A Two-Dimensional Relativistic Field Theory*

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A particular two-dimensional relativistic field theory is considered. In some limit as the masses of the theory go to zero it approaches the Thirring model. By means of a formal transformation of the field operators the Hamiltonian is reduced to that of a free field. An improved perturbation expansion can be written down, necessitating only wave function renormalization, and it appears that the renormalized theory is consistent. The S matrix can be exhibited exactly, and though it leads to no physical scattering, it is not equivalent to the unit matrix. Finally the renormalized current operator is displayed as a suitable limit of products of the renormalized field operators. The form of the result clearly separates the consistency problem in quantum electrodynamics from that of the "photon mass."

THE two-dimensional (one space, one time) theory to be considered is much like the Thirring model. Since it contains nonzero masses none of the infrared problems, possibly unsolvable, of the Thirring model[1] are present. The Lagrangian is the following:

$$L=\psi_1{}^{*T}\sigma_y\left(\sigma_x\frac{\partial}{\partial x}+i\sigma_y\frac{\partial}{\partial t}+M_1\right)\psi_1 +\psi_2{}^{*T}\sigma_y\left(\sigma_x\frac{\partial}{\partial x}+i\sigma_y\frac{\partial}{\partial t}+M_2\right)\psi_2+\lambda J_1{}^{\mu}J_2{}^{\nu}\epsilon_{\mu\nu}, \quad (1)$$

$$J^{1,0}=i\psi^{*T}\sigma_y(\sigma_x,i\sigma_y)\psi,$$

$$\epsilon_{10}=1,\quad \epsilon_{01}=-1,\quad \epsilon_{11}=\epsilon_{00}=0.$$

The σ matrices are the usual 2×2 Pauli spin matrices. The ψ's are two component Fermi fields; the J^{μ}'s, their conserved currents. It is easy to see that the representations of the two dimensional homogeneous Lorentz group contained in the theory are the two one dimensional representations given by $\exp(\pm\tan^{-1}v)$ where v is the velocity given to some reference frame by the particular transformation. There are only one dimensional finite representations (irreducible) of this group. We note that the following theory could be generalized to bosons, or more than two fields similarly coupled, but we will not now consider such possibilities.

First we relate this model to the Thirring model. When both masses become zero the free particle Lagrangian yields equations of motion in which the upper and lower components of the spinors are unrelated. The total Lagrangian with interaction separates into two parts: the upper component of spinor 1 and the lower component of spinor 2 coupled to each other in the same manner as the upper and lower components in the Thirring model, and another similar part with the other components. Thus with zero masses this theory becomes equivalent to two Thirring theories. All of the results of this paper, therefore, can clearly be transcribed into statements about the Thirring model, provided the zero mass limit exists.

Next we consider the renormalization properties of the theory. Counting powers of the momentum, the over-all divergence of a connected graph is $P^{2-\frac{1}{2}n}$, with n the number of external lines. Thus one expects, at worst, a linearly divergent mass renormalization, and a logarithmically divergent wave function and charge renormalization. The perturbation expansion of the theory is straight forward. In particular, regularization would maintain the gauge invariance during renormalization. All the results we will obtain are properties of the perturbation expansion.

The special feature of the coupling to be exploited can be exhibited by studying $\epsilon_{\mu\nu}J^{\nu}$. If, as is the case, J^{ν} is a vector with zero divergence, then $\epsilon_{\mu\nu}J^{\nu}$ is a vector with zero curl, and therefore one can write

$$\epsilon_{\mu\nu}J^{\nu}=\partial_{\mu}O=\partial_{\mu}\left[\int^{x}dx'^{\mu'}\epsilon_{\mu'\nu}J^{\nu}\right]. \quad (2)$$

At first glance, then, it appears as though the total interaction in Eq. (1) disappears upon integration by parts, but since O is a nonlocal operator one cannot neglect the surface term. It is now clear where the two-dimensional property of the theory is used. Can one find such local operators with zero curl, that are not gradients of local operators, in four dimensions?

