where q is the number of excitons, τ_q is the exciton lifetime, and ϵ is the minimum energy to produce an exciton. If it is assumed that the collisions between excitons and trapping centers obey classical kinetic theory, the number of free electrons during x raying is given by

$$n=\tau_n\sigma v_q(N_0-N)q, \tag{6}$$

where τ_n is the lifetime of a conduction electron, σ is the cross section for ionization of a trapping center by an exciton, and v_q is the thermal velocity of the exciton. Substitution of Eq. (5) into Eq. (6) gives the dependence required by Eq. (4).

For a numerical estimate, ϵ may be taken as 10 ev, τ_q as the optical exciton lifetime of 10^{-8} second, and σ as 10^{-15} sq cm. The mean lifetime of a conduction electron against trapping by an F center has been calculated by Redfield[14] as 7×10^{-10} sec at 200°K. If it is assumed that this is the predominant process establishing a steady-state electron concentration at 300°K, τ_n can be taken as 10^{-9} sec. Finally, v_q from the exciton effective mass[15] is about 10^7 cm/sec. Substitution of these quantities gives a value for n of $\approx10^{-26}(dE/dt)(N_0-N)$. This may be compared to the experimental recovery rate if a value is assumed for k', the rate-constant for the reaction between free electrons and negative-ion vacancies. Again from kinetic theory, for a capture cross section of 10^{-14} sq cm and an electron velocity of 10^7 cm/sec, k' equals 10^{-7} cc per electron-sec. Thus, (k/k') equals $\approx10^{-26}$. Although the agreement is fortuitous, these calculations show that an exciton transfer mechanism is consistent with the observed recovery process.

A final point is that the energy required to recover F centers is independent of the extent of optical bleaching. This suggests that the recovered F centers are isolated and not clustered, which is consistent with the conclusion of Konitzer and Markham[11] that "the F center is in a perfect lattice, no matter how it is made."

[14] A. G. Redfield, Phys. Rev. **94**, 537 (1954).

[15] W. R. Heller and A. Marcus, Phys. Rev. **84**, 809 (1951).

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Ultrasonic Attenuation in Normal Metals at Low Temperatures

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Expressions for the attenuation α_d and α_t of plane dilatational and shear sound waves are obtained by solving the Boltzmann transport equation for the electron distribution function f without assuming the existence of a relaxation time τ for the collision term in this equation. Instead the collision integral is considered to arise explicitly from the interaction of electrons with thermal phonons and impurities. Making the usual "ideal metal" assumptions, it is found that the attenuation in general depends on a set of effective relaxation times τ_{LM} which are associated with the various terms in the expansion of f in a series of spherical harmonics $Y_{LM}(\theta,\varphi)$; the τ_{LM} are independent of the subscript M, and hence the same set $\{\tau_L\}$ determines both α_d and α_t. Explicit expressions for τ_L are derived.

For the case in which all the τ_L equal to one another and equal to τ say, the analytical expressions for α_d and α_t obtained here are the same as those of Pippard. However, usually τ_L are not equal to one another. It is then found that when $\lambda\gg l$ (λ is the wavelength of the sound wave and l a mean free path of the electrons), τ in Pippard's expressions must be replaced by τ_2 and, contrary to what is usually assumed, α would not be in general proportional to the electrical conductivity σ ($\sigma\propto\tau_1$). When $\lambda\ll l$, the attenuation, with one exception, is independent of $\{\tau_L\}$ and is the same as that given by Pippard. For $\lambda\sim l$, and τ_L not equal to one another, α may be calculated numerically if the ratios τ_L/τ_1 are known; the results of one such calculation show that the deviations from Pippard's analytical expressions are at most about 20%, provided τ in the latter is identified as τ_2.

Lastly, the possible influence of electron-electron collisions on attenuation is briefly discussed.

INTRODUCTION

BELOW about 20°K, the attenuation of ultrasonic waves in metals arises primarily from their interaction with the conduction electrons. Pippard[1] gave an extensive *kinetic* treatment of this phenomenon in terms of the free-electron model of a metal. His basic idea is that the steady-state distribution of the electrons, determined under the combined influence of the electric field set up by the sound wave passing through the metal and the collisions of the electrons with lattice vibrations and other defects in the metal, is not identical with that which the electrons would have if they were locally in thermal equilibrium. Hence there is a dissipation of sound energy into heat energy. Steinberg,[2] using Pippard's model, formulated the problem in terms of the Boltzmann transport equation. In both these treatments the existence of a relaxation time τ for collisions is assumed. In the Boltzmann formulation this assumption is expressed by the equation

$$[\partial f/\partial t]_{\text{coll}}=-(f-\bar{f})/\tau, \tag{1}$$

[1] A. B. Pippard, Phil. Mag. **41**, 1104 (1955).

[2] M. S. Steinberg, Phys. Rev. **111**, 425 (1958).

where the term on the left-hand side is the collision integral in the transport equation, f is the distribution function of the electrons, and $\bar{f}$ is the distribution function which the electrons would have in the presence of the sound wave if they were locally in thermal equilibrium.

As a next step, it is clearly desirable to ask, firstly, as to what extent Eq. (1) is valid and, secondly, as to what value (or values, if different) of τ should be used in the various expressions for the attenuation of plane dilatational and shear sound waves. The answers to these questions naturally depend on the form of the interactions that give rise to the collision integral. Parenthetically, it may be mentioned that the comparison of theoretical results with experimental data is generally made by assuming that the τ for the attenuation problem is the same as that appearing in the expression for the electrical conductivity of a metal.

In this paper we consider the collision integral to arise explicitly from the interaction of electrons (1) with thermal phonons or lattice vibrations and (2) with any impurities present in the metal, and obtain the relevant solutions of the transport equation which determine the attenuation of plane dilatational and shear sound waves. We make a number of simplifying assumptions and approximations. Principal among these are the use of the free-electron model for the metal, and of the collision integral derived by Bloch[3] for the electron-phonon interaction. We also make all the other usual assumptions on which the celebrated Bloch-Grüneisen formula for the electrical conductivity of a metal is derived.[4]

The starting point of this investigation is the expansion of f in a series of spherical harmonics $Y_{LM}(\theta,\varphi)$. Our results are then most conveniently expressed by formally introducing a set of *effective*[5] relaxation times τ_{LM} which are associated with the various terms in this expansion. For the assumptions mentioned above, the τ_{LM} are found to be independent of the subscript M. The determination of attenuation entails the solving of an infinite set of linear algebraic equations, and analytical expressions for the attenuation are obtained in this paper for the following three cases: (1) when the wavelength λ of the sound wave is much larger than a mean free path l of the electrons; (2) when $\lambda \ll l$; and (3) when there is no restriction on λ but *all* τ_L ($L=1,2,3,\cdots$) are equal to one another. When τ_L are not *all* equal, as is usually the case, and when $\lambda \sim l$, one has to obtain the attenuation numerically. This can be done if the ratios τ_L/τ_1 are known, and the results of one such calculation are discussed in Sec. 6.

A comment on the method of calculating the attenuation may be usefully made here. Pippard calculated the attenuation by essentially calculating the Joule heat associated with the steady-state electronic current. In this method it is the electrons which irreversibly lose some of their energy to the thermal energy of the metal, the sound wave somehow supplying to the electrons an equivalent amount of energy at the same rate. Steinberg, on the other hand, set up the equations of motions for wave propagation in the metal by adding to the elastic stress tensor a kinetic stress tensor associated with the steady-state distribution function f. In this paper, we set up the equations of motion for the ions by considering somewhat more explicitly the forces on the ions which arise from the electron-ion interaction. We assume that the total force on the ions is composed of (a) the force due to the electric field set up by the sound wave, (b) the force due to the transfer of momentum to the ions in electron-ion collisions, and finally (c) an elastic restoring force independent of the electron-ion interaction. If the τ_L are all equal to one another [case (3) above], the results on attenuation by our method are, of course, identical with those obtained by Pippard and Steinberg.

In Sec. 2 of this paper the problem is formulated in terms of the Boltzmann equation and the method of calculating the attenuation is described. In Sec. 3 the collision integrals due to electron-phonon and electron-impurity interactions are analyzed and explicit expressions for τ_L are derived. The relevant solutions of the transport equation are obtained in Sec. 4 and the various expressions for the attenuation in Sec. 5. Finally, the results are discussed in Sec. 6; here the possible influence of electron-electron collisions on attenuation is also briefly discussed.

2. BASIC EQUATIONS

Let an ultrasonic wave of wavelength λ in the metal be characterized by a displacement $\mathbf{s}$ and particle velocity $\mathbf{u}$; for a plane wave travelling in the positive z direction,

$$\mathbf{s}=\mathbf{s}_0 e^{i(\omega t-kz)}, \quad \partial \mathbf{s}/\partial t=\mathbf{u}, \tag{2}$$

where $k=2\pi/\lambda$. For a dilational wave $\mathbf{s}$ and $\mathbf{u}$ are parallel to the z axis. For a shear wave we shall take $\mathbf{s}$ and $\mathbf{u}$ to be along the x axis.

In the presence of a sound wave a certain electric field $\mathbf{F}$ is set up in the metal. The stationary distribution f for the electrons will then be determined by the Boltzmann transport equation:

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\operatorname{grad}_r f+\frac{e\mathbf{F}}{m}\cdot\operatorname{grad}_{\mathbf{v}} f=\left[\frac{\partial f}{\partial t}\right]_{\text{coll}}, \tag{3}$$

where $\mathbf{v}$ is the velocity of an electron, and e and m its charge and mass, respectively. In terms of f the electronic current density $\mathbf{J}_e$ and the number of electrons N per unit volume are given by (h is Planck's constant)

$$\mathbf{J}_e=\left(\frac{2em^3}{h^3}\right)\int \mathbf{v} f(\mathbf{v})d\mathbf{v}; \quad N=\frac{2m^3}{h^3}\int f(\mathbf{v})d\mathbf{v}. \tag{4}$$

[3] F. Bloch, Z. Physik **59**, 208 (1930).
[4] See, for example, A. H. Wilson, *The Theory of Metals* (Cambridge University Press, New York, 1954), 2nd. ed., Chap. 9.
[5] See comments following Eq. (30).

