

## Continuous Absorption Due to Free-Free Transitions in Hydrogen

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The absorption of radiation in the free-free transition of the negative hydrogen ion is the inverse process of bremsstrahlung from an electron in the vicinity of a neutral hydrogen atom. This process is the most important cause of continuous absorption by the atmosphere of the sun and stars in the infrared region. The transition matrix element is expressed, with sufficient accuracy, in terms of the  $s$  phase shifts of electron-hydrogen atom scattering alone. To meet the requirement in astrophysical studies, the continuous absorption coefficients due to the free-free transition have been tabulated for a wide range of wavelengths (4050 Å to infinity) and temperatures (2520 to 10 080°K) of the hydrogen gas, by using the  $s$  phase shifts of  $e$ -H scattering which include exchange and correlation effects between two electrons. The tabulated coefficients are 20–70% less than the ones computed by Chandrasekhar and Breen, who used Hartree functions without exchange. Further improvement of the present result is discussed.

### I. INTRODUCTION

THE photodetachment of an electron of the negative hydrogen ion  $H^-$  is called a “bound-free” transition. This terminology indicates that a bound electron in the initial state becomes a free electron in the final state by the absorption of radiation. Similarly, the absorption of radiation in a “free-free” transition of the negative hydrogen ion is the inverse process of bremsstrahlung from an electron in the vicinity of a neutral hydrogen atom. The possible astrophysical importance of this process was first suggested by Pannekoek.<sup>1</sup> But the subsequent evaluations<sup>2</sup> of the transition rate are too small by a factor 10 to account for the continuous absorption of light in the infrared region. Their calculations are based on a Born approximation. Later, Chandrasekhar and Breen<sup>3</sup> showed that the distortion of the free electron wave function by neutral hydrogen atom makes the free-free transition rates much larger and that the absorption coefficients of the free-free transition of  $H^-$ , together with the bound-free transition, are sufficient to account for the solar continuous spectrum from  $\lambda = 4000$  Å to  $\lambda = 25\,000$  Å. The calculated coefficient seemed to be somewhat too large when compared with that deduced from the observed spectrum, but the deduced spectrum itself has a large uncertainty due to the assumed model of the atmosphere. Chandrasekhar and Breen use Hartree wave functions without exchange for the initial and the final states. For slow  $s$  electrons, however, the effects

of exchange and correlation of the two electrons are not negligible. In fact, these effects are very important for scattering of electrons by hydrogen atoms and for the wave function of the negative hydrogen ion.

It has been shown<sup>4</sup> that the free-free transition matrix element  $M$  can be expressed, with an accuracy of the order of 1%, in terms of the  $s$  phase shifts of the electron-hydrogen atom scattering alone [see Eq. (2) below] in the region we are interested in. A convenient transition matrix element from  $s$ -electron state with wave number  $k_0$  to  $p$ -electron state with wave number  $k_1$  is defined by

$$M^{(\pm)}(0, k_0^2 | 1, k_1^2) = \frac{3k_0k_1(\Delta k^2)^2}{32\pi^2} \int (z_1 + z_2) \phi_1^{(\pm)} \psi_0^{(\pm)} d\tau_1 d\tau_2, \quad (1)$$

where  $\Delta k^2 = |k_0^2 - k_1^2|$ ,  $d\tau_1 = r_1^2 dr_1 \sin\theta_1 d\theta_1 d\varphi_1$ , and the  $s$ -wave function  $\psi_0$  and the  $p$ -plane wave  $\phi_1$  are so normalized that they have the following asymptotic form,

$$\psi_0^{(\pm)} \rightarrow \begin{cases} e^{-r_1} \sin(k_0 r_2 + \delta_0^{(\pm)}) / (k_0 r_2); & r_2 \rightarrow \infty \\ \pm e^{-r_2} \sin(k_0 r_1 + \delta_0^{(\pm)}) / (k_0 r_1); & r_1 \rightarrow \infty, \end{cases} \quad (1a)$$

and

$$\phi_1^{(\pm)} \rightarrow \begin{cases} e^{-r_1} \cos\theta_2 [\sin(k_1 r_2) / (k_1 r_2)^2 \\ - \cos(k_1 r_2) / (k_1 r_2)]; & r_2 \rightarrow \infty \\ \pm e^{-r_2} \cos\theta_1 [\sin(k_1 r_1) / (k_1 r_1)^2 \\ - \cos(k_1 r_1) / (k_1 r_1)]; & r_1 \rightarrow \infty. \end{cases} \quad (1b)$$

