Shell Model Calculations of the Hyperfine Effect in μ-Meson Capture*

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Numerical values for the difference in μ-meson capture rates from the two hyperfine states of a mesonic atom are obtained for F^{19}, Al^{27}, and P^{31}, employing the simple Mayer-Jensen version of the shell model for Al^{27} and P^{31}, and an intermediate coupling version for F^{19}. In all these cases, we get considerably larger values than those found in previous estimates, thus rendering experiments more feasible. In view of the consequences of atomic conversion on the observation of the capture-rate differences, conversion effects in the rate and asymmetry of μ-meson decay electrons are discussed also.

I. INTRODUCTION

THE influence of the hyperfine interaction between nuclear and muon spin on the nuclear capture rate of μ mesons has been pointed out by Bernstein *et al.*,[1] and experiments to measure the resulting difference in the capture rates from the two hyperfine levels of the μ-mesonic atom ground state have been attempted.[2] The reason for the interest in a detection and quantitative analysis of this effect is that it arises from a spin dependence of the basic weak capture interaction, and thus offers a possibility for testing the μ-capture Hamiltonian for such a spin dependence. To give an example, if one takes a simple Fermi and Gamow-Teller interaction with respective coupling constants g_F, g_G, then the muon capture rate by a proton in the triplet state λ_t is proportional to $(g_F+g_G)^2$, in a singlet state λ_s to $(g_F-3g_G)^2$, and a measurement of λ_s, λ_t separately (e.g., by determining the deviation of the decay electron time distribution from a simple exponential) can serve to establish the presence or absence of either interaction. For complex nuclei, on which the experiment has to be performed, nuclear matrix elements will enter. For this case, rough estimates were made[1] on the basis of the Schmidt model; the interpretation of measurements becomes then more uncertain. The purpose of this paper, therefore, is to calculate these nuclear matrix elements more accurately using the shell model. The effect is expected to be largest for nuclei with an unpaired proton (its spin being strongly correlated with the nuclear spin), but will exist in other cases, too. The most suitable isotopes (odd Z, odd A, $Z\sim 10$) would be N^{15}, F^{19}, Na^{23}, Al^{27}, P^{31}, Cl^{35}, Cl^{37}, and others. For most of these, especially the $(2s,1d)$ shell nuclei, accurate configurations are not known. We therefore, after deriving general expressions for the capture rates on the shell model (Sec. II), shall utilize the simple version of the shell model[3] for Al^{27} and P^{31} (Sec. III), assuming a successive filling of the $(1d\,\frac{5}{2})$ and $(2s\,\frac{1}{2})$ shells. We note that such a treatment still consists in an improvement over the Schmidt model insofar as it permits a calculation of the unknown parameter ξ of Bernstein *et al.*,[1] which represents the ratio of the effectiveness of the Pauli principle (excluding the capturing proton turned neutron from the neutron shells) in reducing the rate of μ capture by the zero-spin shells and by the outside proton. For F^{19}, the exact configuration is known,[4,5] and is used for calculating our nuclear matrix elements in Sec. IV. In all three cases studied, we find up to three times larger results than those obtained by tentatively setting $\xi=1$ [6] in the formulas of the work cited in footnote 1. These results are discussed in Sec. V. As pointed out by Telegdi,[2] conversion of the upper to the lower hyperfine state can occur by Auger electron ejection with a rate comparable to the capture rate difference, thereby strongly influencing the decay electron time distribution. This will be discussed in Sec. VI, where the influence of conversion on the decay electron asymmetry is also studied.

II. DERIVATION OF THE GENERAL EXPRESSION FOR THE CAPTURE RATE DIFFERENCE ON THE SHELL MODEL

For calculating the muon capture rate in complex nuclei, we use the effective capture Hamiltonian of Primakoff,[6,7] which contains vector, axial vector, and pion-induced pseudoscalar interactions. In calculating the spin-averaged square of the transition matrix element, we kept the retardation factor of the neutrino space wave function, except in the small pseudoscalar case, where it was replaced by the lowest term of its multipole expansion. By further employing the closure approximation on the sum over final nuclear states (i.e., replacing the neutrino momentum ν by its average

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[1] J. Bernstein, T. D. Lee, C. N. Yang, and H. Primakoff, Phys. Rev. **111**, 313 (1958).

[2] V. L. Telegdi, Phys. Rev. Letters **3**, 59 (1959).

[3] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, New York, 1955), p. 74 ff.

[4] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955).

[5] M. G. Redlich, Phys. Rev. **99**, 1427 (1955).

[6] H. Primakoff, Revs. Modern Phys. **31**, 802 (1959).

[7] See also H. Überall, Phys. Rev. **116**, 218 (1959).

$\bar{\nu}$ in the phase space), we arrive at the result for the muon capture rates (see also work cited in footnote 6):

$$\lambda \equiv \frac{J+1}{2J+1}\lambda_+ + \frac{J}{2J+1}\lambda_- = \frac{1}{2\pi^2 a_Z{}^3}\frac{\bar{\nu}^2}{1+(\bar{\nu}/M)} \times\langle\Psi_{JM_J}|A+A'|\Psi_{JM_J}\rangle, \quad (1a)$$

$$\Delta\lambda \equiv \lambda_+ - \lambda_- = \frac{1}{2\pi^2 a_Z{}^3}\frac{\bar{\nu}^2}{1+(\bar{\nu}/M)}\frac{2J+1}{J(J+1)} \times\langle\Psi_{JM_J}|(\mathbf{B}+\mathbf{B}')\cdot\mathbf{J}|\Psi_{JM_J}\rangle. \quad (1b)$$