Integrating (3) over all $\mathbf{v}$, one obtains

$$\frac{\partial N}{\partial t}+(1/e)\,\mathrm{div}\mathbf{J}_e=\int\left[\frac{\partial f}{\partial t}\right]_{\mathrm{coll}}d\mathbf{v}=0. \tag{5}$$

In (5) the second equality arises from the fact that the number density of electrons does not change due to collisions. Equation (5) is just the equation of continuity.

Next one relates $\mathbf{F}$ to the total current density $\mathbf{J}=\mathbf{J}_e-N_0e\mathbf{u}$ by Maxwell's electromagnetic equations (in Gaussian units):

$$\mathrm{curl}\,\mathrm{curl}\mathbf{F}=-c^{-2}\left[4\pi\left(\frac{\partial\mathbf{J}_e}{\partial t}-N_0e\frac{\partial\mathbf{u}}{\partial t}\right)+\frac{\partial^2\mathbf{F}}{\partial t^2}\right], \tag{6}$$

where N_0 is the number density of electrons in the undisturbed metal, and where we have neglected—as we shall do throughout this paper—powers of u higher than the first. If the collision term in Eq. (3) is known, Eqs. (2) to (6) are sufficient to determine completely f and $\mathbf{F}$ in terms of the particle velocity $\mathbf{u}$ associated with the sound wave.

In accordance with the remarks made in the Introduction, the equations of motion for wave propagation in the z direction may be written as

$$\rho\frac{\partial^2\mathbf{s}}{\partial t^2}=\rho\beta\frac{\partial^2\mathbf{s}}{\partial z^2}-N_0e\mathbf{F}+\mathbf{I}_c, \tag{7}$$

where ρ is the density of the metal. The first term on the right-hand side of (7) arises from the elastic restoring force and β is the square of the velocity of the wave arising from this force, i.e., in absence of the electron-ion interaction or the terms $-N_0e\mathbf{F}$ and $\mathbf{I}_c$; the second term $-N_0e\mathbf{F}$ is the force per unit volume on the ions due to the electric field $\mathbf{F}$ set up by the wave, and finally $\mathbf{I}_c$ is the momentum imparted to the ions per unit time per unit volume by the electrons in collisions with the ions; $\mathbf{I}_c$ is given by

$$\mathbf{I}_c=-\left(\frac{2m^3}{h^3}\right)\int m\mathbf{v}\left[\frac{\partial f}{\partial t}\right]_{\mathrm{coll}}d\mathbf{v}. \tag{8}$$

The last two terms in (7) lead to a force which is out of phase with the displacements of the ions and hence are responsible for the attenuation of the wave.

On substituting (2) into (7) and putting $k=k_1-ik_2$, one can obtain the velocity $V=\omega/k_1$, and the intensity attenuation α in nepers/cm as $2k_2$. When $k_2/k_1\ll1$, as is the case in the problem considered here, we may proceed as follows: Let $\omega^2/k^2=Z=Z_1+iZ_2$, where Z_1 and Z_2 are real. Then, if powers of (k_2/k_1) higher than the second are neglected, it may be shown that

$$\alpha\simeq(\omega/V^3)Z_2, \tag{9a}$$

and

$$V^2\simeq Z_1[1+3\alpha^2/4k_1^2]. \tag{9b}$$

3. ANALYSIS OF THE COLLISION INTEGRALS AND AN INTEGRATION OF THE BOLTZMANN EQUATION

For convenience, we first introduce the following symbols, etc.:

The wave vector of an electron will be denoted by $\mathbf{K}$, and its energy by $E_{\mathbf{K}}$. In the free electron approximation ($\hbar=h/2\pi$), $\hbar\mathbf{K}=m\mathbf{v}$, $E_{\mathbf{K}}=\hbar^2|\mathbf{K}|^2/2m$. The values of v, K, and E at the Fermi surface will be denoted by v_0, K_0, and E_0, respectively. v_0 and N_0 are related by

$$mv_0=h(3N_0/8\pi)^{\frac{1}{3}}. \tag{10}$$

f_0, the thermal equilibrium distribution function for the electrons in an undisturbed metal, and $\bar{f}$ are given by

$$f_0(\mathbf{K})=\{\exp[(E_{\mathbf{K}}-E_0)/\kappa T]+1\}^{-1}, \tag{11}$$

$$f(\mathbf{K})=\left\{\exp\left[\left(\frac{\hbar^2}{2m}\left|\mathbf{K}-\frac{m\mathbf{u}}{\hbar}\right|^2-E_0-E_1\right)\Big/\kappa T\right]+1\right\}^{-1}, \tag{12}$$

where E_1 is introduced in (12) to include the changes in the number density of the electrons associated with the passage of a dilatational wave. ($E_1=0$ for shear waves.)

The direction of $\mathbf{K}$ will be specified by the polar angles (θ,φ); we take the direction of propagation of the sound wave (z axis) as the polar axis and measure the azimuthal angle φ from the z-x plane. For convenience the spherical harmonics $Y_{LM}(\theta,\varphi)$ will be defined as

$$Y_{LM}(\theta,\varphi)=P_L{}^M(\cos\theta)\cos M\varphi \quad \text{for} \quad 0\leqslant M\leqslant L,$$

and

$$Y_{LM}(\theta,\varphi)=P_L{}^{|M|}(\cos\theta)\sin M\varphi \quad \text{for} \quad -L\leqslant M<0,$$

where $P_L{}^M(\cos\theta)$ are the associated Legendre polynomials. The expansion of f in spherical harmonics will be written as

$$f=f_0-u\frac{\partial f_0}{\partial E_{\mathbf{K}}}\sum_{LM}C_{LM}(K)Y_{LM}(\theta,\varphi), \tag{13}$$

where the summation, in general, is over all integral values of L, including zero, and over all M such that $-L\leqslant M\leqslant L$. The coefficients in a similar expansion for $\bar{f}$ will be denoted by $\bar{C}_{LM}(K)$. To the first power in u, the only nonzero $\bar{C}_{LM}$ are $\bar{C}_{10}=mv$ and $\bar{C}_{00}=E_1/u$ for the dilatational waves and $\bar{C}_{11}=mv$ for the shear waves. We shall also write (13) in the form

$$f=f-u\frac{\partial f_0}{\partial E_{\mathbf{K}}}\sum_{LM}c_{LM}(K)Y_{LM}(\theta,\varphi), \tag{14}$$

with

$$c_{LM}(K)=C_{LM}(K)-\bar{C}_{LM}(K). \tag{15}$$

3.1. Electron-Phonon Interaction

The Bloch collision integral is discussed in detail in Wilson[4] and is given by [Wilson, Eq. (9.33.6)]

$$\left[\frac{\partial f}{\partial t}\right]_{\text{coll}} = \frac{\mathfrak{C}^2}{\mathcal{V} n_a M_a h} \sum_{\mathbf{q}} \frac{|\mathbf{q}^2|}{\nu_{\mathbf{q}}} \{[(\mathfrak{N}_{\mathbf{q}}+1) f(\mathbf{K}+\mathbf{q})(1-f(\mathbf{K})) - \mathfrak{N}_{\mathbf{q}} f(\mathbf{K})(1-f(\mathbf{K}+\mathbf{q}))] \times \Omega(E_{\mathbf{K}}-E_{\mathbf{K}+\mathbf{q}}+h\nu_{\mathbf{q}}) + [\mathfrak{N}_{-\mathbf{q}} f(\mathbf{K}+\mathbf{q})(1-f(\mathbf{K})) - (\mathfrak{N}_{-\mathbf{q}}+1) f(\mathbf{K}) \times (1-f(\mathbf{K}+\mathbf{q}))] \Omega(E_{\mathbf{K}}-E_{\mathbf{K}+\mathbf{q}}-h\nu_{\mathbf{q}})\}, \quad (16)$$

where $\mathbf{q}$ is the wave vector of a phonon of frequency $\nu_{\mathbf{q}}$ ($=w_0|\mathbf{q}|/2\pi$, w_0 being an *average* velocity of sound) and energy $h\nu_{\mathbf{q}}$, $\mathfrak{C}$ is the Sommerfeld-Bethe electron-phonon interaction constant, $\mathcal{V}$ is the volume of the crystal under consideration, M_a is the mass of an atom, n_a the number density of atoms, and $\mathfrak{N}_{\mathbf{q}}$ is the phonon distribution function.

$$\Omega(x) = \frac{\sin(xt/\hbar)}{x/\hbar}, \qquad \int_{-\infty}^{+\infty} \Omega(x)dx = \pi\hbar. \quad (17)$$

$\Omega(x)$ acts as a kind of δ function differing from zero only when $x=0$. Finally the summation in (16) is over all values of $\mathbf{q}$, lying within a sphere of radius $q_0=(6\pi^2 n_a)^{\frac{1}{3}}$ (Debye spectrum for phonons).