(+) and (−) correspond to the singlet and the triplet systems, respectively. Atomic units will be employed throughout this paper unless otherwise stated.  $M^{(\pm)}$  is approximated with sufficient accuracy<sup>4</sup> by (see also Sec. IV)

$$M^{(\pm)}(0, k_0^2 | 1, k_1^2) = \frac{1}{2} k_1^2 \sin\delta_0^{(\pm)}. \quad (2)$$

If the accurate  $s$  phase shifts  $\delta_0^{(+)}$  and  $\delta_0^{(-)}$  which in-

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<sup>1</sup> A. Pannekoek, *Monthly Notices Roy. Astron. Soc.* **91**, 162 (1931).

<sup>2</sup> L. Nedelsky, *Phys. Rev.* **42**, 641 (1932); D. H. Menzel and C. L. Pekeris, *Monthly Notices Roy. Astron. Soc.* **96**, 77 (1935); J. A. Wheeler and R. Wildt, *Astrophys. J.* **95**, 281 (1942).

<sup>3</sup> S. Chandrasekhar and F. H. Breen, *Astrophys. J.* **104**, 430 (1946).

<sup>4</sup> T. Ohmura and H. Ohmura, *Astrophys. J.* **131**, 8 (1960).

TABLE I.  $\sin^2\delta_0^{(\pm)}$  calculated by (3) to (7). The listed values in parentheses are obtained by (6) and (7) but with  $a_- = 2.33$  replaced by  $a_- = 1.91$ .

$k^2$	$\sin^2\delta_0^{(+)}$	$\sin^2\delta_0^{(-)}$
0.01	0.311	0.0529 (0.0360)
0.02	0.521	0.1032 (0.0710)
0.03	0.667	0.1509 (0.1051)
0.04	0.770	0.1962 (0.1383)
0.06	0.898	0.280 (0.202)
0.08	0.962	0.356 (0.262)
0.10	0.991	0.424 (0.318)
0.12	1.000	0.486 (0.372)
0.16	0.985	0.593 (0.469)
0.20	0.950	0.679 (0.554)
0.24	0.909	0.750 (0.628)
0.32	0.824	0.854 (0.748)
0.40	0.748	0.921 (0.837)

clude the exchange and correlation effects are once known, the free-free absorption rate can be calculated without the use of the wave functions of the initial and the final states. This simplifies the calculation greatly. The most accurate solutions of the two-electron system (with an infinitely heavy proton) in the low-energy region have been obtained by variational methods,<sup>5</sup> in which the accuracy of the phase shift is clearly greater than the corresponding accuracy of the wave function itself. This is the second advantage of using (2) rather than using the wave function. The accuracy of the phase shifts may be, at present, of the order of 10%. It has been shown<sup>4</sup> that the inclusion of the exchange and the correlation effects will reduce the transition rate considerably and that the new values of the free-free coefficient derived from (2) are 40–50% less than the ones<sup>3</sup> computed by Chandrasekhar and Breen for the temperature  $T = 6300^\circ\text{K}$ .

In the present paper an extensive tabulation of the free-free coefficients is attempted on the present knowledge of the  $s$  phase shifts of electron-hydrogen scattering. It is hoped that this table provides a better basis for astrophysical calculations on the atmosphere of the sun and stars.

TABLE II. The ratio  $R$  of the present result for the absorption coefficient to that of Chandrasekhar and Breen. The calculation is based on (3)–(9). The values in parentheses are obtained in the same way, only with  $a_- = 2.33$  replaced by  $a_- = 1.91$ .  $\theta = 5040/(\text{absolute temperature of the gas})$ . The wavelength  $\lambda$  (in Å)  $= 911.3/(\Delta k^2)$ .