Here, $\lambda_\pm$ are the capture rates in the two hyperfine states with spin $F=J\pm\frac{1}{2}$, J being the nuclear spin. M is the mass of the nucleus; the muon Bohr radius is given by

$$a_Z = (137/\mu Z)(1+\mu/M) \quad (2)$$

(we use units with $\hbar=c=1$ and electron mass $m_e=1$); $\bar{\nu}$ is the average momentum of the emitted neutrino, μ the muon mass, and the matrix elements are taken over the nuclear ground state. We have further:

$$A = a\sum_i \tfrac{1}{2}(1+\tau_z{}^i)[\psi(\mathbf{r}_i)]^2, \quad (3a)$$

$$A' = \sum_{i\neq j}(a'+a''\boldsymbol{\sigma}^i\cdot\boldsymbol{\sigma}^j)\tau_+{}^i\tau_-{}^j\psi(\mathbf{r}_i)\psi(\mathbf{r}_j)j_0(\bar{\nu}r_{ij}), \quad (3b)$$

$$\mathbf{B} = b\sum_i \boldsymbol{\sigma}^i\tfrac{1}{2}(1+\tau_z{}^i)[\psi(\mathbf{r}_i)]^2, \quad (3c)$$

$$\mathbf{B}' = \sum_{i\neq j}(b'\boldsymbol{\sigma}^i+b'^*\boldsymbol{\sigma}^j+ib''\boldsymbol{\sigma}^i\times\boldsymbol{\sigma}^j) \times\tau_+{}^i\tau_-{}^j\psi(\mathbf{r}_i)\psi(\mathbf{r}_j)j_0(\bar{\nu}r_{ij}), \quad (3d)$$

the sums over i and j running over all nucleons in the nucleus. The muon space wave function $\psi(\mathbf{r}_i)$ is normalized to 1 in the limit of $Z\to 0$; j_0 is the spherical Bessel function. The primed operators, which connect different particles, represent the effect of the exclusion principle on the neutron produced from the proton which captured the muon. Finally, we have the combinations of effective coupling constants[6,7]:

$$a = |G_V|^2+3|G_A|^2-2\,\mathrm{Re}G_A{}^*G_P+|G_P|^2, \quad (4a)$$

$$a' = |G_V|^2, \quad (4b)$$

$$a'' = |G_A|^2-\tfrac{2}{3}\,\mathrm{Re}G_A{}^*G_P+\tfrac{1}{3}|G_P|^2, \quad (4c)$$

$$b = 2\,\mathrm{Re}G_A{}^*(G_V-G_A)-\tfrac{2}{3}\,\mathrm{Re}G_V{}^*G_P+\tfrac{4}{3}\,\mathrm{Re}G_A{}^*G_P, \quad (4d)$$

$$b' = G_V{}^*(G_A-\tfrac{1}{3}G_P), \quad (4e)$$

$$b'' = \mathrm{Re}G_A{}^*(G_A-\tfrac{2}{3}G_P). \quad (4f)$$

For calculating the nuclear matrix elements appearing in Eqs. (1a) and (1b), we have to use a completely antisymmetrized nuclear wave function (spin indices are suppressed for the time being),

$$\Psi(1,\cdots A) = N\sum_P \epsilon_P\Phi(P1,\cdots PA), \quad (5)$$

where Φ will be taken as a product shell-model wave function,

$$\Phi(1,\cdots A) = \Psi_1(1,\cdots n_1)\Psi_2(n_1+1,\cdots n_1+n_2)\cdots \Psi_r(\cdots A), \quad (6)$$

consisting of completely antisymmetric wave functions of the shells Ψ_q, each containing n_q particles. $\Psi_r(1,\cdots n_r)$ is considered the unfilled shell, the only one which possesses nonzero spin, and consists of both protons and neutrons, whereas the other Ψ_q are either pure proton or pure neutron shells with zero spin. The operators (3) in (1) which have to be taken between states (5) are either one- or two-particle operators (in case b multiplied by the operator $\mathbf{J}=\mathbf{J}_1+\cdots+\mathbf{J}_A$, symmetric in all particles) of the form

$$Q^{(1)} = \sum_i Q(i,i), \quad Q^{(2)} = \sum_{i\neq j} Q(ij,i'j') \quad (7)$$

(in matrix component notation, i represents the space, spin, and isospin coordinate of particle i). We can then reduce our nuclear matrix elements to those taken between states Φ, by using the following two theorems:

Theorem I: $\langle\Psi|Q^{(1)}|\Psi\rangle = \langle\Phi|Q^{(1)}|\Phi\rangle$, (8a)

Theorem II: $\langle\Psi|Q^{(2)}|\Psi\rangle = \langle\Phi|Q_a{}^{(2)}|\Phi\rangle$, (8b)

where

$$Q_a{}^{(2)} = Q^{(2)} \quad (9a)$$

if i, j lie in the same shell Ψ_q,

$$Q_a{}^{(2)} = \sum_{i\neq j}[Q(ij,i'j')-Q(ij,j'i')] \quad (9b)$$

if i, j lie in different shells Ψ_p, Ψ_q. The proof of these two theorems is straightforward, using general symmetry properties of Ψ and the orthogonality of different shells Ψ_p, Ψ_q. The same theorems hold also if $Q^{(k)}$ is multiplied by $\mathbf{J}$, as in case (1b). By applying Eqs. (8a) and (8b) to our Eqs. (1a) and (1b) now, we obtain the general results for the muon capture rates on the shell model:

$$\lambda = K[(Z_{\mathrm{eff}}{}^4/Z^3)a+\langle\Omega'\rangle], \quad (10a)$$

$$\Delta\lambda = K[(2J+1)/J(J+1)](\langle\boldsymbol{\Gamma}\cdot\mathbf{J}\rangle+\langle\boldsymbol{\Gamma}'\cdot\mathbf{J}\rangle), \quad (10b)$$

with

$$K = (1/2\pi^2 a_Z{}^3)[\bar{\nu}^2/(1+(\bar{\nu}/M))] = [\gamma\lambda(1,1)/a]Z^3, \quad (11a)$$

$$Z_{\mathrm{eff}}{}^4/Z^3 = \langle\Psi|\sum_i \tfrac{1}{2}(1+\tau_z{}^i)[\psi(\mathbf{r}_i)]^2|\Psi\rangle. \quad (11b)$$

Here, the parameter $\gamma\lambda(1,1)$ was used by Sens[8] to replace the (unknown) $\bar{\nu}^2$, and was determined by him experimentally by fitting measured capture rates to Primakoff's[6] formula for λ. The effective nuclear charge $Z_{\mathrm{eff}}{}^4$ was introduced by Wheeler,[9] and was accurately calculated by Sens[8] for a variety of nuclei.