The collision integral (16) vanishes if we substitute into it for $f(\mathbf{K})$ and $\mathfrak{N}_{\mathbf{q}}$, respectively, the corresponding thermal equilibrium distribution functions, namely, $f_0(\mathbf{K})$ from (11) and $\mathfrak{N}_{\mathbf{q}}^0$, where

$$\mathfrak{N}_{\mathbf{q}}^0 = [\exp(h\nu_{\mathbf{q}}/\kappa T)-1]^{-1}. \quad (18)$$

The collision integral also vanishes for $f(\mathbf{K})=\bar{f}(\mathbf{K})$ and $\mathfrak{N}_{\mathbf{q}}=\bar{N}_{\mathbf{q}}$, where

$$\bar{N}_{\mathbf{q}} = \{\exp[(h\nu_{\mathbf{q}}-\hbar\mathbf{q}\cdot\mathbf{u})/\kappa T]-1\}^{-1}. \quad (19)$$

However, the collision integral (16) does not vanish if we set $f(\mathbf{K})=\bar{f}(\mathbf{K})$ and $\mathfrak{N}_{\mathbf{q}}=\mathfrak{N}_{\mathbf{q}}^0$. This shows that when the electron gas is at rest, $f(\mathbf{K})$ and $\mathfrak{N}_{\mathbf{q}}$ relax due to collisions towards $f_0(\mathbf{K})$ and $\mathfrak{N}_{\mathbf{q}}^0$; on the other hand, when the electron gas contained in a *macroscopic* element of volume possesses a mass motion (as in the presence of an impressed sound wave), we may assume that $f(\mathbf{K})$ relaxes due to collisions towards $\bar{f}(\mathbf{K})$, provided we assume also that $\mathfrak{N}_{\mathbf{q}}$ relaxes towards $\bar{N}_{\mathbf{q}}$. We note that the phonon gas represented by $\bar{N}_{\mathbf{q}}$ also has a finite macroscopic momentum; this is not unreasonable since the sound wave passing through a metal, imparts momentum to an element of the metal as a whole.[6]

Now in order to determine the steady-state distribution function $f(\mathbf{K})$ from (16) and the transport Eq. (3), we should, to be consistent, write similar equations for $\mathfrak{N}_{\mathbf{q}}$ and solve the two transport equations simultaneously for $f(\mathbf{K})$ and $\mathfrak{N}_{\mathbf{q}}$. Instead of doing this, for simplicity we shall determine $f(\mathbf{K})$ by setting[7] in (16) $\mathfrak{N}_{\mathbf{q}}=\bar{N}_{\mathbf{q}}$. Let us now substitute into (16) $\mathfrak{N}_{\mathbf{q}}=\bar{N}_{\mathbf{q}}$ and for $f(\mathbf{K})$ from (14), and simplify the resulting expression by using the condition that (16) vanishes for $f=\bar{f}$ and $\mathfrak{N}_{\mathbf{q}}=\bar{N}_{\mathbf{q}}$. The remaining expression is proportional to u and, in addition, contains higher powers of u through its dependence on $\bar{f}(\mathbf{K})$ and $\bar{N}_{\mathbf{q}}$. Since we neglect powers of u higher than the first, we may replace $\bar{f}(\mathbf{K})$ and $\bar{N}_{\mathbf{q}}$ by $f_0(\mathbf{K})$ and $\mathfrak{N}_{\mathbf{q}}^0$, respectively. Finally, we substitute for $\mathfrak{N}_{\mathbf{q}}^0$ from (18), remember that $\mathfrak{N}_{\mathbf{q}}^0=\mathfrak{N}_{-\mathbf{q}}^0$, and replace the summation over $\mathbf{q}$ by an integral (for this step see Wilson,[4] p. 259). We then obtain for the collision integral (16) the expression

$$\left[\frac{\partial f}{\partial t}\right]_{\text{coll}} = \frac{\mathfrak{C}^2 u}{8\pi^3 n_a M_a h k T} \sum_{LM} \int \frac{|\mathbf{q}|^2}{\nu_{\mathbf{q}}} \frac{1}{\exp(h\nu_{\mathbf{q}}/\kappa T)-1} \times \{f_0(E_{\mathbf{K}})[1-f_0(E_{\mathbf{K}+\mathbf{q}})]\Omega(E_{\mathbf{K}}-E_{\mathbf{K}+\mathbf{q}}+h\nu_{\mathbf{q}}) + f_0(E_{\mathbf{K}+\mathbf{q}})[1-f_0(E_{\mathbf{K}})]\Omega(E_{\mathbf{K}}-E_{\mathbf{K}+\mathbf{q}}-h\nu_{\mathbf{q}})\} \times \{c_{LM}(|\mathbf{K}+\mathbf{q}|)Y_{LM}(\theta',\varphi') - c_{LM}(|\mathbf{K}|)Y_{LM}(\theta,\varphi)\} \times q^2 dq \sin\delta d\delta d\varpi, \quad (20)$$

where (δ,ϖ) are the polar coordinates of $\mathbf{q}$ with $\mathbf{K}$ as the polar axis and (θ',φ') are the polar coordinates of $\mathbf{K}+\mathbf{q}$ in the frame of reference in which the polar coordinates of $\mathbf{K}$ are (θ,φ). With $\mathbf{K}$ as the polar axis, the polar coordinates of $\mathbf{K}+\mathbf{q}$ may be written as (μ,ϖ). Now in (20), $Y_{LM}(\theta',\varphi')$ is the only term which depends on the azimuth ϖ and it can be readily integrated over ϖ by using the group-theoretic properties of the three-dimensional rotation group. The result is

$$\int_0^{2\pi} Y_{LM}(\theta',\varphi')d\varpi = 2\pi Y_{L0}(\mu)Y_{LM}(\theta,\varphi). \quad (21)$$

The integration over δ is carried out by using (17) and

[6] It will be realized that the above argument is, by no means, rigorous, since we have assumed that the collision integral itself is not modified due to the impressed sound wave and that the collision events can be considered, to some extent at least, localized in space and time, even though the collision probabilities in (16) are calculated quantum mechanically. These interrelated questions have been discussed by Holstein [T. Holstein, Phys. Rev. 113, 479 (1959)]. He has shown that when the interaction between the thermal phonons and the impressed sound wave is taken into account, the collision integral (16) is so modified that it vanishes for $f=\bar{f}$ and $\mathfrak{N}_{\mathbf{q}}=\mathfrak{N}_{\mathbf{q}}^0$. To the first power in u, Holstein's collision integral leads to the same results as the procedure adopted in this paper.

[7] We note that in solving the electrical conductivity problem, one sets $\mathfrak{N}_{\mathbf{q}}=\mathfrak{N}_{\mathbf{q}}^0$ in (16). The error involved in such a procedure is not appreciable so long as one is not considering second order phenomena like the thermoelectric power; see, for example, J. M. Ziman, *Electrons and Phonons* (Oxford University Press, New York, 1960), Sec. 9.13.

is analogous to the corresponding step in Wilson,[4] p. 261. Using (21) and integrating (20) over δ and ϖ, one obtains[8]

$$\left[\frac{\partial f}{\partial t}\right]_{\text{coll}} = u\left(\frac{\partial f_0}{\partial E_K}\right)\frac{(\frac{1}{2}m)^{\frac{1}{2}}}{\Lambda\hbar^2 E_K^{\frac{1}{2}}}\left(\frac{T}{\Theta}\right)^3 \times\int_{-\Theta/T}^{+\Theta/T}\frac{z^2dz(e^\eta+1)}{|1-e^{-z}|(e^{\eta+z}+1)} \times\{\sum_{LM}[c_{LM}(\eta)-c_{LM}(\eta+z)Y_{L0}(\mu)]Y_{LM}(\theta,\varphi)\}, \quad (22)$$

$$\equiv u\sum_{LM}\vartheta_{LM}Y_{LM}(\theta,\varphi), \quad (23)$$

where $z=h\nu_q/\kappa T$, $\eta=(E_K-E_0)/\kappa T$, $\kappa\Theta=\hbar w_0 q_0$, and

$$\Lambda=\left(\frac{4\pi}{3n_a}\right)^{\frac{1}{3}}\frac{4M_a\kappa\Theta}{3h^2\mathfrak{C}^2}, \quad \frac{D}{E_0}=2^{-\frac{1}{3}}\left(\frac{n_a}{N_0}\right)^{\frac{2}{3}}, \quad (24)$$

and

$$\cos\mu=\left(1+\frac{\kappa Tz}{E_K}\right)^{-\frac{1}{2}}\left[1+\frac{\kappa Tz}{2E_K}-\frac{D}{E_K}\left(\frac{T}{\Theta}\right)^2 z^2\right]. \quad (25)$$

We next substitute (13) into the left-hand side of Eq. (3), note (a) that our considerations apply only to isotropic metals (i.e., for which the electrical conductivity tensor is a scalar) so that $\mathbf{F}\|\mathbf{u}$, and (b) that $\partial u/\partial t=i\omega u$, and $\partial u/\partial z=-iku$, and equate the resulting expression[9] to the right-hand side of (23). One then obtains for $C_{LM}(E_K)$ the integral equation

$$-\left(\frac{\partial f_0}{\partial E_K}\right)\left\{(i\omega-ikv Y_{10}(\theta,\varphi)) \times[\sum_{LM}C_{LM}(E_K)Y_{LM}(\theta,\varphi)]-\frac{eFv}{u}Y_{1M'}(\theta,\varphi)\right\} =\sum_{LM}\vartheta_{LM}Y_{LM}(\theta,\varphi), \quad (26)$$

where we have to put $M'=0$ for the dilatational waves and $M'=1$ for the shear waves.