$\theta$ $\Delta k^2$	0.5	0.8	1.4	2.0
0.00	0.79 (0.69)	0.66 (0.57)	0.53 (0.46)	0.46 (0.41)
0.06	0.71	0.57	0.43	0.37
0.20	0.60	0.48 (0.42)	0.37	0.32

<sup>5</sup> H. S. W. Massey and B. L. Moiseiwitsch, Proc. Roy. Soc. (London) A205, 483 (1951); S. Borowitz and H. Greenberg, Phys. Rev. 108, 716 (1957); M. J. Seaton, Proc. Roy. Soc. (London) A241, 522 (1957); B. H. Bransden, A. Dalgarno, T. L. John, and M. J. Seaton, Proc. Phys. Soc. (London) A71, 877 (1958); S. Geltman, Phys. Rev. 119, 1283 (1960); L. Rosenberg, L. Spruch, and T. F. O'Malley, Phys. Rev. 119, 164 (1960); Y. Hara, T. Ohmura, and T. Yamanouchi, Progr. Theoret. Phys. (to be published).

The phase-shift values available at present are discussed in Sec. II and the adopted phase shifts are given in Table I. With the phase shifts given in Table I, the continuous absorption coefficients are calculated in Sec. III by Eqs. (8) and (9), and compared with the values of Chandrasekhar and Breen in Table II. Then the coefficients are tabulated in Table III for a wide range of wavelengths ranging from 4050 Å to infinity, and for temperatures  $T$  of the gas, 2520°K to 10 080°K. Further possible improvements are discussed in Sec. IV.

*Note added in proof.* S. Geltman and J. B. Martin have recently calculated the same absorption coefficients by the same method employed in the present paper

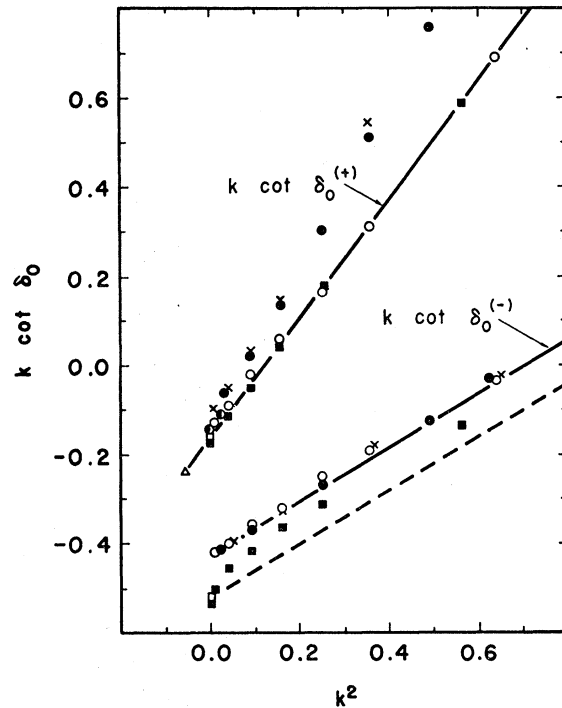


FIG. 1. The values of  $k \cot \delta_0$  computed by various methods: the Hartree-Fock approximation (closed circles), Massey-Moiseiwitsch (open circles), Bransden *et al.* (half-closed circles), Geltman (crosses), Rosenberg-Spruch-O'Malley (open square), the point of bound state of  $\text{H}^-$  (triangle), and Temkin-Lamkin (closed squares). The solid line is drawn by use of Eqs. (3) to (7). The dashed line corresponds to  $a_- = 1.91$  and  $r_0 = 1.22$ .

[see Eq. (9) below], but by using Geltman's  $s$  phase shifts.<sup>5</sup> Their values are somewhat larger than our values tabulated in Table III (private communication).

## II. PHASE SHIFT IN ELECTRON-HYDROGEN SCATTERING

The phase shifts of  $e\text{--H}$  scattering including electron exchange have been obtained by many authors.<sup>6</sup> These

<sup>6</sup> P. M. Morse and W. P. Allis, Phys. Rev. 44, 269 (1933); M. J. Seaton, Proc. Phys. Soc. (London) A241, 522 (1957); K. Omidvar, New York University, Institute of Mathematical Sciences Report No. CX-37, 195 (unpublished); E. Trefftz (quoted by F. B. Malik, dissertation, 1958, Max-Planck Institute, Munich); T. L. John (to be published).

results do not take into account the electron-electron correlation, and will be called the "Hartree-Fock" approximation.