[8] J. C. Sens, Phys. Rev. **113**, 679 (1959).
[9] J. Tiomno and J. A. Wheeler, Revs. Modern Phys. **21**, 153 (1949).

The remaining matrix elements are:

$$\langle\Omega'\rangle = -\sum_{q\neq q',1}^{r-1} n_q n_{q'} X_{qq'} - n_r \sum_{q=1}^{r-1} n_q T_{qr} + n_r(n_r-1)S_r, \quad (12a)$$

$$\langle \mathbf{\Gamma}\cdot\mathbf{J}\rangle = n_r b \langle \Psi_r(1,\cdots n_r)|\tfrac{1}{2}(1+\tau_z^1)[\psi(\mathbf{r}_1)]^2 \boldsymbol{\sigma}^1\cdot\mathbf{J}| \times \Psi_r(1,\cdots n_r)\rangle, \quad (12b)$$

$$\langle \mathbf{\Gamma}'\cdot\mathbf{J}\rangle = -n_r \sum_{q=1}^{r-1} n_q D_{qr} + n_r(n_r-1)E_r, \quad (12c)$$

with the further notations:

$$X_{qq'} = \langle \Psi_q(1,\cdots n_q)\Psi_{q'}(1',\cdots n_{q'}')|\tau_+^1\tau_-^{1'}\omega(1,1') \times |\Psi_q(1',2,\cdots n_q)\Psi_{q'}(1,2',\cdots n_{q'}')\rangle, \quad (13a)$$

$$T_{qr} = \langle \Psi_q(1,\cdots n_q)\Psi_r(1',\cdots n_r')|(\tau_+^1\tau_-^{1'}+\tau_-^1\tau_+^{1'}) \times\omega(1,1')|\Psi_q(1',2,\cdots n_q)\Psi_r(1,2',\cdots n_r')\rangle, \quad (13b)$$

$$S_r = \langle \Psi_r(1,\cdots n_r)|\tau_+^1\tau_-^2\omega(1,2)|\Psi_r(1,\cdots n_r)\rangle, \quad (13c)$$

$$D_{qr} = \langle \Psi_q(1,\cdots n_q)\Psi_r(1',\cdots n_r')| \times[(\tau_+^1\tau_-^{1'}+\tau_-^1\tau_+^{1'})\gamma_s(1,1')+(\tau_+^1\tau_-^{1'} - \tau_-^1\tau_+^{1'})\gamma_a(1,1')]\cdot(\mathbf{J}_1+\mathbf{J}_{2'}+\cdots+\mathbf{J}_{n_r'}) \times|\Psi_q(1',2,\cdots n_q)\Psi_r(1,2',\cdots n_r')\rangle, \quad (13d)$$

$$E_r = \langle \Psi_r(1,\cdots n_r)|\tau_+^1\tau_-^2[\gamma_s(1,2) + \gamma_a(1,2)]\cdot\mathbf{J}|\Psi_r(1,\cdots n_r)\rangle, \quad (13e)$$

where finally

$$\omega(1,2) = \psi(\mathbf{r}_1)\psi(\mathbf{r}_2)j_0(\bar{\nu}r_{12})(a'+a''\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2), \quad (14a)$$

$$\gamma_s(1,2) = \psi(\mathbf{r}_1)\psi(\mathbf{r}_2)j_0(\bar{\nu}r_{12})\,\mathrm{Re}b'(\boldsymbol{\sigma}^1+\boldsymbol{\sigma}^2), \quad (14b)$$

$$\gamma_a(1,2) = i\psi(\mathbf{r}_1)\psi(\mathbf{r}_2)j_0(\bar{\nu}r_{12}) \times[\mathrm{Im}b'(\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^2)+b''\boldsymbol{\sigma}^1\times\boldsymbol{\sigma}^2]. \quad (14c)$$

Actually, only $\Delta\lambda$ will be calculated in the following.

A simplification can be made in Eq. (13) for matrix elements taken between filled shells. The wave function of such a shell

$$q = (nlj)_{n(p)}{}^{n_q}, \quad (15)$$

with $q\neq r$, $n_q = 2j+1$, is a Slater determinant

$$\Psi_q(1,\cdots n_q) = (n_q!)^{-\frac{1}{2}}\textstyle\sum_P \epsilon_P \psi_{jj}(P1) \times\psi_{jj-1}(P2)\cdots\psi_{j-j}(Pn_q), \quad (16)$$

and the matrix element of an operator $O(1,1')$ becomes simply

$$\langle \Psi_q(1,\cdots n_q)|O(1,1')|\Psi_q(1,\cdots n_q)\rangle = n_q^{-1}\textstyle\sum_m \langle \psi_{jm}(1)|O(1,1')|\psi_{jm}(1')\rangle, \quad (17)$$

i.e., the matrix element taken over the single orbitals ψ_{jm}, averaged over all their orientations.

