$$\epsilon = \hbar^2k^2/2m, \qquad \epsilon' = \hbar^2(k^2+2kQ'\cos\theta+Q'^2)/2m. \quad (68)$$

An important special case of (67) is for small Q', namely

$$\hbar v_0 Q' \ll I, \quad (69)$$

and either

$$v_0Q'\ll\omega \quad \text{or} \quad v_0Q'\ll\tau^{-1}. \quad (70)$$

Then we may expand in a power series in Q':

$$E' = E + \frac{\hbar^2}{m}kQ'\cos\theta\frac{\epsilon}{E} + \sim Q'^2 + \cdots, \quad (71)$$

and

$$M(\epsilon,\epsilon',\epsilon') = M(\epsilon,\epsilon,\epsilon) + \left[\frac{\partial M}{\partial E'}(\epsilon,\epsilon',\epsilon')\right]_{\epsilon'=\epsilon} \times(E'-E) + \sim Q'^2 + \cdots. \quad (72)$$

The first term does not contribute to the current since

$$\sum_{\mathbf{k}} k_y^2 k_z M(\epsilon,\epsilon,\epsilon) = 0. \quad (73)$$

Using

$$\left[\frac{\partial p^2}{\partial E'}(k,Q')\right]_{\epsilon=\epsilon'} = \left[\frac{\partial l^2}{\partial E'}(k,Q')\right]_{\epsilon=\epsilon'} = 0, \quad (74)$$

$$\left[\frac{\partial M}{\partial E'}(\epsilon,\epsilon',\epsilon')\right]_{\epsilon'=\epsilon} = \frac{2}{\omega - is}\frac{d^2f(E)}{dE^2}, \quad (75)$$

One finds that the current is

$$j_y(Q') = -\frac{\alpha^2 e\hbar^3 iH_0}{\pi^2m^2(\omega-is)}\int_0^\pi \sin^3\theta\cos^2\theta d\theta \times\int_0^\infty k^6dk\frac{\epsilon}{E}\frac{d^2f(E)}{dE^2}. \quad (76)$$

Use

$$k^5 = k_F^5(1+\tfrac{5}{2}\epsilon/E_F), \quad (77)$$

then

$$\int_0^\infty k^6dk\frac{\epsilon}{E}\frac{d^2f(E)}{dE^2} = \frac{m}{\hbar^2}\int_{-\infty}^\infty d\epsilon\, k_F^5\frac{\epsilon}{E}\frac{d^2f(E)}{dE^2} + 5\left(\frac{m}{\hbar^2}\right)^2 k_F^3\int_{-\infty}^\infty d\epsilon\frac{\epsilon^2}{E}\frac{d^2f(E)}{dE^2}, \quad (78)$$

and integrating by parts

$$\int_0^\infty dk\, k^6\frac{\epsilon}{E}\frac{d^2f(E)}{dE^2} = 5\left(\frac{m}{\hbar^2}\right)^2 k_F^3\left\{-2\int_0^\infty \frac{df(E)}{dE}d\epsilon\right\} = 5(m/\hbar^2)^2k_F^3\{1-\Lambda/\Lambda_T\}, \quad (79)$$

where $(1-\Lambda/\Lambda_T)$ corresponds to the microscopic analog of the fraction of normal electrons ρ_n/ρ of a two fluid model.[28] Thus,

$$j_y(Q') = \frac{R_0\sigma_0^2H_0}{(1+i\omega\tau)^2}\frac{\rho_n}{\rho}\left[\frac{-i}{c}(\omega-i\tau^{-1})a_\omega(-Q')\exp(i\tilde{\omega}t)\right], \quad (80)$$

where we have again identified the phenomenological[29] relaxation time s^{-1} with τ. Since the last factor is $E(-Q')$, the Hall conductivity for long wavelengths (small Q') is the first factor of Eq. (80).

At $T=T_c$, this gives the usual Hall conductivity for the normal state in the long-wavelength limit. The result (80) may be interpreted in a qualitative manner from a two-fluid model viewpoint. The normal component of the primary current flow is proportional to ρ_n/ρ in a two-fluid model [primary current denotes the current proportional to $E(-Q')$]. If we make the additional assumption that only the normal component and

[28] J. Bardeen, Phys. Rev. Letters **1**, 399 (1958).