Several variational calculations with 3-5 adjustable parameters, which take care of correlation to some extent, have been carried out by Massey and Moiseiwitch,<sup>5</sup> Bransden *et al.*,<sup>5</sup> and Geltman<sup>5</sup> for various energies. Temkin and Lamkin<sup>7</sup> obtained some values by the polarized orbital method. Finally we mention the calculation, with 7-9 parameters, at zero energy by Rosenberg, Spruch, and O'Malley.<sup>5</sup> All these results are shown in Fig. 1. We may see from Fig. 1 that, roughly speaking, all the points for the singlet (and also similarly for the triplet) state are on a straight line.

We now assume that  $k \cot \delta_0$  is a linear function of  $k^2$ . It is known<sup>8</sup> that the curve  $k \cot \delta_0^{(+)}(k^2)$  in the singlet state goes through the point,  $k^2 = -\gamma^2$ ,  $k \cot \delta_0 = -\gamma$ , which is marked by a triangle in Fig. 1.  $\gamma^2/2$  is the electron affinity of the negative hydrogen ion  $H^-$ . Then  $k \cot \delta_0^{(+)}$  is expanded in powers of  $(k^2 + \gamma^2)$  around  $k^2 = -\gamma^2$ , and up to the first term,

$$k \cot \delta_0^{(+)} = -\gamma + (\rho/2)(\gamma^2 + k^2). \quad (3)$$

The constant  $\rho$  in (3) can be calculated<sup>8</sup> from the asymptotic amplitude of the  $H^-$  wave function and has been evaluated<sup>9</sup> as

$$\rho = 2.464. \quad (4)$$

The most accurate value<sup>10</sup> for  $\gamma$  is

$$\gamma = 0.23559. \quad (5)$$

The straight line in Fig. 1 for  $k \cot \delta_0^{(+)}$  is drawn according to (3) with the constants (4) and (5) deduced from the  $H^-$  wave function. The straight line is well fitted to the value of Rosenberg, Spruch, and O'Malley at  $k^2 = 0$ , and to the Massey and Moiseiwitch values and the Temkin and Lamkin values for all energies. Thus it would be reasonable to assume at present that (3) is approximately correct with the constants (4) and (5). Some values of the phase shift  $\delta_0^{(+)}$  thus computed are tabulated in Table I. In the triplet state we also assume a linear relation between  $k \cot \delta_0^{(-)}$  and  $k^2$ :

$$k \cot \delta_0^{(-)} = -(1/a_-) + (r_0/2)k^2. \quad (6)$$

The solid line for  $k \cot \delta_0^{(-)}$  in Fig. 1 is drawn with the constants<sup>9</sup>

$$a_- = 2.33, \quad r_0 = 1.22. \quad (7)$$

The line is also well fitted to a group of points with important exceptions—the point 1.91 (as an upper bound of  $a_-$ ) due to Rosenberg, Spruch, and O'Malley, and the Temkin and Lamkin values. We now assume that  $\delta_0^{(-)}$  is well reproduced by (6) and (7). The com-

TABLE III. The continuous absorption coefficient of the negative hydrogen ion per neutral hydrogen atom and per unit electron pressure due to the free-free transitions for various temperatures and wavelengths, after allowing for the stimulated emission factor. The numbers in parentheses give the powers of 10 by which the corresponding entries should be multiplied to get the coefficient in units of  $\text{cm}^4/\text{dyne}$ .