III. EVALUATION USING THE MAYER-JENSEN MODEL

The configurations of the nuclides investigated by us are according to the Mayer-Jensen shell model (apart from the common $(1s_{\frac{1}{2}})$, $(1p_{\frac{3}{2}})$, and $(1p_{\frac{1}{2}})$ shells):

$$\mathrm{F}^{19}: \quad (2s_{\frac{1}{2}})_n{}^2(2s_{\frac{1}{2}})_p{}^1, \qquad J=\tfrac{1}{2}; \quad (18a)$$

$$\mathrm{Al}^{27}: \quad (1d_{\frac{5}{2}})_n{}^6(1d_{\frac{5}{2}})_p{}^5, \qquad J=\tfrac{5}{2}; \quad (18b)$$

$$\mathrm{P}^{31}: \quad (1d_{\frac{5}{2}})_n{}^6(1d_{\frac{5}{2}})_p{}^6(2s_{\frac{1}{2}})_n{}^2(2s_{\frac{1}{2}})_p{}^1, \quad J=\tfrac{1}{2}. \quad (18c)$$

Note that all these configurations suggest a state $J=l+\frac{1}{2}$ of the Schmidt model. F^{19} actually represents an anomaly of the Mayer-Jensen model, the $2s$ shell being filled before the $1d$ shell; moreover, the unfilled neutron shell is not determined, our assumption of $2s$ only being justified by the more exact evaluations of the configuration.[4,5]

The normalized radial wave functions used are

$$R_{1l}(r) = \frac{\alpha^{\frac{3}{2}}}{\pi^{\frac{1}{4}}}\left(\frac{2^{l+2}}{(2l+1)!!}\right)^{\frac{1}{2}}(\alpha r)^l \exp[-\tfrac{1}{2}(\alpha r)^2], \quad (19a)$$

$$R_{2l}(r) = \frac{\alpha^{\frac{3}{2}}}{\pi^{\frac{1}{4}}}\left(\frac{2^{l+3}}{(2l+3)!!}\right)^{\frac{1}{2}}(\alpha r)^l\left[\frac{2l+3}{2}-(\alpha r)^2\right] \times\exp[-\tfrac{1}{2}(\alpha r)^2], \quad (19b)$$

giving rise to a mean square nuclear radius

$$\frac{1}{A}\sum_{q=1}^{r} n_q\langle r^2\rangle_q = \frac{1}{A\alpha^2}\sum_{q=1}^{r} n_q[2(n-1)+l+\tfrac{3}{2}], \quad (20)$$

and by equating this to the nuclear radii quoted by Sens[8] in his Table II, we were able to determine the parameter $\alpha = 188$ (F^{19}), 178 (Al^{27}), 173 (P^{31}). The K-shell muon wave functions were calculated numerically by Ford and Wills.[10] From the tables given by them, we could obtain an analytic expression of the wave functions, valid only *inside the nucleus* (i.e., in a region where $r^2R_{nl}(r)$ is appreciable) of the form (large component only):

$$\psi(r) = N(1-ar^2+br^3), \quad (21)$$

with the coefficients given in Table I. The normalization is as mentioned after Eq. (3), namely,

$$4\pi\int_0^\infty \psi^2(r)r^2dr = \pi a_Z{}^3. \quad (22)$$

The factor N represents the reduction of the wave function from 1 due to the nuclear extension.

TABLE I. Parameters of the K-shell muon wave function inside the nucleus.

	Z	N	a/α^2	b/α^3
F^{19}	9	0.88351	0.02134	0.002958
Al^{27}	13	0.82724	0.03405	0.004623
P^{31}	15	0.79615	0.03970	0.005517

[10] K. W. Ford and J. G. Wills, Los Alamos Scientific Laboratory Report LAMS-2387, 1960 (unpublished); and private communication.

The simple configurations all have only a single particle in the unfilled shell (Al^{27} a single hole, which can be treated exactly like a particle). Thus, the angular part of $\langle\mathbf{\Gamma}\cdot\mathbf{J}\rangle$ will give the same results as those obtained from the Schmidt model, and $\langle\mathbf{\Gamma}'\cdot\mathbf{J}\rangle$ becomes fairly simple, too (e.g., $E_r=0$), and is further simplified by expanding $j_0(\bar{\nu}r_{12})$ in a power series and only keeping the first three terms (the next higher term was estimated to be no larger than 2%). The results are:

$$F^{19}:\quad \langle\mathbf{\Gamma}\cdot\mathbf{J}\rangle=\tfrac{3}{2}bI(2s), \tag{23a}$$

$$\langle\mathbf{\Gamma}'\cdot\mathbf{J}\rangle=-3(b'-b'')[A_0(1s,2s)+\tfrac{1}{3}A_2(1s,2s)+A_1(1p,2s)+A_0(2s,2s)+\tfrac{1}{3}A_2(2s,2s)], \tag{23b}$$

$$Al^{27}:\quad \langle\mathbf{\Gamma}\cdot\mathbf{J}\rangle=\tfrac{7}{2}bI(1d), \tag{24a}$$

$$\langle\mathbf{\Gamma}'\cdot\mathbf{J}\rangle=-\{(14/15)(b'-b'')[A_2(1s,1d)+3A_1(1p,1d)]+7(b'-\tfrac{1}{5}b'')A_0(1d,1d)+((17/5)b'-b'')A_2(1d,1d)\}, \tag{24b}$$

$$P^{31}:\quad \langle\mathbf{\Gamma}\cdot\mathbf{J}\rangle=\tfrac{3}{2}bI(2s), \tag{25a}$$

$$\langle\mathbf{\Gamma}'\cdot\mathbf{J}\rangle=-3(b'-b'')[A_0(1s,2s)+\tfrac{1}{3}A_2(1s,2s)+A_1(1p,2s)+\tfrac{2}{5}A_2(1d,2s)+A_0(2s,2s)+\tfrac{1}{3}A_2(2s,2s)], \tag{25b}$$

with the notation:

$$I(nl)=\int_0^\infty R_{nl}^2(r)\psi^2(r)r^2dr, \tag{26a}$$

$$A_0=L_0^2-\tfrac{1}{3}\bar{\nu}^2L_0L_2+(1/60)\bar{\nu}^4(L_0L_4+L_2^2),$$
$$A_1=\tfrac{1}{3}\bar{\nu}^2L_1^2-(1/15)\bar{\nu}^4L_1L_3, \tag{26b}$$
$$A_2=(1/30)\bar{\nu}^4L_2^2,$$