We now note that ϑ_{LM} [see the expression within in the square bracket of (28)] are different from zero only for values of E_K such that $|E_K-E_0|\simeq\kappa Tz<\kappa\Theta$; we, therefore, write $C_{LM}(E_K)$ in the form $C_{LM}(E_K)=C_{LM}(E_0)+d_{LM}(E_K-E_0)$, and assume that in this small neighborhood of E_0, $d_{LM}(E_K-E_0)\ll C_{LM}(E_0)$. Then integrating (26) over E_K, we obtain to zero order in $\kappa T/E_0$ the following equation for $C_{LM}(E_0)$:

$$(i\omega-ikv_0Y_{10}(\theta,\varphi))[\sum_{LM}C_{LM}(E_0)Y_{LM}(\theta,\varphi)] -\frac{eFv_0}{u}Y_{1M'}(\theta,\varphi)=-\sum_{LM}{}'\frac{C_{LM}(E_0)-\bar{C}_{LM}(E_0)}{\tau_L}, \quad (27)$$

where we have used (15) to eliminate $c_{LM}(E_0)$, and where

$$\frac{1}{\tau_L}=\left(\frac{m}{2E_0}\right)^{\frac{1}{2}}\frac{1}{\Lambda\hbar^2}\left(\frac{T}{\Theta}\right)^3\int_{-\infty}^{+\infty}d\eta \times\int_{-\Theta/T}^{+\Theta/T}\frac{z^2dz(1-Y_{L0}(\mu))}{|1-e^{-z}|(e^z-1)}\left[\frac{1}{e^\eta+1}-\frac{1}{e^{\eta+z}+1}\right] \quad (28)$$

$$=\left(\frac{m}{2E_0}\right)^{\frac{1}{2}}\frac{1}{\Lambda\hbar^2}\left(\frac{T}{\Theta}\right)^3\int_{-\Theta/T}^{+\Theta/T}\frac{z^3dz(1-Y_{L0}(\mu))}{|1-e^{-z}|(e^z-1)}. \quad (29)$$

The prime on the summation sign on the right-hand side of (27) implies that the term $L=0$ is to be excluded from this sum. This is because the integral $\int\vartheta_{00}dE$ vanishes, as it should [see, Eq. (5)]. Finally, since $\kappa Tz/E_0\lesssim 10^{-2}$, the expression (25) for $\cos\mu$ may be approximated to

$$\cos\mu\simeq[1-(D/E_0)(T/\Theta)^2z^2]. \quad (30)$$

It will be observed that Eq. (27) determines $C_{LM}(E_K)$ at $E_K=E_0$ only. This is, however, sufficient to determine $\mathbf{J}_e$ and hence attenuation. It should be mentioned that in order to completely justify the step leading to Eq. (27) from Eq. (26), one must still verify that $d_{LM}(E_K-E_0)\ll C_{LM}(E_0)$ for $|E_K-E_0|\lesssim\kappa Tz<\kappa\Theta$. We shall not take up the verification of this inequality here; but we mention that one may convince oneself of its validity by noting that all the integrals ϑ_{LM} are of the same form and that for the conductivity problem, for which the collision integral is just (22) with all $c_{LM}\equiv 0$ except c_{10}, it has been found to be true at all temperatures.[10] Finally we remark that the τ_L, as defined in (29), may be formally regarded as a set of *effective* relaxation times; for example, when the expression for τ_1 from (29) is substituted for τ in the expression $\sigma=(Ne^2/m)\tau$ for the electrical conductivity one obtains just the Bloch-Grüneisen formula. (Note that τ_L are undefined as a function of the energy E_K of an electron and hence are not the relaxation times in the usual sense of the term.)

Evaluation of τ_L

To evluate τ_L from (29), we note that $Y_{L0}(\mu)$ is identical with the hypergeometric series[11] $F(L+1,-L;$

[8] The limits of integration $\pm\Theta/T$ in (22) are on the assumption that $N_0>0.25n_a$. When $N_0<0.25n_a$, the limits of integration become $\pm(\Theta/T)(4N_0/n_a)^{\frac{1}{3}}(E/E_0)^{\frac{1}{2}}$.

[9] It has been assumed here that changes, if any, in the temperature of the electron gas can be neglected. We note that the attenuation per wavelength resulting from heat flow (either among the electrons or between the electrons and the lattice) is at most of the order $[(C_P/C_V)-1]$ or 10^{-7} at liquid helium temperatures; here C_P and C_V are the specific heats at constant pressure and constant volume, respectively.

[10] See, for example, W. Ehrenberg, *Electric Conduction in Semiconductors and Metals* (Oxford University Press, New York, 1958), p. 168; also reference 4, pp. 278, 310.

[11] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1946), 4th ed., p. 312.

$1; \frac{1}{2}(1-\cos\mu)$). Hence (29) may be immediately written as

$$\frac{1}{\tau_L}=\left(\frac{m}{2E_0}\right)^{\frac{3}{2}}\frac{2}{\Lambda\hbar^2}\left(\frac{D}{E_0}\right)\left(\frac{T}{\Theta}\right)^5\sum_{r=1}^{L}\left\{J_{2r+3}(\Theta/T)(-1)^{r-1}\left[\frac{D}{E_0}\left(\frac{T}{\Theta}\right)^2\right]^{r-1}\times\frac{(L+1)(L+2)\cdots(L+r)L(L-1)(L-2)\cdots(L-r+1)}{(r!)^2 2^r}\right\}, \quad (31)$$

where

$$J_{2r+3}\left(\frac{\Theta}{T}\right)=\int_0^{\Theta/T}\frac{z^{2r+3}dz}{(e^z-1)(1-e^{-z})}. \quad (32)$$

At very low temperatures, $T/\Theta\ll 1$, the integrals (32) become independent of T and the series in (31) is in increasing powers of $(T/\Theta)^2$. Hence for sufficiently low temperatures all but the first term in the series may be ignored. One then has

$$\tau_1/\tau_L=\tfrac{1}{2}L(L+1), \quad T\ll\Theta. \quad (33)$$

Using the table of integrals for $J_{2r+3}(\Theta/T)$ given in Wilson[4] (p. 337), one finds that for $L=2$ expression (33) is correct to within 10% for $T/\Theta<1/20$; and for L up to 6 it is correct to within 10% for $T/\Theta<1/40$. For larger L, the approximation (33) for τ_1/τ_L is a good one only at still lower temperatures. Because of this, it is of interest to obtain here the values of τ_1/τ_L in the high-temperature limit $T\gg\Theta$ (although the attenuation of sound waves due to electrons is negligible at these temperatures). Noting that for $T\gg\Theta$, the factor $[\exp(z)-1][1-\exp(-z)]$ in (29) may be replaced by z^2, and transforming the integration variable from z to $\cos\mu$, we may obtain from (29)

$$\tau_1/\tau_L=2(E_0/D)^2\int_0^{\cos^{-1}(1-D/E_0)}(1-Y_{L0}(\mu))\times\sin\mu d\mu. \quad (34)$$

Since $D/E_0\simeq 1$, for simplicity we evaluate (34) at $D/E_0=1$. One then has $\tau_1/\tau_L=2$ for all even L. For $L=3$ and 5, $\tau_1/\tau_L=7/4$ and 17/8, respectively; and as $L\to\infty$, $\tau_1/\tau_L\to 2$. Therefore, whereas τ_1/τ_2 changes in the whole temperature range by only about 50%, the variation of τ_1/τ_L with temperature for larger L is much greater.

3.2 Electron-Impurity Scattering

The collision integral due to the scattering of electrons by the impurities may be written in the form [Wilson, Eq. (9.4.1)]

$$\left[\frac{\partial f}{\partial t}\right]_{\text{coll}}=\frac{\mathcal{U}}{\pi\hbar^2}\int[f(\mathbf{K}')-f(\mathbf{K})]\times\Omega(E_{\mathbf{K}}-E_{\mathbf{K}'})|\langle\mathbf{K}'|\Delta V|\mathbf{K}\rangle|^2 d\mathbf{K}'. \quad (35)$$

The factor $\Omega(E_{\mathbf{K}}-E_{\mathbf{K}'})$ in (35) ensures that the collisions of electrons with impurities are elastic. Now, since the momentum of the electrons is not conserved in collisions and since the right-hand side of (35) is obtained on the assumption that impurities are at rest, expression (35) vanishes for $f=f_0$, but not for the distribution function $\bar{f}(\mathbf{K})$ which represents electrons with a mean momentum $m\mathbf{u}$ per electron. Since in the presence of a sound wave impurities also move with a velocity $\mathbf{u}$, we consider the scattering of electrons from states $\mathbf{K}_1\to\mathbf{K}_1'$, where $\hbar\mathbf{K}_1$ and $\hbar\mathbf{K}_1'$ are the momentum of the electrons in the reference frame in which impurities are at rest and thus *heuristically* modify the collision integral by making the replacement:

$$\Omega(E_{\mathbf{K}}-E_{\mathbf{K}'})|\langle\mathbf{K}'|\Delta V|\mathbf{K}\rangle|^2\to\Omega(E_{K_1}-E_{K_1'})|\langle\mathbf{K}_1'|\Delta V|\mathbf{K}_1\rangle|^2.$$

Since $\hbar\mathbf{K}_1=\hbar\mathbf{K}-m\mathbf{u}$, $\hbar\mathbf{K}_1'=\hbar\mathbf{K}'-m\mathbf{u}$, the collision integral so obtained vanishes for $f=\bar{f}$.