$\Delta k^2$	$\lambda$ (Å)	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$	$\theta = 0.9$	$\theta = 1.0$	$\theta = 1.2$	$\theta = 1.4$	$\theta = 1.6$	$\theta = 1.8$	$\theta = 2.0$
0.005	182 300	2.31 (-29)	2.81 (-29)	3.30 (-29)	3.77 (-29)	4.23 (-29)	4.68 (-29)	5.56 (-29)	6.64 (-29)	7.73 (-29)	8.82 (-29)	9.89 (-29)
0.010	91 130	9.16 (-25)	1.12 (-24)	1.31 (-24)	1.49 (-24)	1.67 (-24)	1.85 (-24)	2.20 (-24)	2.53 (-24)	2.87 (-24)	3.19 (-24)	3.52 (-24)
0.015	60 750	2.28 (-25)	2.78 (-25)	3.25 (-25)	3.70 (-25)	4.15 (-25)	4.59 (-25)	5.54 (-25)	6.63 (-25)	7.71 (-25)	8.79 (-25)	9.87 (-25)
0.020	45 560	1.01 (-25)	1.23 (-25)	1.44 (-25)	1.64 (-25)	1.84 (-25)	2.03 (-25)	2.41 (-25)	2.79 (-25)	3.15 (-25)	3.51 (-25)	3.87 (-25)
0.025	36 450	5.69 (-26)	6.92 (-26)	8.11 (-26)	9.32 (-26)	1.03 (-25)	1.15 (-25)	1.36 (-25)	1.57 (-25)	1.77 (-25)	1.97 (-25)	2.17 (-25)
0.030	30 380	3.64 (-26)	4.43 (-26)	5.19 (-26)	5.92 (-26)	6.67 (-26)	7.43 (-26)	8.87 (-26)	1.00 (-25)	1.13 (-25)	1.26 (-25)	1.38 (-25)
0.040	22 780	2.54 (-26)	3.08 (-26)	3.60 (-26)	4.12 (-26)	4.61 (-26)	5.09 (-26)	6.61 (-26)	0.69 (-25)	0.79 (-25)	0.87 (-25)	0.96 (-25)
0.050	18 230	1.43 (-26)	1.74 (-26)	2.04 (-26)	2.32 (-26)	2.60 (-26)	2.87 (-26)	3.39 (-26)	3.90 (-26)	4.44 (-26)	4.99 (-26)	5.54 (-26)
0.060	15 190	0.92 (-26)	1.12 (-26)	1.31 (-26)	1.49 (-26)	1.67 (-26)	1.84 (-26)	2.17 (-26)	2.48 (-26)	2.79 (-26)	3.08 (-26)	3.38 (-26)
0.070	13 020	0.47 (-27)	0.78 (-27)	0.91 (-27)	1.04 (-27)	1.16 (-27)	1.28 (-27)	1.50 (-27)	1.71 (-27)	1.92 (-27)	2.12 (-27)	2.32 (-27)
0.080	11 390	0.37 (-27)	0.45 (-27)	0.51 (-27)	0.59 (-27)	0.65 (-27)	0.72 (-27)	0.84 (-27)	0.95 (-27)	1.06 (-27)	1.18 (-27)	1.29 (-27)
0.090	10 130	0.29 (-27)	0.35 (-27)	0.40 (-27)	0.46 (-27)	0.52 (-27)	0.56 (-27)	0.66 (-27)	0.75 (-27)	0.84 (-27)	0.93 (-27)	1.01 (-27)
0.100	9113	0.23 (-27)	0.28 (-27)	0.32 (-27)	0.37 (-27)	0.41 (-27)	0.45 (-27)	0.53 (-27)	0.61 (-27)	0.68 (-27)	0.75 (-27)	0.82 (-27)
0.120	7594	1.65 (-27)	1.99 (-27)	2.31 (-27)	2.60 (-27)	2.89 (-27)	3.17 (-27)	3.70 (-27)	4.42 (-27)	5.17 (-27)	5.92 (-27)	6.67 (-27)
0.140	6509	1.22 (-27)	1.47 (-27)	1.70 (-27)	1.92 (-27)	2.13 (-27)	2.34 (-27)	2.74 (-27)	3.13 (-27)	3.55 (-27)	3.99 (-27)	4.43 (-27)
0.160	5695	0.94 (-27)	1.13 (-27)	1.31 (-27)	1.48 (-27)	1.65 (-27)	1.81 (-27)	2.13 (-27)	2.45 (-27)	2.77 (-27)	3.10 (-27)	3.44 (-27)
0.180	5063	0.74 (-27)	0.90 (-27)	1.04 (-27)	1.18 (-27)	1.32 (-27)	1.46 (-27)	1.73 (-27)	2.01 (-27)	2.28 (-27)	2.56 (-27)	2.84 (-27)
0.200	4556	0.59 (-28)	0.73 (-27)	0.85 (-27)	0.98 (-27)	1.10 (-27)	1.22 (-27)	1.46 (-27)	1.70 (-27)	1.95 (-27)	2.20 (-27)	2.45 (-27)
0.225	4050	0.47 (-28)	0.58 (-27)	0.69 (-27)	0.79 (-27)	0.90 (-27)	1.00 (-27)	1.21 (-27)	1.43 (-27)	1.65 (-27)	1.96 (-27)	2.11 (-27)