$$L_m(nl,n'l')=\int_0^\infty r^mR_{nl}(r)R_{n'l'}(r)\psi(r)r^2dr. \tag{26c}$$

These radial integrals are easily evaluated using Eqs. (19) and (21) together with Table I. We then get

$$F^{19}:\quad \langle\mathbf{\Gamma}\cdot\mathbf{J}\rangle=1.061b=-5.703,$$
$$\langle\mathbf{\Gamma}'\cdot\mathbf{J}\rangle=-1.271(b'-b'')=3.416, \tag{27a}$$

$$Al^{27}:\quad \langle\mathbf{\Gamma}\cdot\mathbf{J}\rangle=2.028b=-10.90,$$
$$\langle\mathbf{\Gamma}'\cdot\mathbf{J}\rangle=-2.529b'+0.9669b''=4.529, \tag{27b}$$

$$P^{31}:\quad \langle\mathbf{\Gamma}\cdot\mathbf{J}\rangle=0.7975b=-4.288,$$
$$\langle\mathbf{\Gamma}'\cdot\mathbf{J}\rangle=-1.022(b'-b'')=2.746. \tag{27c}$$

The numerical values of b, b', and b'' were obtained using Primakoff's[6] coupling constants, Eqs. (1b) and (1c) of his paper; these are valid under the assumptions of universal Fermi interaction for the bare couplings, Gell-Mann's[11] conserved vector current hypothesis, and an induced pseudoscalar interaction according to Wolfenstein.[12]

[11] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

TABLE II. Fractional hyperfine differences of muon capture rates in Al^{27}, P^{31}.

		$\Delta\lambda/\lambda$	$(\Delta\lambda/\lambda)_{\xi=1}$	$\Delta\lambda/\Lambda$
Al^{27}	(α)	−0.53	−0.17	−0.30
	(β)	−0.46		−0.28
P^{31}	(α)	−0.46	−0.25	−0.32
	(β)	−0.44		−0.31

For obtaining $\Delta\lambda/\lambda$, we take the values of Eq. (27) and Eqs. (10b) and (11a). We have not calculated λ, but will use the following procedure for getting its magnitude. The experimental capture rates of Sens[8] can in the average be fitted by Primakoff's formula[6,8]:

$$\lambda(A,Z)=\gamma\lambda(1,1)Z_{\text{eff}}^4[1-\delta(A-Z)/2A], \tag{28}$$

with the parameter values $\gamma\lambda(1,1)=188$ sec^{-1}, $\delta=3.15$. The deviation from this fitted λ of the experimental capture rates in individual nuclides[8] can be interpreted in two extreme cases as (α) deviation of $\gamma\lambda(1,1)$ alone (resulting from a different average square neutrino momentum $\bar{\nu}^2$, to which γ is proportional), (β) deviation of δ alone. Deviations of both $\gamma\lambda(1,1)$ and δ will give results in between. In these two cases (which are considered here simply to show us the effect of the uncertainty of $\bar{\nu}^2$ on our results), we get slightly different values of K, except for F^{19}, where the deviation between experimental and fitted λ is quite large. This case will therefore be treated separately in the next section. The results for Al^{27} and P^{31} are shown in Table II. For comparison, the second column presents the values $(\Delta\lambda/\lambda)_{\xi=1}$, obtained by using the Schmidt model and assuming an equal effect of the exclusion principle on the spinless nuclear core and the outside proton, a procedure tentatively suggested by Primakoff.[6] The last column shows the experimentally important quantity $\Delta\lambda/\Lambda$, designating the total muon disappearance rate $\Lambda=\lambda+\lambda_{\text{dec}}$, which contains the decay rate $\lambda_{\text{dec}}=4.505\times10^5$ sec^{-1}. Note that the ratio between the first and the last column also depends on the case, (α) or (β). As mentioned earlier, the more exact values of column 1 are much larger than those of column 2.

IV. EVALUATION FOR F^{19}

In the case of F^{19}, the Mayer-Jensen model does not really establish the configuration (18a) uniquely. Detailed theoretical investigations on the configurations of this nucleus, however, have been made,[4,5] and we shall use the results of Redlich[5] for our calculations. He has determined the ground-state ($T=\frac{1}{2}$, $J=\frac{1}{2}$) configuration of F^{19} to be a mixture of various ($2s_{\frac{1}{2}}$),

[12] L. Wolfenstein, Nuovo cimento **8**, 882 (1958).

($1d_{\frac{5}{2}}$), and ($1d_{\frac{3}{2}}$) orbitals in the three-particle unfilled shell, with amplitudes presented in his Table VI. This table is not, however, complete, as far as the phases of the j^3 states are concerned. The full specification is shown[13] in our Table III, with amplitudes in the column labeled R. The relevant unfilled-shell wave function to be used in our matrix elements (12) is given by

$$\Psi_r(1,2,3)=\sum_\alpha c_\alpha \Psi_{r\alpha}(1,2,3) \tag{29a}$$

(the sum over α including the primes on c_α), and

$$\Psi_{r\alpha}(1,2,3)=\sum_l{}' \sum_{T''J''}(j_ij_k,T''J'';j_l|\}T'J',TJ) \times|(j_i(1)j_k(2),[T''J''])_a j_l(3),TT_z,JJ_z\rangle, \tag{29b}$$

where $(ikl)=(123)$, the sum over l runs over distinct j_l only, and particles 1, 2 add up to the antisymmetric parent state $[T'',J'']$, which in turn adds up with 3 to TT_z, JJ_z. The fractional parentage coefficients