If we now substitute for f from (14) into the modified collision integral, use the condition that it vanishes for $f=\bar{f}$, and then ignore powers of u higher than the first, we obtain

$$\left[\frac{\partial f}{\partial t}\right]_{\text{coll}}=\frac{\mathcal{U}}{\pi\hbar^2}u\left(\frac{\partial f_0}{\partial E_{\mathbf{K}}}\right)\times\sum_{LM}'\left[c_{LM}(E_{\mathbf{K}})\int[Y_{LM}(\theta,\varphi)-Y_{LM}(\theta',\varphi')]\times\Omega(E_{\mathbf{K}}-E_{\mathbf{K}'})|\langle\mathbf{K}'|\Delta V|\mathbf{K}\rangle|^2 d\mathbf{K}'\right]. \quad (36)$$

We assume now that the matrix elements $\langle\mathbf{K}'|\Delta V|\mathbf{K}\rangle$ depend only on the angle between $\mathbf{K}$ and $\mathbf{K}'$ and not *individually* on the directions of $\mathbf{K}$ and $\mathbf{K}'$ in the metal. Then taking the polar coordinates of $\mathbf{K}'$ as (μ,ϖ) with $\mathbf{K}$ as the polar axis, and integrating (36) over $E_{\mathbf{K}'}$ and ϖ with the help of Eqs. (17) and (21), we may write (36) in the form

$$\left[\frac{\partial f}{\partial t}\right]_{\text{coll}}=u\left(\frac{\partial f_0}{\partial E_{\mathbf{K}}}\right)\sum_{LM}'\frac{c_{LM}(E_{\mathbf{K}})Y_{LM}(\theta,\varphi)}{\tau_L(E_{\mathbf{K}})}, \quad (37)$$

with

$$\frac{1}{\tau_L(E_{\mathbf{K}})}=\frac{\mathcal{U}K^2}{h}\left(\frac{dK}{dE_{\mathbf{K}}}\right)\int_0^{\pi}[1-Y_{L0}(\mu)]\times|\langle\mathbf{K}'|\Delta V|\mathbf{K}\rangle|^2\sin\mu d\mu. \quad (38)$$

The integration of (37) over E_K is straightforward and we thus see that for impurity scattering also $C_{LM}(E_0)$ are determined by Eq. (27). Further, to the approximations made in this paper, when the electrons are scattered by both phonons and impurities, one has only to write for $1/\tau_L$ in Eq. (27) the expression

$$\frac{1}{\tau_L}=\frac{1}{\tau_L{}^{(p)}}+\frac{1}{\tau_L{}^{(i)}}, \tag{39}$$

where $\tau_L{}^{(p)}$ are given by (29) and $\tau_L{}^{(i)}[\equiv\tau_L{}^{(i)}(E_0)]$ are the effective relaxation times for the electron-impurity scattering.

Evaluation of $\tau_L{}^{(i)}$

We first rewrite (38) in a more convenient form. Let the fraction of impurity atoms, all of the same kind, be denoted by χ, and let $\chi\ll 1$. We suppose the impurities to be distributed at random and we denote by $U(r)$ the difference in potential between the dissolved and solvent atoms. Further let $P(\mu)$ be the probability that an electron (energy E_0) in a unit volume (wave functions ψ_K and $\psi_{K'}$ of the electron normalized to unit volume) is scattered through an angle μ per unit solid angle per unit time. Then (38) for $E_K=E_0$ may be written in the form

$$\frac{1}{\tau_L{}^{(i)}}=2\pi\int^{\pi}[1-Y_{L0}(\mu)]P(\mu)\sin\mu d\mu, \tag{40}$$

with

$$P(\mu)=n_a\chi v_0\left(\frac{2\pi m}{h^2}\right)^2\left|\int\psi_{K'}{}^*U(r)\psi_K d\mathbf{r}\right|^2. \tag{41}$$

The ratios $\tau_1{}^{(i)}/\tau_L{}^{(i)}$ would naturally depend on the form of $U(r)$.

As an example, we consider here $U(r)$ to be a screened Coulomb potential: $U(r)\propto(r^{-1})\exp(-\mathfrak{q}r)$. Mott[12] has shown that the potential of a polyvalent impurity dissolved in a monovalent metal is approximately of this type. In the Born approximation $P(\mu)$ is proportional to $[(1-\cos\mu)+\mathfrak{q}^2/(2K_0{}^2)]^{-2}$, and the ratios $\tau_1{}^{(i)}/\tau_L{}^{(i)}$ may be easily found to be given by

$$\frac{\tau_1{}^{(i)}}{\tau_L{}^{(i)}}=\frac{Q_L'(\vartheta)-Q_0'(\vartheta)}{Q_1'(\vartheta)-Q_0'(\vartheta)};\quad \vartheta=1+\frac{\mathfrak{q}^2}{2K_0{}^2}, \tag{42}$$

where $Q_L(\vartheta)$ is the Legendre function of degree L of the second kind,[13] and $Q'(\vartheta)=dQ(z)/d\vartheta$. From (42) we see that $\tau_1{}^{(i)}/\tau_L{}^{(i)}$ depend on the range $\mathfrak{q}^{-1}$ of the potential and on the wave-number K_0 of the Fermi electron. For $L=2$, using (42), one finds that $\tau_1{}^{(i)}/\tau_L{}^{(i)}$ monotonically decreases from the value 3 to unity as ϑ increases from 1 to ∞. For a polyvalent impurity dissolved in copper,[14] $\mathfrak{q}^{-1}=0.55\times10^{-8}$ cm, and one has $\vartheta=1.87$ and $\tau_1{}^{(i)}/\tau_2{}^{(i)}\simeq1.4$. A similar calculation for a polyvalent impurity dissolved in silver or gold gives $\tau_1{}^{(i)}/\tau_2{}^{(i)}\simeq1.37$. We have not made a similar calculation for other values of L.

[12] See, for example N. F. Mott and H. Jones, *The Theory of Properties of Metals and Alloys* (Oxford University Press, New York, 1936 and Dover Publications, Inc., New York, 1959), p. 87.

[13] See reference 11, pp. 318–320. The first three $Q_L(\vartheta)$ are:
$Q_0(\vartheta)=\frac{1}{2}\ln[(\vartheta+1)(\vartheta-1)^{-1}]$, $Q_1(\vartheta)=\vartheta Q_0(\vartheta)-1$,
$Q_2(\vartheta)=\frac{1}{2}(3\vartheta^2-1)Q_0(\vartheta)-\frac{3}{2}\vartheta$.

4. SOLUTIONS OF EQ. (27)

By rearranging the terms and remembering that $\tilde{C}_{11}(E_0)=mv_0$ for the shear waves and $\tilde{C}_{10}(E_0)=mv_0$ for the dilatational waves, we may rewrite Eq. (27) in the form

$$(i\omega-ikv_0Y_{10}(\theta,\varphi))C_{00}+\sum_{LM}{}'\left[i\omega-ikv_0Y_{10}(\theta,\varphi)+\frac{1}{\tau_L}\right]C_{LM}Y_{LM}(\theta,\varphi)=\left(\frac{eF}{u}+\frac{m}{\tau_1}\right)v_0Y_{1M'}(\theta,\varphi). \tag{43}$$

For convenience, we suppress from now on the argument E_0 in $C_{LM}(E_0)$. Since in the absence of the sound wave *all* $C_{LM}\equiv0$, it follows from the orthogonality of $Y_{LM}(\theta,\varphi)$ and $Y_{LM'}(\theta,\varphi)$ that the only nonzero C_{LM} are C_{L1} for the shear waves and C_{L0} for the dilatational waves. Of these nonzero C_{LM}, we have actually only to determine C_{11} and C_{10}, as these coefficients determine completely $\mathbf{J}_e$ and hence $\mathbf{F}$ through the electromagnetic Eqs. (6).

4.1 Determination of C_{11} for Shear Waves

Using the identity

$$(2L+1)Y_{10}(\theta,\varphi)Y_{L1}(\theta,\varphi)=(L+1)Y_{L-1,1}(\theta,\varphi)+LY_{L+1,1}(\theta,\varphi),$$

and equating the coefficients of different spherical harmonics on the two sides of (43), we obtain the following infinite set of equations for C_{L1}: $(L=1, 2, 3, \cdots)$.

$$C_{L1}-i\gamma_L(b_{L,L-1}C_{L-1,1}+b_{L,L+1}C_{L+1,1})=(1+i\omega\tau_1)^{-1}A(\tau_1)\delta_{L,1}, \tag{44}$$

where $\delta_{L,1}=1$ for $L=1$ and is equal to zero otherwise, and where

$$\gamma_L=kl_L(1+i\omega\tau_L)^{-1},\quad l_L=v_0\tau_L, \tag{45}$$

$$b_{L,L-1}=(L-1)/(2L-1),\qquad b_{L,L+1}=(L+2)/(2L+3), \tag{46}$$

and

$$A(\tau_1)=mv_0[1+\sigma F/(N_0eu)], \tag{47}$$

with

$$\sigma=(Ne^2/m)\tau_1. \tag{48}$$

[14] Values of $\mathfrak{q}^{-1}$ are taken from reference 12. It should be mentioned that the values of $\tau_1{}^{(i)}/\tau_2{}^{(i)}$ obtained here have a semi-quantitative significance only, since the Born approximation for $P(\mu)$ for the short-range interaction under consideration ($\mathfrak{q}/K_0\sim1$) is not a very good one.