<sup>a</sup> The entries in this line when divided by  $(\Delta k^2)^2$  will give the corresponding absorption coefficients to a sufficient accuracy for all  $\Delta k^2 < 0.005$ .

<sup>7</sup> A. Temkin and J. C. Lamkin, Phys. Rev. (to be published).

<sup>8</sup> T. Ohmura, Y. Hara, and T. Yamanouchi, Progr. Theoret. Phys. (Kyoto) 20, 82 (1958).

<sup>9</sup> T. Ohmura and H. Ohmura, Phys. Rev. 118, 154 (1960).

<sup>10</sup> C. L. Pekeris, Phys. Rev. 112, 1649 (1958).

puted values by (6) and (7) are tabulated in Table I. The values of  $\delta_0^{(-)}$  computed by (6) but with  $a_- = 1.91$  and  $r_0 = 1.22$  are also included in parentheses in Table I for comparison. The latter values correspond to the dashed line in Fig. 1, and may be considered to be, roughly speaking, a lower bound on  $k \cot \delta_0^{(-)}$ . The dashed line in Fig. 1 does not seem to represent accurate phase shifts for  $k^2 \gtrsim 0.1$ .

Recently Spruch, O'Malley, and Rosenberg have pointed out<sup>11</sup> that the linear approximation, (3) and (6), is not valid if we consider the long-range polarization effect properly in a system consisting of an electron plus the target atom. The situation, however, has not yet been completely clarified. This point will be discussed in Sec. IV.

### III. EVALUATION OF FREE-FREE ABSORPTION COEFFICIENT

The continuous absorption coefficient of the negative hydrogen ion due to the free-free transition per neutral hydrogen atom and per unit electron pressure with the stimulated emission factor  $[1 - \exp(-31.32\theta\Delta k^2)]$  is evaluated by the formula<sup>3</sup>

$$\kappa(\Delta k^2) = 7.25 \times 10^{-27} \theta^3 [1 - \exp(-31.32\theta\Delta k^2)] (\Delta k^2)^{-3} \times \int d(k_0^2) \frac{\exp(-31.32\theta k_0^2)}{k_0 k_1} \{ |M(0, k_0^2 | 1, k_1^2)|^2 + |M(0, k_1^2 | 1, k_0^2)|^2 \} \text{ cm}^4/\text{dyne}, \quad (8)$$

where  $|M|^2$  is obtainable<sup>4</sup> from (2) as

$$|M(0, k^2 | 1, k_1^2)|^2 = \frac{1}{16} k_1^4 [3 \sin^2 \delta_0^{(-)}(k) + \sin^2 \delta_0^{(+)}(k)]. \quad (9)$$

The ratio  $R$  of some of the present values using (3) to (9) to those of Chandrasekhar and Breen is tabulated in Table II. It will be seen from Table II that the new absorption coefficients are quite small (about 30–40%) compared with the values of Chandrasekhar and Breen for low temperatures and high-frequency photons. The ratio increases up to 0.8.

Several values of  $R$  have been computed using  $a_- = 1.91$  and  $r_0 = 1.22$  (corresponding to the dashed line in Fig. 1 and to the values in parentheses in Table I) instead of using (7). The result is shown in parentheses in Table II. These values may be compared with the main body of Table II in order to see the extent of the dependence of the result on the  $s$  phase shift.

The absorption coefficient  $\kappa$  computed by the formulas (3)–(9) is tabulated in Table III. Table III should be compared with Table VI of reference 3.