$$(j_ij_k,T''J'';j_l|\}T'J',TJ)$$

can be found in Redlich's paper,[5] expressed in terms of Racah coefficients, the latter ones being given in tables.[14] This representation of the wave function by fractional parentage coefficients has the advantage that three-particle matrix elements can be directly expressed by one- and two-particle ones. Nevertheless, use of the complete wave function would lead to excessive labor in calculating our matrix elements (12), (13). For this reason, and also because we did not expect them to depend very critically on different configurations, we proceeded in three consecutive steps of approximations, labeled A_1, A_2, A_3 in Table III, by retaining successively only the largest configurations, with amplitudes kept in the same ratio as in column R. The first approximation A_1 should then—and did—give the same results as the simple model for F^{19} of Sec. III. By comparing the results of these successive steps, we also get an idea about the actual dependence on configurations, which indeed turned out not to be too critical, and gave justification to our approach. A_3 is already expected to give very accurate results, as the neglected amplitudes, except possibly c_6' and c_1, are much smaller than those which were kept. We shall also subdivide our matrix element $\langle \mathbf{\Gamma}'\cdot\mathbf{J}\rangle$ of Eq. (12c) into the parts coming from D_{qr} and from E_r (physical meaning of D_{qr}: the capturing proton turned neutron tries to go from a filled shell q to the unfilled shell r, or vice versa; of E_r: the particle goes from the unfilled shell to the unfilled shell; this matrix element should be largest because it has the best overlap), and finally get the numerical results:

$$\begin{aligned} A_1:\quad &\langle \mathbf{\Gamma}\cdot\mathbf{J}\rangle=1.061b=-5.703,\\ &\langle \mathbf{\Gamma}'\cdot\mathbf{J}\rangle_D=-0.195(b'-b'')=0.524,\\ &\langle \mathbf{\Gamma}'\cdot\mathbf{J}\rangle_E=-1.076(b'-b'')=2.892, \end{aligned} \tag{30a}$$

$$\begin{aligned} A_2:\quad &\langle \mathbf{\Gamma}\cdot\mathbf{J}\rangle=0.894b=-4.806,\\ &\langle \mathbf{\Gamma}'\cdot\mathbf{J}\rangle_D=-0.195b'+0.134b''=0.435,\\ &\langle \mathbf{\Gamma}'\cdot\mathbf{J}\rangle_E=-0.737b'+0.592b''=1.770, \end{aligned} \tag{30b}$$

$$\begin{aligned} A_3:\quad &\langle \mathbf{\Gamma}\cdot\mathbf{J}\rangle=0.879b=-4.724,\\ &\langle \mathbf{\Gamma}'\cdot\mathbf{J}\rangle_D=-0.159b'+0.187b''=0.467,\\ &\langle \mathbf{\Gamma}\cdot\mathbf{J}\rangle_E=-0.844b'+0.249b''=1.405; \end{aligned} \tag{30c}$$

the numerical values of the b are again obtained as in Eq. (27). For finding $\Delta\lambda/\lambda$, we compare the experimental capture rate of Sens[8] with Burkhardt's[15] calculated capture rate, obtained by using the shell model for F^{19} as given by Elliott and Flowers,[4] which is similar to Redlich's model. Burkhardt fits his results to the experiments by assuming a neutrino momentum $\bar{\nu}=85$ Mev/$c=0.80\,\mu$, a value which is no more than a reasonable guess,[16] and then finds $y\equiv a''/a'=1.7$. We shall assume here that $a''/a'=1.53$ as given by the universal Fermi interaction,[6] and then obtain from Burkhardt's formula and the experimental λ the value $\bar{\nu}=0.84\,\mu$, which we use in Eq. (11a) to calculate K. This is a unique way for finding $\Delta\lambda$, Eq. (10b), and for λ we take the experimental result[8] 2.54×10^5 sec^{-1}. The results for $\Delta\lambda/\lambda$ are then presented in Table IV, in our three successive approximations. Again, the Schmidt model value and the experimentally important quantity $\Delta\lambda/\Lambda$ are shown too, and again our values are much larger than those of the Schmidt model.

TABLE III. Three-particle configurations of the F^{19} unfilled-shell ground state, with amplitudes c_α, parent state quantum numbers $[T',J']$, and j^3 state normalization N', and approximations A_1, A_2, A_3 used in this work.

State	$[T',J']$	N'	c_α	R	A_1	A_2	A_3
$(\frac{5}{2})^2\,\frac{5}{2}$	12	$\sqrt{2}/3$	c_1	0.30	0	0	0
$(\frac{5}{2})^2\,\frac{3}{2}$	12		c_2	0.17	0	0	0
	01		c_2'	−0.08	0	0	0
$(\frac{3}{2})^2\,\frac{5}{2}$	12		c_3	0.12	0	0	0
	03		c_3'	0.01	0	0	0
$(\frac{3}{2})^2\,\frac{3}{2}$	12	$\sqrt{2}/3$	c_4	0.03	0	0	0
$(\frac{5}{2})^2\,\frac{1}{2}$	10		c_5	0.52	0	0.69	0.61
	01		c_5'	−0.37	0	0	−0.44
$(\frac{5}{2}\,\frac{3}{2})\,\frac{1}{2}$	11		c_6	0.01	0	0	0
	01		c_6'	−0.33	0	0	0
$(\frac{3}{2})^2\,\frac{1}{2}$	10		c_7	0.22	0	0	0
	01		c_7'	0.09	0	0	0
$(\frac{1}{2})^2\,\frac{1}{2}$	10	$\sqrt{2}/3$	c_8	0.55	1.00	0.72	0.65

[13] M. G. Redlich (private communication).

[14] A. Simon, J. H. Van der Sluis, and L. C. Biedenharn, Oak Ridge National Laboratory Report ORNL-1679, 1954 (unpublished).

[15] G. H. Burkhardt and C. A. Caine, Phys. Rev. **117**, 1375 (1960).

[16] The value of $\bar{\nu}$ used in our Eqs. (4) to obtain b, b', and b'' numerically is $0.75\,\mu$, following a reasonable guess of Primakoff.[6] However, the G are quite insensitive to changes in $\bar{\nu}$.