If ψ_2 were an unquantized field one could immediately write down a solution for the quantized field ψ_1.

$$\psi_1(x)=\tilde{\psi}_1(x)\exp\left(i\lambda\int^{x}dx'^{\mu}\epsilon_{\mu\nu}J_2{}^{\nu}\right), \quad (3)$$

with $\tilde{\psi}_1$ a solution of the free field equation. One is led to consider the following transformation when both fields are quantized:

$$\begin{aligned}\psi_1&=\tilde{\psi}_1\exp\left[i\frac{\lambda}{2}\left(\int_{-\infty}^{x}-\int_{x}^{\infty}\right)\epsilon_{\mu\nu}J_2{}^{\nu}dx'^{\mu}\right],\\ \psi_2&=\tilde{\psi}_2\exp\left[i\frac{\lambda}{2}\left(\int_{-\infty}^{x}-\int_{x}^{\infty}\right)\epsilon_{\mu\nu}J_1{}^{\mu}dx'^{\nu}\right].\end{aligned} \quad (4)$$

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[1] W. E. Thirring, Ann. Phys. 3, 91 (1958).

The integrals are understood to be along a constant time surface (the operators may be considered in the Schrödinger representation). The fields $\tilde{\psi}$ have the same commutation properties as the fields ψ.

$$\{\psi,\psi^{*T}\}=\{\tilde{\psi},\tilde{\psi}^{*T}\}=\delta. \tag{5}$$

Note that in these few paragraphs we are dealing with expressions such as (4) which are not really defined. However, as will be seen, the formal manipulations lead to correct conclusions. The Hamiltonian may also be expressed in terms of the fields $\tilde{\psi}$. It results that the interacting Hamiltonian expressed in terms of the fields $\tilde{\psi}$ has the form of the free Hamiltonian.

$$H(\psi)=H_0(\tilde{\psi}). \tag{6}$$

This amazing transformation is possible for any theory in terms of "in" or "out" fields. We deduce that there are no energy shifts (or mass renormalizations), and that all the states of the theory are known in terms of the $\tilde{\psi}$ fields.

As an exercise we consider the two-point function.

$$G_1(x,y)=\langle 0|\psi_1{}^{*T}(x)\psi_1(y)|0\rangle. \tag{7}$$

We need only consider points x and y along a constant time surface, and can express the ψ fields in terms of the $\tilde{\psi}$ fields by (4).

$$G_1(x_1y)=\left\langle 0\left|\tilde{\psi}_1{}^{*T}(x)\times\exp\left(i\lambda\int_x^y \tilde{J}_2{}^{\nu}\epsilon_{\mu\nu}dx'^{\mu}\right)\tilde{\psi}_1(y)\right|0\right\rangle_{(0)}. \tag{8}$$

The subscript (0) reminds us that the operators and states may now be treated as free.

$$G_1(x,y)=\langle 0|\tilde{\psi}_1{}^{*T}(x)\tilde{\psi}_1(y)|\rangle_{(0)}\times\left\langle 0\left|\exp\left(i\lambda\int_x^y \tilde{J}_2{}^{\nu}\epsilon_{\mu\nu}dx'^{\mu}\right)\right|0\right\rangle_{(0)}. \tag{9}$$

There follows the symbolic expression for the wave function renormalization constant

$$Z_2=\lim_{a\to\infty}\left\langle 0\left|\exp\left(i\lambda\int_0^a \tilde{J}_2{}^{\nu}\epsilon_{\mu\nu}dx'^{\mu}\right)\right|0\right\rangle_{(0)} \tag{10}$$

and from (9) also, as predicted, the lack of mass renormalization. The second term in the product in Eq. (9) may be expanded in perturbation theory. By well known juggling it can be expressed in terms of a linked cluster expansion.

$$G_1(x,y)=G_1(x,y)_{(0)}\exp\left(\sum_{n=1}^{\infty}\frac{\lambda^{2n}}{2n}M_{2n}(x,y)\right). \tag{11}$$

Each $M_{2n}(x,y)$ represents the corrections to the propagator of particle 1 due to closed loops of particle type 2 that interact $2n$ times with the 1 line. The M_{2n} are logarithmically divergent; the renormalization is explicitly multiplicative in terms of the improved perturbation theory in the exponent. It appears that: (a) The wave function renormalization actually diverges as a power, as it is the exponent of logarithmically divergent constants.

$$Z_2\sim 1/\Lambda^{\alpha} \tag{12}$$

Λ is the regularization mass, or some other cutoff parameter. No other infinite renormalizations occur in the theory. (b) The renormalized series converges. There is only one "diagram" to each order in λ^2 in the expansion of the exponent of Eq. (11). (c) The nature of the singularity as x approaches y is:

$$G(x,y)\sim G(x,y)_{(0)}\frac{1}{|x-y|^{\alpha}}\sim\frac{1}{x-y}\frac{1}{|x-y|^{\alpha}},\qquad G(p)\sim\int\frac{d\kappa}{\kappa}\frac{\kappa^{\alpha}}{\sigma\cdot p+\kappa}. \tag{13}$$

$G(p)$ is the Fourier transform of the time-ordered two-fold function. The singularity is clearly worse than any order of perturbation theory.