### IV. POSSIBLE FURTHER IMPROVEMENTS

The possible errors in the calculated absorption coefficient in Table III may be enumerated as follows:

- (a) Contribution from other than the electric dipole transition.
- (b) Contribution of the electric dipole transition from other than  $s \rightarrow p$ , and  $p \rightarrow s$  transitions.
- (c) Approximation given by (2) or (9).
- (d) Use of the plane wave solution for  $p$  states.
- (e) Use of the approximate phase shifts (see Table I).

The magnitude of errors coming from (a), (b),<sup>3</sup> and (c)<sup>4</sup> has been evaluated to be of the order of one percent for the energy range we are interested in. But it will be shown that the long-range polarization effect will give a correction term to the formula (2).

The approximate expression (2) is obtained<sup>4</sup> by substituting the asymptotic form (1a) and (1b) into (1) for  $\psi_0$  and  $\phi_1$ , respectively. Thus the evaluation of  $M$  originates from the integral  $I_0$ ,

$$I_0 \equiv \int_0^\infty r \sin(k_0 r + \delta_0) \left\{ \frac{\sin k_1 r}{k_1 r} - \cos k_1 r \right\} dr = 2k_1^2 \sin \delta_0 / (k_0^2 - k_1^2)^2. \quad (10)$$

According to Castillejo, Percival, and Seaton,<sup>12</sup> the  $s$  radial wave function  $u(r)$ , for sufficiently large  $r$ , satisfies the equation

$$\left( \frac{d^2}{dr^2} + k_0^2 + \frac{4.5}{r^4} \right) u(r) = 0, \text{ for large } r \text{ and } k_0^2 < \frac{3}{4}.$$

The potential  $-2.25/r^4$  (in atomic units) is the Van der Waals force due to the polarization of the target hydrogen by the incident electron. Therefore the radial wave function will deviate from the asymptotic form,  $\sin(k_0 r + \delta_0)$ , even for very large  $r$ . Since the contribution to the integral (10) comes from larger  $r$ , the deviation at large  $r$  will affect the integral (10).

Consider the limit of  $k_0 \rightarrow 0$  for simplicity. The integral becomes

$$I_0 = -k_0 \int_0^\infty r(r-a) \left\{ \frac{\sin k_1 r}{k_1 r} - \cos k_1 r \right\} dr = \frac{2k_0 a}{k_1^2}, \quad (11)$$

where  $a$  is the scattering length which is defined by

$$\lim_{k_0 \rightarrow 0} \tan \delta_0 = -k_0 a.$$

Let us improve the approximate radial function  $r-a$  in (11) by replacing it by a more accurate function<sup>11</sup>:

$$r \left( 1 - \frac{4.5}{2r^2} + \dots \right) - a \left( 1 - \frac{4.5}{6r^2} + \dots \right). \quad (12)$$

The integral now has extra terms,

$$I = I_0 \left( 1 + \frac{4.5\pi k_1}{8a} - \frac{4.5k_1^2}{12} + \dots \right). \quad (13)$$

<sup>11</sup> The authors wish to thank Professor Spruch and Mr. O'Malley for conversation on this point.

<sup>12</sup> L. Castillejo, I. C. Percival, and M. J. Seaton, Proc. Roy. Soc. (London) **254**, 259 (1960).

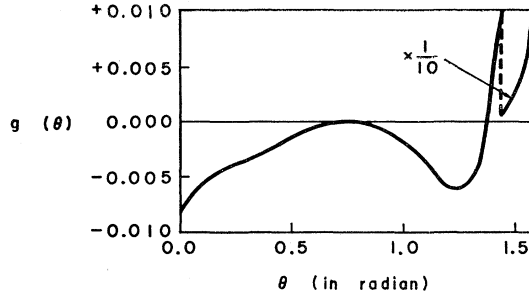


Fig. 2. The function  $g(\theta) \equiv f(k_1, k_2)/(k_0^2 + k_1^2)$ ,  $\theta \equiv \tan^{-1}(k_1^2/k_0^2)$ .

$I$  is thus expressed by a series in powers of  $k_1$  corresponding to the expansion in (12). The second and third terms in (13) come, respectively, from the terms  $-4.5/(2r)$  and  $4.5a/(6r^2)$  in (12). The use of higher terms in (12) will not give a reasonable result, because the higher terms have a stronger singularity at  $r=0$ . The second term in (13) is the main correction which amounts to 8% for the singlet system and 28% for the triplet system if  $k_1^2$  is taken as 0.09 which corresponds to  $\lambda=10^5$  Å.