TABLE IV. Fractional hyperfine differences of muon capture rates in F^{19}, three successive approximations.

		$\Delta\lambda/\lambda$	$(\Delta\lambda/\lambda)_{\xi=1}$	$\Delta\lambda/\Lambda$
F^{19}	A_1	-0.62		-0.22
	A_2	-0.70	-0.42	-0.25
	A_3	-0.77		-0.28

V. DISCUSSION OF THE RESULTS

In looking first at the configuration dependence of the F^{19} results, Eq. (30), we notice that $\langle\mathbf{\Gamma}\cdot\mathbf{J}\rangle$, which represents the capture rate if there were no exclusion principle, decreases by only 17% in going from A_1 to A_3, whereas the dominant exclusion principle effect, given by $\langle\mathbf{\Gamma}'\cdot\mathbf{J}\rangle_E$, decreases by 51%. This means that if the configuration becomes more mixed, the final neutron has many more states to which to go, and the exclusion principle effect in the unfilled shell is reduced, whereas the main part of the hyperfine effect, $\langle\mathbf{\Gamma}\cdot\mathbf{J}\rangle$, has a weaker configuration dependence. If this were true more generally, then, considering our one-configuration results of Al^{27} and P^{31}, it could be said that the values of Table II, large as they are, probably still underestimate the actual hyperfine effect (and so may A_3 of Table III, although to a lesser extent).

The hyperfine effect, being essentially a spin-dependent effect, stems mainly from the outer protons in unfilled shells. As our results are so much larger than those with $\xi=1$, which means equal Pauli principle reduction in muon capture by inner and outer shell protons, we can conclude that there should be a predominance in capture by the protons in the outer regions of the nucleus. Indeed, $\Delta\lambda/\lambda$ is thus a measure of such a nuclear structure dependent effect as the relative capture rate by intranuclear and surface protons—as was recently also stated by Lubkin[17]—but for drawing quantitative conclusions on this point, assumptions on the structure of the basic weak interaction have to be made (as we did in obtaining the numerical values of Tables II and IV). Conversely, if we accept our numerical results for the nuclear matrix elements, measurements of $\Delta\lambda/\lambda$ then become a tool for finding out the relative magnitude and sign of the spin-independent and spin-dependent parts of the weak interaction Hamiltonian, as mentioned in the Introduction.

VI. DISCUSSION OF CONVERSION EFFECTS

The foregoing considerations have to be modified if there is an appreciable conversion from the upper to the lower hyperfine state, as noted by Telegdi.[2] Such a conversion will, in the region of Z around Al, be caused predominantly by Auger electron ejection,[18] and its rate is sufficient to cause the effect discussed in the work cited in footnote 2; instead of the positive curvature[1] in the logarithmic time distribution of μ-decay electrons created by the hyperfine effect [which curvature is incidentally only dependent on $(\Delta\lambda)^2$, so that, from experiment, one would not be able to tell which of the two hyperfine states absorbs faster], a negative curvature can be caused by conversion[2] (dependent on $\Delta\lambda$, whose sign could then be measured and thus more information on the relative sign of G_V, G_A be given). Indeed, this negative curvature *will* be found if the conversion rate $R>|\Delta\lambda|$ (which seems to be the case[18] in the region around Al), *and* if $\Delta\lambda<0$ (this is also true for F^{19}, Al^{27}, and P^{31} with the universal Fermi interaction, which was assumed in obtaining our numerical results). The latter condition is valid only under the assumption that the nuclear magnetic moment is positive, so that the state $F=J+\frac{1}{2}$ is the higher lying one; but this is again satisfied for the nuclides considered by us. Such a negative curvature has been found.[19]

In the light of these remarks, our discussion should then be as follows. The decay electron time distribution to be measured depends on the conversion rate R and on $\Delta\lambda/\Lambda$, the latter quantity containing the interaction constants and also the nuclear matrix elements (which can be considered as known, for example, from the present calculation). A measurement will thus allow us either to determine the structure of the weak interaction, if an independent value of R is known[18]; or, alternatively, if the universal Fermi interaction is assumed (as in this work), to make predictions on the conversion rate, and even on the density of conduction electrons near the nuclei (see below) which enters as a factor in R, thus essentially an atomic and solid state effect.

To conclude our discussion, we remark that the time-dependent decay electron rate is not the only observable quantity containing $\Delta\lambda$ and R; there is also the decay electron asymmetry from polarized muons.[20] These asymmetries are measurable, but quite small.[21] Nevertheless, we thought it worth while to state explicitly the decay electron rate and angular distribution as a function of time. The nucleus-muon wave function as a function of time is given by

$$\varphi_M(t)=a_+^M(t)\mathcal{Y}_{F=J+\frac{1}{2}}{}^M+a_-^M(t)\mathcal{Y}_{F=J-\frac{1}{2}}{}^M, \quad (31a)$$

where[22]

$$|a_+^M(0)|^2=\frac{J+\frac{1}{2}+M}{(2J+1)^2}, \quad |a_-^M(0)|^2=\frac{J+\frac{1}{2}-M}{(2J+1)^2}, \quad (31b)$$

if we assume a muon spin in the $+z$ direction. The

[17] E. Lubkin, Phys. Rev. **119**, 815 (1960).
[18] H. Primakoff,[6] reference 20; and unpublished.