C_{11} may be readily determined from Eqs. (44) for the following three cases:

Case I: $|\gamma_1|$, $|\gamma_2|$, $|\gamma_3|$, etc., Much Less Than Unity[15]

Under these conditions $C_{11}, C_{21}, C_{31}, \cdots$ form a rapidly decreasing sequence. Hence putting $C_{L1}=0$ for $L\geqslant 4$ into (44), we obtain for C_{11} the expression

$$C_{11}=\frac{A(\tau_1)}{1+i\omega\tau_1}\left[1-\frac{1}{5}\gamma_1\gamma_2+\frac{8}{175}\gamma_1\gamma_2^2\gamma_3+\frac{1}{25}\gamma_1^2\gamma_2^2+\cdots\right]. \tag{49}$$

Case II: $|\gamma_1|$, $|\gamma_2|$, etc., Much Greater Than Unity[15]

For this case C_{11} is given by[16]

$$C_{11}=\tfrac{3}{4}\pi A(\tau_1)(1+i\omega\tau_1)^{-1}\gamma_1^{-1}. \tag{50}$$

Case III: All τ_L Equal to One Another

Let $\tau_1=\tau_2=\tau_3\cdots=\tau$ say, and $\gamma_1=\gamma_2=\gamma_3\cdots=\gamma$ say. In this case an analytical solution of Eqs. (44) is possible for the whole range of values of $|\gamma|$. C_{L1} are given by

$$C_{L1}=\frac{(2L+1)A(\tau)}{2\pi L(L+1)(1+i\omega\tau)}\int_0^{\pi}\int_0^{2\pi}\frac{Y_{11}(\theta,\varphi)Y_{L1}(\theta,\varphi)}{1-i\gamma Y_{10}(\theta,\varphi)}\times\sin\theta d\theta d\varphi. \tag{51}$$

Integrating (51) for $L=1$, one obtains for C_{11} the expression

$$C_{11}=\tfrac{3}{2}A(\tau)(1+i\omega\tau)^{-1}\gamma^{-2}[(1+\gamma^2)(1-g(\gamma))-1], \tag{52}$$

where

$$g(\gamma)=1-\frac{1}{2}\int_{-1}^{1}\frac{dx}{1-i\gamma x}=1-\gamma^{-1}\tan^{-1}\gamma. \tag{53}$$

In the limit $|\gamma|\gg 1$, Eq. (52) reduces to

$$C_{11}=\tfrac{3}{4}\pi A(\tau)(1+i\omega\tau)^{-1}\gamma^{-1}. \tag{54}$$

4.2 Determination of C_{10} for Dilatational Waves

From Eq. (43), one readily obtains the following set of equations for C_{L0}:

$$i\omega C_{00}-\tfrac{1}{3}ikv_0C_{10}=0, \tag{55a}$$

$$C_{10}-i\gamma_1(b_{10}C_{00}+b_{12}C_{20})=A(\tau_1)(1+i\omega\tau_1)^{-1}, \tag{55b}$$

and for $L\geqslant 2$

$$C_{L0}-i\gamma_L(b_{L,L-1}C_{L-1,0}+b_{L,L+1}C_{L+1,0})=0, \tag{55c}$$

where now

$$b_{L,L+1}=L/(2L-1), \quad b_{L,L+1}=(L+1)/(2L+3). \tag{56}$$

Equation (55a) may be seen to be equivalent to the equation of continuity (5). Eliminating C_{00} between (55a) and (55b), we obtain

$$C_{10}(1-\tfrac{1}{3}i\gamma_1kv_0/\omega)-i\gamma_1b_{12}C_{20}=A(\tau_1)(1+i\omega\tau_1)^{-1}. \tag{55d}$$

Analytical expressions for C_{10}, by solving Eqs. (55c) and (55d), are readily obtained for the three cases discussed in Sec. 4.1.

Case I: First Few $|\gamma_L|$ Much Less Than Unity[15]

In this case C_{10} is given by

$$C_{10}=A(\tau_1)(1+i\omega\tau_1)^{-1}\times\left[1-\frac{1}{3}i\gamma_1k(v_0/\omega)+\frac{4}{15}\gamma_1\gamma_2-\frac{12}{175}\gamma_1\gamma_2^2\gamma_3+\cdots\right]^{-1}. \tag{57}$$

Case II: First Few $|\gamma_L|$ Much Greater Than Unity[15]

C_{10} for this case is given by (see footnote 16)

$$C_{10}=A(\tau_1)(1+i\omega\tau_1)^{-1}\gamma_1^{-1}[-\tfrac{1}{3}ik(v_0/\omega)+\tfrac{1}{6}\pi]^{-1}. \tag{58}$$

When the first few $|\gamma_L|\sim 1$, one has to solve Eqs. (44) for C_{11} or Eqs. (55c) and (55d) for C_{10} numerically.

Case III: All τ_L Equal to One Another

In this case C_{10} is given by

$$C_{10}=\frac{A(\tau)}{(1+i\omega\tau)}\left[-\tfrac{1}{3}i\gamma\frac{kv_0}{\omega}+\tfrac{1}{3}\gamma^2\left(\frac{1-g(\gamma)}{g(\gamma)}\right)\right]^{-1}. \tag{59}$$

Expression (59) is valid for *all* values of γ. For $|\gamma|\gg 1$, (59) reduces to

$$C_{10}=A(\tau)(1+i\omega\tau)^{-1}\gamma^{-1}[-\tfrac{1}{3}ik(v_0/\omega)+\tfrac{1}{6}\pi]^{-1}. \tag{60}$$

5. EXPRESSIONS FOR THE ATTENUATION

5.1 Shear Waves

Writing $C_{11}\equiv AB$, where A is given by (47), and using Eqs. (4), (10), and (13), we obtain for $\mathbf{J}_e$ the expression:

$$\mathbf{J}_e=(\sigma\mathbf{F}+N_0e\mathbf{u})B. \tag{61}$$

Substituting (61) in (6) we obtain for $\mathbf{F}$ the expression

$$\mathbf{F}=-\frac{4\pi i\omega N_0e\mathbf{u}(B-1)}{c^2(k^2+Bk_0^2)}, \tag{62}$$

where $k_0=(4\pi i\omega\sigma/c^2)^{\frac{1}{2}}$ is the reciprocal of the classical skin depth of the electromagnetic waves of frequency ω.

[15] One may readily convince oneself that in order that (49) and (50) form a good approximation, it is sufficient that the first few $|\gamma_L|$ in the sequence $\gamma_1, \gamma_2, \gamma_3, \cdots$ satisfy these conditions.

[16] Divide each of the equations in (44) by the respective γ_L appearing in it and write C_{11} as a ratio of two determinants D_1 and D_2: $C_{11}=(A(\tau_1)/kl_1)(D_1/D_2)$. If one now expands D_1 and D_2 in series in which successive terms contain the inverse of the products of more and more γ's, one finds that the leading term in each of these series is independent of γ's. Hence in the limit $|\gamma_L|\to\infty$, D_1/D_2 would be independent of γ's and would have the same value as for the case where all the γ_L are equal to one another. Equation (50) then follows by comparing the above expression for C_{11} with Eq. (54).

In (62) a term ω^2/c^2 has been neglected in comparison with k^2.

Using Eqs. (2), (8), (61), and (62) in the equations of motion (7), one obtains, after some algebraic manipulation, for $Z=\omega^2/k^2$ the expression

$$Z=\beta_t+\frac{i\omega N_0 m(1-B)}{\rho k^2\tau_1}\left[\frac{k_0^2+k^2}{Bk_0^2+k^2}\right]. \tag{63}$$

The attenuation and the velocity of shear waves may now be readily calculated from (63) by using Eqs. (9) and different expressions for C_{11} or AB obtained in Sec. 4; the expressions for the attenuation for the three cases discussed in Sec. 4 are given below. We note that in this calculation it is sufficient to regard the k appearing in (63) to be real; in what follows we denote the real part of the wave number by k instead of k_1.

Case I: $kl_1\ll 1$

In this case C_{11} is given by (49), and the inequalities $k^2\ll|k_0^2|$ and $k^2\ll|Bk_0^2|$ are also satisfied. Hence the expression within the square bracket of (63) may be replaced by B^{-1}. One then obtains for the attenuation α_t of the shear waves the expression

$$\alpha_t=\frac{mv_0^2N_0\omega^2}{5\rho V_t^3}\tau_2\left[1-\frac{8}{35}k^2l_2l_3+\cdots\right], \tag{64}$$

where V_t is the velocity of the shear waves.

Case II: $kl_1\gg 1$

Now C_{11} is given by (50) so that $B=3\pi/(4kl_1)$. We have to subdivide this case into two:

IIa: When $kl_1\gg 1$, but $k^2\ll|Bk_0^2|\ll|k_0^2|$, α_t is given by

$$\alpha_t=\frac{4mv_0N_0\omega}{3\pi\rho V_t^2}. \tag{65}$$

IIb: When $kl_1\gg 1$, $k^2\gg|k_0^2|$, $\omega\tau_1\gg 1$; in this case $k^2\gg|Bk_0^2|$ also, and

$$\alpha_t=\frac{mN_0}{\rho V_t\tau_1}. \tag{66}$$

We note that the frequency ω_c at which $k^2=|Bk_0^2|$ is given by, using (50), $\omega_c=2\pi(3V_t^3N_0e^2/(4mc^2v_0))^{\frac{1}{3}}$. For most metals $(\omega_c/2\pi)\sim 10^9$ sec^{-1}. Since (66) is valid only for $\omega\gg\omega_c$, this case is not experimentally realizable at the present time. (It will be noticed that for high frequencies the attenuation is determined by τ_1, whereas at low frequencies it is determined by τ_2.)

Case III: All τ_L Equal to One Another

For this case the preceding cases (I) and (IIa) can be combined together. Using for C_{11} the expression from (52), we obtain for the attenuation

$$\frac{\alpha_t}{\omega}=\frac{mv_0N_0}{\rho V_t^2}G_t(kl), \tag{67}$$

where

$$G_t(kl)=\frac{g_0+(kl)^{-2}g_0-\frac{1}{3}}{kl[1-g_0-g_0(kl)^{-2}]}, \tag{68a}$$

and

$$g_0\equiv g(kl)=1-(kl)^{-1}\tan^{-1}(kl). \tag{68b}$$

Here $l=v_0\tau$. For $kl\gg 1$, (67) and (68) lead to the expression (65) for the attenuation; for $kl\ll 1$, the expression obtained from (67) and (68) is the same as that obtained from (64) by putting in it $\tau=\tau_1=\tau_2=\tau_3$. Finally the expression for the attenuation corresponding to case (IIb) above is given by (66) with τ_1 replaced by τ.