The correction will be much smaller for  $k_0 \neq 0$ . Similar calculations lead to the following expression for  $I$ :

$$I = I_0 [1 - k_0 \cot \delta_0 \times 4.5\pi (k_1^2 - k_0^2)^2 (k_1 - k_0)^2 / 8k_1^5 - 4.5f(k_0, k_1) + \dots], \quad \text{for } k_0 < k_1, \quad (14)$$

$$= I_0 [1 - 4.5f(k_0, k_1) + \dots], \quad \text{for } k_0 > k_1.$$

The function  $-4.5f(k_0, k_1)$  is identical with the third term in (13) for  $k_0=0$ , and  $f(k_0, 0) = -k_0^2/120$ . The function  $g(\theta) \equiv (k_0^2 + k_1^2)^{-1}f(k_0, k_1)$  is shown in Fig. 2, with  $\tan \theta \equiv k_1^2/k_0^2$ .  $f(k_0, k_1)$  is very small for small  $k_0^2$  and  $k_1^2$  except for  $k_0 \approx 0$ . Therefore, the correction can be safely neglected for  $k_0 > k_1$ . Even for  $k_0 < k_1$  the correction is small. For example, if we take  $\Delta k^2 = k_1^2 - k_0^2 = 0.9$  (e.g.,  $\lambda \sim 10^5$  Å) and  $k_0^2 = 0.04$ , the correction is 0.7% for the singlet system and 2.8% for the triplet system. The correction becomes smaller for smaller  $\Delta k^2$ .

The formula for  $u(r)$ , Eq. (12), is only valid for large  $r$ . But it seems certain that the correction to the approximate equation (9) due to the long-range correlation makes the value of  $|M|^2$  larger. We expect that the formulas (14) with  $f=0$  will give an expression for  $M$  with an accuracy of a few percent.

The uncertainty due to (d) and (e) is rather difficult

to estimate. The magnitude and the sign of the  $p$  phase shifts is largely dependent on the details of the calculation, but it seems very probable that the  $p$  phase shifts are very small<sup>6,7</sup> in magnitude. The uncertainty in the  $s$  phase shifts seems to be the largest source of error in the present calculation.

We already know that the correct scattering length for the triplet system must be smaller<sup>13</sup> than 1.91, while the adopted value in the present calculation is 2.33. Temkin and Lamkin's result<sup>7</sup> also suggest a possible improvement over the approximation, (6) and (7). This has been calculated by the "polarized orbital method,"<sup>14</sup> and it is the only result available at the present time which takes into account the long-range polarization effect systematically. If we use their values of  $s$  phase shifts (see Fig. 1), the value of the matrix element  $M$  of Eq. (2), in the triplet system with the constants (6) and (7), is reduced by 18% for  $k_0^2=0$ , 10% for  $k_0^2=0.04$ , 5% for  $k_0^2=0.16$ . The reduction in the singlet system is small, that is, the value of  $M$  with the constants (3), (4), and (5) is reduced by 6% for  $k_0^2=0$ , 2% for  $k_0^2=0.04$ , and -4% for  $k_0^2=0.16$ .

Finally we may conclude as follows: The correction to the expression for  $M$ , Eq. (2), due to the polarization may be within a few percent except for the triplet system where  $k_0^2$  is less than 0.04. The expected improvement of the  $\sin \delta_0$  value in (2) itself will be of the order of 10% for the triplet system, and less than several percent for the singlet system. Two kinds of corrections cancel out partially. The contributions of the triplet and the singlet systems to the absorption coefficient are of the same order [see Eq. (9) and Table I]. Thus it seems possible that  $|M|^2$  will be further reduced by about 10% from the values listed in Table III with further improvements of the theoretical calculations on the electron-hydrogen atom scattering. When sufficient new information is available, we shall revise the tabulation of the free-free absorption coefficient given here.

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<sup>13</sup> L. Rosenberg, L. Spruch, and T. F. O'Malley, see reference 5.

<sup>14</sup> A. Temkin, Phys. Rev. **107**, 1004 (1957); **116**, 358 (1959).