[19] V. L. Telegdi, work cited in footnote 2, and private communication.
[20] This was pointed out to the author by Dr. J. Bernstein.
[21] L. B. Egorov, A. E. Ignatenko, and D. Chultém, Zhur. Eksp. i Teoret. Fiz. **37**, 1517 (1959) [translation: Soviet Phys.—JETP **37** (**10**), 1077 (1960)].
[22] H. Überall, Phys. Rev. **114**, 1640 (1959).

probability density $\sum_M\langle\varphi_M(t)|\varphi_M(t)\rangle$ and muon polarization $\sum_M\langle\varphi_M(t)|\sigma_z^\mu|\varphi_M(t)\rangle$ can then be calculated if we know the time development of $n_+^M=|a_+^M(t)|^2$, $n_-^M=|a_-^M(t)|^2$. These quantities, however, obey the equations

$$\begin{aligned} dn_+/dt&=-\Lambda_+n_+-\textstyle\sum_{M'} R_{M'}'n_+,\\ dn_-^{M'}/dt&=-\Lambda_-n_-^{M'}+R_{M'}'n_+, \end{aligned} \tag{32}$$

where $\Lambda_\pm=\lambda_\pm+\lambda_{\rm dec}$, $n_+=\sum_M n_+^M$, and $R_{M'}'=\sum_M p_M'R_{MM'}$. We denote by $R_{MM'}$ the conversion rate from the Mth magnetic sublevel of the higher hyperfine state to the M'th sublevel of the lower state, and $R_{M'}'$ is obtained by averaging $R_{MM'}$ over all initial states with a weight $p_M'=(J+M+\frac{1}{2})(J+1)^{-1}(2J+1)^{-1}$ corresponding to the muon arriving in the K shell polarized with spin in the $+z$ direction. This conversion rate can be obtained simply by a perturbation calculation using a Fermi hyperfine interaction,

$$H=(8\pi/3)\boldsymbol{\mu}_\mu\cdot\boldsymbol{\mu}_e\delta(\mathbf{r}_\mu-\mathbf{r}_e), \tag{33}$$

between the magnetic moments of the muon and of the electron to be Auger ejected (a $3s$ conduction electron in the case of Al^{27}); the result is

$$\begin{aligned} R_{MM'}=[R/J(2J+1)][&(J+\tfrac{1}{2}-M')(J+\tfrac{1}{2}+M')\delta_{M'M}\\ &+\tfrac{1}{2}(J+\tfrac{1}{2}-M')(J+\tfrac{3}{2}-M')\delta_{M'M+1}\\ &+\tfrac{1}{2}(J+\tfrac{1}{2}+M')(J+\tfrac{3}{2}+M')\delta_{M'M-1}]; \end{aligned} \tag{34a}$$

here

$$R=\frac{64\pi}{9}\frac{Z-1}{137^6}\left(\frac{m_e}{\mu}\right)^3\frac{\mu c^2}{\hbar}y\frac{J}{2J+1}, \tag{34b}$$

which represents the conversion rate $R_{MM'}$, averaged over M and summed over M'. The quantity y is defined by

$$|\psi_{3S\ \rm cond\ el}(0)|^2=\frac{1}{\pi}\frac{1}{137^3}y, \tag{35}$$

and can be found from Knight shift data[18,23] to be $y=2.7$, for Al. This gives a value of $R=8.4\times10^5$ sec^{-1}, to be compared with our value $\Delta\lambda=-3.3\times10^5$ sec^{-1} for Al^{27}. Now Eq. (32) can be solved, and, using Eq. (34), we find that if the decay electrons from muons in the K shell of spinless nuclei had a rate and angular distribution with respect to the muon spin,

$$w=\alpha_0+\alpha_1\cos\vartheta, \tag{36a}$$

then the corresponding rate and angular distribution from muons bound by similar nuclei with spin would be given by

$$w=A_0+A_1\cos\vartheta, \tag{36b}$$

$$\begin{aligned} A_0=\alpha_0\bigg\{&\frac{J+1}{2J+1}e^{-(\Lambda_++R)t}+\frac{J}{2J+1}e^{-\Lambda_-t}\\ &+\frac{J+1}{2J+1}\frac{R}{R+\Delta\lambda}[e^{-\Lambda_-t}-e^{-(\Lambda_++R)t}]\bigg\}, \end{aligned} \tag{36c}$$

$$\begin{aligned} A_1=\alpha_1\bigg\{&\frac{J+1}{2J+1}\frac{1}{3}\left(1+\frac{2}{2J+1}\right)e^{-(\Lambda_++R)t}\\ &+\frac{J}{2J+1}\frac{1}{3}\left(1-\frac{2}{2J+1}\right)e^{-\Lambda_-t}\\ &-\frac{J+1}{2J+1}\frac{1}{3}\left(1+\frac{2}{2J+1}\right)\left(1-\frac{2}{2J+1}\right)\frac{R}{R+\Delta\lambda}\\ &\times[e^{-\Lambda_-t}-e^{-(\Lambda_++R)t}]\bigg\}, \end{aligned} \tag{36d}$$

dropping a term in A_1 which oscillates too rapidly with time for being observable. Equation (36c) represents just the conversion effect discussed by Telegdi,[2] and Eq. (36d) represents for $R=0$ the hyperfine depolarization effect considered by the author[22] (for times t small compared to the muon lifetime) and by Lubkin.[17] A_0 as well as A_1 depends on R and on $\Delta\lambda$, which can thus both be measured. Equation (36) shows that for large conversion rates R, not the "slow" exponential with Λ_+ will survive at large t, but the "fast" one with Λ_-. This is really the reason for the negative curvature in $\ln A_0$. In the same case, the surviving term in the asymmetry will be (assuming, for example, that $R\gg|\Delta\lambda|$)

$$A_1\to-\alpha_1\frac{1}{3}\left(1-\frac{2}{2J+1}\right)\frac{4J+3}{(2J+1)^2}e^{-\Lambda_-t},\quad t\gg\Lambda_\pm^{-1}, \tag{37}$$

i.e., the sign of the asymmetry will be reversed; but it is doubtful to what extent this effect or even much of a detail in Eq. (36d) would be measureable.[24]

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[23] W. D. Knight, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, Inc., New York, 1956), Vol. 2.

[24] For a measurement of the time-dependent asymmetry, magnetic spin rotations may not be advisable, as the difference in magnetic moments of the two hyperfine states[22] would complicate the situation.