5.2 Dilatational Waves

Writing again $C_{10}\equiv AB$, and using the same procedure as outlined above for shear waves, we obtain for Z the expression

$$Z=\beta_d+\frac{i\omega mN_0}{\rho k^2\tau_1}\left(\frac{1-B}{B}\right)\left[\frac{1+i\omega/(\tau_1\omega_p^2)}{1+i\omega/(\tau_1\omega_p^2B)}\right], \tag{69}$$

where $\omega_p^2=(4\pi e^2N_0/m)$ is the square of the plasma frequency and β_d is the square of the velocity of the dilatational waves in the absence of the electron-ion interaction. For most metals $\omega_p\sim 10^{16}$ sec^{-1}, and it is easily verified that the terms $(\omega/\tau_1\omega_p^2)$ and $(\omega/\tau_1\omega_p^2B)$ are negligibly small compared to unity at the ultrasonic frequencies currently available. Hence the expression within the square brackets of (69) may be replaced by unity. The various expressions for the attenuation corresponding to different C_{10} evaluated in Sec. 4 are then given by the following:

Case I: $kl_1\ll 1$

$$\alpha_d=\frac{4mv_0^2N_0\omega^2}{15\rho V_d^3}\tau_2\left[1-\frac{9}{35}k^2l_2l_3+\cdots\right], \tag{70}$$

where V_d is the velocity of the dilatational waves.

Case II: $kl_1\gg 1$

For this case C_{10} is given by (58); the attenuation is independent of the τ_L and is given by

$$\alpha_d=\frac{\pi mv_0N_0\omega}{6\rho V_d^2}. \tag{71}$$

This expression for α_d is valid for all values of $kl_1\gg 1$, irrespective of whether $\omega\tau_1$ is less or greater than unity.

Case III: All τ_L Equal to One Another

By substituting (59) into (69), we may obtain

$$\frac{\alpha_d}{\omega}=\frac{mv_0N_0}{\rho V_d^2}G_d(kl), \tag{72}$$

where

$$G_d(kl)=[\tfrac{1}{3}kl(1-g_0)(g_0)^{-1}-(kl)^{-1}]. \tag{73}$$

Expression (72) for α_d is valid for *all* values of kl. It reduces to (71) for $kl\gg1$, and to (70) for $kl_1\ll1$, when the substitution $\tau=\tau_1=\tau_2=\tau_3$ etc., is made in (70).

It will be observed that for $\omega\tau_1\gg1$, the attenuation of dilatational and shear waves, given, respectively, by (71) and (66), have different qualitative behaviors as regards to their dependence on ω and τ; further (66) is smaller than (71) by at least a factor $(V_t/v_0)\approx10^{-3}$. Strictly speaking our treatment is invalid for $\omega\tau_1\gg1$, since here (following Pippard and Steinberg) the strain field of the impressed sound wave is regarded as classical. In the free-electron approximation a quantum mechanical calculation for the attenuation[17] of dilatational waves gives just the expression (71); for quantized shear waves, the attenuation is zero since these waves do not interact with the electrons in the free-electron approximation.[18]

A comment on the velocity of the dilatational waves is also necessary here. In terms of Z the velocity V_d is given by (9b). For every case discussed in the preceding paragraphs, we have for V_d an expression of the form

$$V_d^2=\beta_d+\tfrac{1}{3}(mv_0^2N_0/\rho)+\mathcal{O}(V_d^2mN_0/\rho). \tag{74}$$

The last term, which is of the order $10^{-5}V_d^2$, is associated with the attenuation and is frequency dependent. The second term has a simple interpretation since $\frac{1}{3}mv_0^2N_0$ is just the bulk modulus K_e of the electron gas; we note that K_e/ρ is of the same order of magnitude as β_d. The experimentally observed velocity is, of course, that given by (74) and not just $\beta_d^{\frac{1}{2}}$ which is in contrast to the case of shear waves where $V_t^2\simeq\beta_t$.

Finally we mention that for the case all τ_L equal to one another, (Case III above), the results obtained here are identical with those obtained by Pippard[1] and Steinberg.[2] These results have been included here to facilitate comparison of our results with theirs and with experimental data.

6. DISCUSSION

The results on attenuation obtained in this paper possess certain features which are missed in the more phenomenological treatments of the problem where one assumes the existence of a single relaxation time in accordance with Eq. (1). Some of these features are discussed below.

6.1 $kl_1\ll1$

In this case the attenuation of dilatational waves[19] is given by (70) and is proportional to the effective relaxation time τ_2. In the temperature region in which both the electron-phonon and electron-impurity scatterings are significant, τ_2 is in general, not proportional to τ_1. Hence the attenuation α_d also as a rule will not be proportional to the electrical conductivity σ. Neglecting the $k^2l_2l_3$ terms, etc., compared to unity in (70) and making use of (39) and (48), we may write for α_d/σ, the expression

$$\frac{\alpha_d}{S\sigma}\equiv\frac{\tau_2}{\tau_1}=\frac{(\tau_1^{(p)})^{-1}+(\tau_1^{(i)})^{-1}}{(\tau_2^{(p)})^{-1}+(\tau_2^{(i)})^{-1}}, \tag{75}$$

where

$$S=4m^2v_0^2\omega^2/(15\rho V_d^3e^2). \tag{76}$$

Let now T_0 denote the temperature at which $\tau_1^{(p)}=\tau_1^{(i)}$, i.e., at which the electrical resistance due to impurities equals that due to the electron-phonon interaction. Taking $\tau_1^{(p)}=3\tau_2^{(p)}$ and $\tau_1^{(i)}=1.4\tau_2^{(i)}$, as obtained in Sec. 3, one then sees from (75) that for $T\ll T_0$, $\alpha_d/S\sigma\approx0.71$, and for $T\gg T_0$, $\alpha_d/S\sigma\approx0.33$. A plot of $\alpha_d/S\sigma$ against T/T_0 is given in Fig. 1, which has been drawn for $T\ll\Theta$, so that $\tau_1^{(p)}\propto T^{-5}$. We note that most of the variation of $\alpha_d/S\sigma$ with T is confined to the temperature region $0.5T_0\leqslant T\leqslant1.5T_0$.

It will be realized that the numerical values in the above example cannot be taken too literally, since they are based on the values of τ_1/τ_2 for the *specific* electron-phonon and electron-impurity interactions, and also since $P(\mu)$ for the latter has been calculated in the Born approximation (see reference 14). Nevertheless, one may expect that, in general, $\tau_1^{(p)}/\tau_2^{(p)}$ would be sufficiently different from $\tau_1^{(i)}/\tau_2^{(i)}$ to make the deviations from proportionality of α to σ experimentally observable. Experimental measurements on aluminum by Filson[20] and Lax[21] and on silver by Lax,[22] however, show that α is proportional to σ. The reason for this discrepancy is not known. It would be of interest to have experimental measurements of α and σ on specimens having different and varying amounts of impurities.

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[17] See, for example, R. W. Morse, *Progress in Cryogenics* (Heywood, London, 1959), Vol. I, pp. 219 ff.

[18] It may be of interest here to mention that the first and second terms in the numerators within the square brackets of both (63) and (69) arise, respectively, from the electric force $-N_0e\mathbf{F}$ and the impulsive force $\mathbf{I}_c$ in the equations of motion (7). In all cases for the dilatational waves and shear waves, except the case IIb, ($\omega\tau_1\gg1$), for shear waves, the attenuation arises solely from the electric force term; in the case IIb, however, it is the term $\mathbf{I}_c$ which determines attenuation.

[19] The discussion for shear waves is similar since the attenuation of both these types of waves is determined by the same set of τ_L. We add that, as pointed out by Pippard (reference 1, p. 1110) in metals in which the electrons occupy more than one Brillouin zone, there may occur for dilatational waves an additional relaxation effect (and hence attenuation) due to a periodic transfer of electrons from one zone to another. This effect is not considered in our work and has yet not been experimentally observed.

[20] D. H. Filson, Phys. Rev. **115**, 1516 (1959).

[21] E. Lax, Phys. Rev. **115**, 1591 (1959).

[22] E. Lax, Technical Report No. XVII, Office of Naval Research, December, 1959 (unpublished).

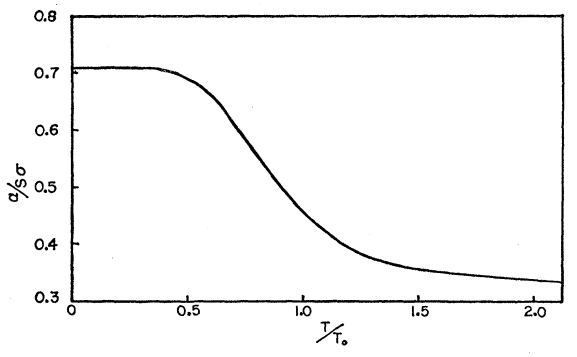

FIG. 1. Typical variation of $\alpha/S\sigma$ against T/T_0 in the temperature region where $T \ll$ Debye temperature Θ. S is a constant given by (76), and T_0 is the temperature at which the electrical resistance due to electron-phonon interaction equals that due to the impurities in the specimen.

As regards the absolute magnitude of the attenuation, Filson[20] and Lax[21,22] find that if one uses τ_1 in expression (70) for attenuation, $\alpha_{\text{exp}}/\alpha_{\text{theor}}$ is about 1.5 for aluminum and 1.8 for silver. On the other hand, if we use τ_2 in (70), as one should, then $\alpha_{\text{exp}}/\alpha_{\text{theor}}$ is even larger since according to the calculations of Sec. 3 $\tau_2 < \tau_1$. In connection with these discrepancies between theory and experiment, we make here the following two comments:

First, the reason why the values of both $\tau_2^{(p)}/\tau_1^{(p)}$ and $\tau_2^{(i)}/\tau_1^{(i)}$ given in Sec. 3 are less than unity is that for the particular electron-phonon and electron-impurity interactions chosen, the scattering or $P(\mu)$ is predominantly in the forward directions. As may be verified by using (40), if the scattering were relatively more in the backward directions or if there were a minimum in $P(\mu)$ around $\mu = 90°$, then τ_2/τ_1 would be greater than unity. Such a situation, for example, can arise for the electron-phonon interaction if the "Umklapp" processes (ignored in the calculation of Sec. 3) contribute significantly to the collision integral at the temperature under consideration, since for these processes the scattering is predominantly in the backward directions.[23] At room temperature the "Umklapp" processes contribute to σ as much as 80%; and, although their relative contribution decreases with decreasing temperatures, there seem to be both experimental evidence and theoretical arguments to indicate that they continue to contribute significantly down to very low temperatures.[24] A further discussion on this point will not be given here, since it is rather involved, and also because there exist other effects, like those due to deviation of Fermi surface from spherical shape, which may be of comparable magnitude and which have not been considered in this paper.