[29] The identification of s^{-1} with τ may not be completely valid in the superconducting case since the relaxation time of quasi particles depends on their excitation energy. However no such question arises in the important special case of $l\to\infty$.

not the supercurrent is acted on by the magnetic field, then we expect a Hall conductivity proportional to ρ_n/ρ. We note that in order to get the constant of proportionality in (80) from a two-fluid model, we must make a specific assumption about the conductivity of the normal electrons in the two-fluid model; in the absence of a detailed theory such an assumption is little more than guesswork.[30] To get the same constant of proportionality as in (80), one must take for the conductivity of the normal electrons,

$$j_n=\frac{\sigma_0}{1+i\omega\tau}\frac{\rho_n}{\rho}E. \tag{81}$$

An important application of the general result (67) is to small superconducting specimens such as thin films. Let d be the thickness of the film or more generally d is the small dimension of the superconducting specimen. Then for very thin films such as those used by Ginsberg and Tinkham, we have that[24]

$$l\gg d,\quad d/\xi_0\ll 1,\quad d\omega\ll v_0. \tag{82}$$

The electric field is uniform inside such a film and H_0 normal to the film surface will also be uniform inside the film under the appropriate conditions on the sample dimensions discussed earlier. We assume that random scattering takes place at the surface of the specimen; then the important Q' in the film are of order d^{-1} so that $Q'\xi_0\gg 1$ and one needs to evaluate the integral of Eq. (67) in the extreme anomalous limit.

Since both the Lewis experiment and the Spiewak experiment give longitudinal Hall currents, we may only make a qualitative comparison with Eq. (80).[9–11]

APPENDIX A. DIAMAGNETIC HALL CURRENT

The general diamagnetic current operator is

$$\mathbf{J}_D(\mathbf{r})=\frac{-e^2}{mc\Lambda}\sum_{\mathbf{k},\mathbf{q},\sigma} c_{k+Q,\sigma}{}^*c_{k,\sigma}e^{-i\mathbf{q}\cdot\mathbf{r}}[\mathbf{A}_{H0}(r)+\mathbf{A}_\omega(r)], \tag{1}$$

where Λ is the volume. Since

$$J(Q)=(2\pi)^{-\frac{3}{2}}\int \mathbf{J}(r)\exp(i\mathbf{Q}\cdot\mathbf{r})d^3r, \tag{2}$$

$$\mathbf{A}(\mathbf{Q})=(2\pi)^{-\frac{3}{2}}\int \mathbf{A}(\mathbf{r})\exp(-i\mathbf{Q}\cdot\mathbf{r})d^3r, \tag{3}$$

then,

$$\mathbf{J}_D(\mathbf{q}')=\frac{-e^2}{mc\Lambda}\sum_{k,q,\sigma} c_{k+q,\sigma}{}^*c_{k,\sigma}\times[\mathbf{a}_{H_0}(\mathbf{q}-\mathbf{q}')+\mathbf{a}_\omega(\mathbf{q}-\mathbf{q}')], \tag{4}$$

or in terms of quasi-particle operators

$$\begin{aligned}\mathbf{J}_D(\mathbf{q}')=-(e^2/mc\Lambda)\sum_{k,q} m(k,q)\\ \times(\gamma_{k+q\sigma}{}^*\gamma_{k1}{}^*+\gamma_{k+Q1}\gamma_{k0})+n(k,q)\\ \times(\gamma_{k+q1}{}^*\gamma_{k1}+\gamma_{k+q0}{}^*\gamma_{k0})\\ \times[\mathbf{a}_{H_0}(q-q')+\mathbf{a}_\omega(q-q')].\end{aligned} \tag{5}$$

Since we only have $a_{H_0}(-Q)$ and $a_\omega(-Q')$ present the only component of current excited in $J_D(Q'')$

$$\begin{aligned}\mathbf{J}_D(\mathbf{Q}'')=\frac{-e^2}{mc\Lambda}\{\sum_{\mathbf{k}}[m(k,Q')(\gamma_{k+Q'0}{}^*\gamma_{k1}{}^*+\gamma_{k+Q'1}\gamma_{k0})\\ +n(k,Q')(\gamma_{k+Q'1}{}^*\gamma_{k1}+\gamma_{k+Q'0}{}^*\gamma_{k0})]\mathbf{a}_{H_0}(-Q)\\ +[m(k,Q)(\gamma_{k+Q0}{}^*\gamma_{k1}{}^*+\gamma_{k+Q1}\gamma_{k0})\\ +n(k,Q)(\gamma_{k+Q1}{}^*\gamma_{k1}+\gamma_{k+Q0}{}^*\gamma_{k0})]\mathbf{a}_\omega(-Q')\}.\end{aligned} \tag{6}$$

We need the expectation values to first order in $a_{H_0}(-Q)$ of operators which differ by momentum $\mathbf{Q}$ (i.e., $\gamma_{k+Q0}{}^*\gamma_{k1}{}^*$) and to first order in $a_\omega(-Q')$ of operators which differ by momentum Q' (i.e., $\gamma_{k+Q'1}{}^*\gamma_{k1}{}^*$). These results are given in Sec. IV by Eqs. (43) and (45). These equations show that $\gamma_{k+Q0}{}^*\gamma_{k1}{}^*$ is proportional to an odd power of k_x and hence the sum over $\mathbf{k}$ in Eq. (6) clearly yields zero for our choice of gauge,

$$\mathbf{J}_D(\mathbf{Q}'')=0. \tag{7}$$

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[30] D. Shoenberg, *Superconductivity* (Cambridge University Press, New York, 1960).