Second, we mention that in an "effective mass" approximation in which the Fermi surface is spherical but the energy of an electron is given by $\hbar^2|\mathbf{K}|^2/2m^*$, where m^* is the effective mass of an electron, the various expressions for the attenuation given in Sec. 5 have to be modified as follows: First eliminate the Fermi velocity v_0 from these expressions by means of (10), and then replace any m appearing explicitly in these expressions by m^*. Thus, for example, in expressions (64) and (70) for α_t and α_d for $kl_1 \ll 1$, the factor mv_0^2 outside the square brackets is to be replaced by $(3N_0h^3/8\pi)^{\frac{2}{3}} \times (1/m^*)$. When the factor mv_0 is eliminated from expressions (65) and (71) for α_t and α_d for $kl_1 \gg 1$, the resulting expressions become independent of the effective mass parameter.[25] We see also that the factor S in (76) is independent of m^* and thus the effective-mass approximation cannot account for the observed magnitude of attenuation in silver and aluminum. (Note that τ_L also contain m^*—but the ratios $\tau_L^{(p)}/\tau_1^{(p)}$ or $\tau_L^{(i)}/\tau_1^{(i)}$ are independent of m^*. The dependence of $\tau_1^{(p)}$ and $\tau_1^{(i)}$ on m^* may be found in reference 4 and will not be given here.)

Electron-Electron Collisions

It is well known that in the free electron approximation the interelectronic collisions do not contribute to the electrical resistance of a metal. However, they would influence the attenuation of sound waves. For a degenerate Fermi gas the relaxation time τ_2^{ee} for interelectronic collisions is proportional to T^{-2}. Hence, writing $(\tau_2)^{-1} = (\tau_2^{(p)})^{-1} + (\tau_2^{(i)})^{-1} + (\tau_2^{ee})^{-1}$, we see that when $kl_1 \ll 1$ and $T \ll \Theta$, α^{-1} will depend on temperature in the manner

$$\alpha^{-1} = a_i + a_{ee}T^2 + a_pT^5. \tag{77}$$

Thus there exists the possibility of determining τ_2^{ee} from the attenuation measurements. A rough theoretical estimate of τ_2^{ee} shows that the last two terms in (77) should be of the same order of magnitude at about 3°K in copper.

6.2 Attenuation over a Wide Range of Frequencies of Sound Waves

Mason[26] has compared Pippard's analytical expression for α_d with experimental results on tin over a wide range of values of kl_1 as follows: Assuming that $\tau = \tau_1$ in (72) and using the values of τ_1 from the electrical conductivity measurements, he multiplies (72) by a

[23] See, for example, Fig. 113 of reference 7 or Tables II to V in a paper by A. B. Bhatia and K. S. Krishnan, Proc. Roy. Soc. (London) **194**, 185 (1948).

[24] See reference 7, p. 370.

[25] Recently Blount [E. I. Blount, Phys. Rev. **114**, 418 (1959)] has obtained attenuation [assuming Eq. (1) for the collision integral] in such a way that his expressions contain not only m^* but also an interaction parameter A (=1 in the free-electron approximation) for the direct interaction between electrons and the impressed sound wave. If we take this latter interaction to be the same as that between longitudinal thermal phonons of long wavelengths and electrons, then in the spherical band approximation $|A| = m^*/m$. If we substitute this value of A and $v_0 = (h/m^*) \times (3N_0/8\pi)^{\frac{1}{3}}$ in Blount's expressions, the effective-mass dependence of his expressions becomes identical with that given above.

[26] W. P. Mason, *Physical Acoustics and the Properties of Solids* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1958), p. 329.

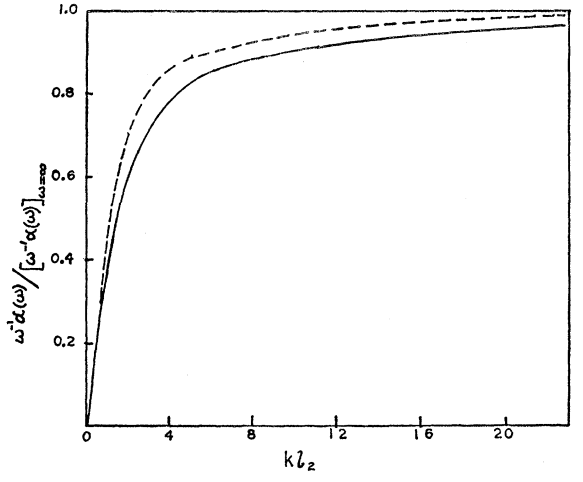

FIG. 2. Attenuation as a function of kl_2. Here $k=2\pi/$(wavelength of sound wave) and $l_2=v_0\tau_2$, a mean free path of the electrons. – – – Numerical calculations of this paper using for τ_L/τ_1 the values from Eq. (33). —— Pippard's analytical expression (72) with $\tau\equiv\tau_2$.

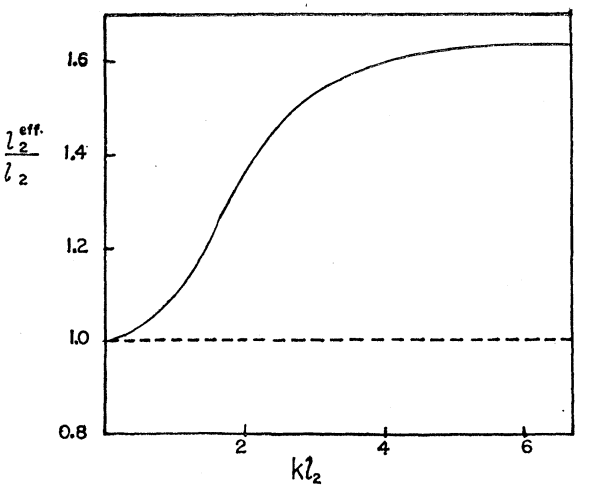

FIG. 3. l_2^{eff}/l_2 as a function of kl_2. —— based on the numerical calculations of this paper. – – – Pippard; ($l_2^{\text{eff}}\equiv l_2$ for all kl_2 in this case by definition). For explanation of l_2^{eff} see text.

normalization factor $\mathfrak{M}_P$ such that the theoretical values of attenuation agree with experiment for $kl_1\ll 1$. He then finds that the theoretical limiting value of α_d/ω for $kl_1\gg 1$ is only about 40% of the observed value. If we make a similar comparison with the theoretical results of this paper, we would have to multiply our expressions by a factor $\mathfrak{M}_P\times(\tau_1/\tau_2)$ to obtain numerical agreement with experiment for $kl_1\ll 1$; and hence our limiting value of α_d/ω for $kl_1\gg 1$ would be τ_1/τ_2 times greater than Pippard's. Since as a rule $\tau_1>\tau_2$, we see that experimental results on tin are in better agreement if τ in Pippard's expression is identified as τ_2 rather than τ_1. It should be mentioned that in view of the rather idealized assumptions made in this paper, our considerations hardly apply to tin, and it would be desirable to have measurements over a wide frequency range on some of the alkali and noble metals, particularly in view of the results already discussed for silver and aluminum for the case $kl_1\ll 1$.

We have seen above that if we identify τ appearing in Pippard's analytical expression (72) as τ_2, then for both $kl_2\ll 1$ and $kl_2\gg 1$, (72) reduces to the expressions obtained in this paper. For the intermediate values of kl_2, however, the attenuation calculated from our Eqs. (55) will in general be different than that calculated from (72), if the various τ_L appearing in (55) are different from one another. As an example, we have calculated numerically the attenuation for dilatational waves for several values of kl_2 assuming that the impurity scattering can be neglected and that the ratios τ_1/τ_L are given by (33). The results of these calculations are shown in Fig. 2, where α_d/ω is plotted against kl_2. A plot of α_d/ω as obtained from Pippard's expression (72) with $\tau\equiv\tau_2$ is also given. We note that the maximum difference between the two curves is about 20%.

A more sensitive way to compare the results given in Fig. 2 would be as follows.[27] Take the numerically calculated value (or the experimental value) of α_d/ω at a given frequency and equate it to (72). This would determine a value of kl_2, and hence of l_2. We designate this value of l_2 as an effective l_2 at this frequency and denote it by l_2^{eff}. Doing this at several frequencies would give a value of l_2^{eff} at each frequency. Now plot l_2^{eff}/l_2 against ω or kl_2. (Note that $l_2^{\text{eff}}\equiv l_2$ for $kl_2\ll 1$.) Such a plot based on the numerical calculations outlined in the preceding paragraph is given in Fig. 3. If all the τ_L were equal to one another, one would just obtain for l_2^{eff}/l_2 a horizontal straight line of ordinate unity. It should be mentioned that a comparison such as above would be meaningful only if the experimental data on attenuation (after allowing for the background attenuation due to causes other than the interaction of sound waves with electrons) is accurate to within a few percent.

[27] This procedure is reminiscent of the manner in which the experimental data on specific heats C_V is compared with theories; here instead of plotting C_V against T, one plots an effective Debye Θ against T.