Expressions similar to (8) may be derived for products of arbitrarily many field operators. Analytic continuation from a constant time surface determines only the two-fold function in two dimensions. However, the expressions can be generalized to any space-like surface; such generalizations formally satisfy the field equations. We now pass to a study of the theory in terms of diagrams, that complements the understanding achieved by studying the transformation (4).

Each diagram consists of lines that carry particle number through them and do not terminate, but enter and leave the diagram, and closed loops. All interactions couple currents of field 1 with currents of field 2. Consider a closed loop:

$$\epsilon_{\alpha_1\mu_1}\cdots\epsilon_{\alpha_n\mu_n}T\langle 0|J^{\mu_1}(x_1)\cdots J^{\mu_n}(x_n)|0\rangle_{(0)}{}^{\text{linked}}=\partial_{\alpha_1}{}^1\cdots\partial_{\alpha_n}{}^n\left[\int^{x_1}d\tilde{x}_1{}^{\beta_1}\cdots\times\int^{x_n}d\tilde{x}_n{}^{\beta_n}\cdot\epsilon_{\beta_1\gamma_1}\cdots\epsilon_{\beta_n\gamma_n}\right]\times T\langle 0|J^{\nu_1}(\tilde{x}_1)\cdots J^{\nu_n}(\tilde{x}_n)|0\rangle_{(0)}{}^{\text{linked}}. \tag{14}$$

Couplings to a closed loop are essentially derivative couplings. Three rules deduced from this greatly simplify calculations:

(1) No diagrams contribute in which two closed loops interact with each other, i.e., touch at a vertex. Such a rule, of course, only holds if all diagrams of a given class are considered at once, as the cancellation between diagrams is important.

(2) Closed loops that interact only with a single current line serve to multiply the diagram by a factor depending only on the difference $x-y$ of the end points of the line. More generally, all closed loops of type 1 (2)

multiply the diagram by a factor depending only on the endpoints of lines of type 2 (1).

Rules 1 and 2 essentially yield (11). It is convenient to have both these rules, and expressions of type (8) in general considerations. The factoring and exponentiation as in Eq. (9) and its generalizations is clumsy to exhibit by combinatorics alone.

(3) In calculating the S matrix no closed loops need be considered; as the current operator has zero divergence, calculated in an external line on the mass shell, between true particle spinors.

The diagrams that contribute to the S matrix, not having closed loops, can be shown to lead to the classical result for the scattering of type 1 particles by particles of type 2. Referring to Eq. (3) for example, one sees that the S matrix receives a factor $e^{i\lambda}$ every time a 1 particle moving in the positive x direction passes a 2 particle moving in the negative x direction (only the relative motion matters) and a factor of $e^{-i\lambda}$ if the sense of motion or charge changes. There is no real scattering or production only a phase change. This is completely nonmeasurable even with the addition of more interactions provided they are gauge invariant. Such nonmeasurable energy-independent phase shifts have previously been claimed for the Thirring model.

One further source of investigation is the evaluation of vacuum matrix elements. Consider the four-point function $\langle 0|\psi_1{}^{*T}(x)\psi_2{}^{*T}(y)\psi_2(2)\psi_1(w)|0\rangle$, with x, y, z, and w points on a single space-like surface. By a factorization of the type leading to Eq. (9) one is led to consider the matrix element

$$\left\langle 0\left|\tilde{\psi}_1{}^{*T}(x)\exp\left(i\lambda\int_y^z dx'^{\mu}\epsilon_{\nu\mu}\tilde{J}_1{}^{\nu}\right)\tilde{\psi}_1(w)\right|0\right\rangle_{(0)}. \quad (15)$$

This can be further factored

$$\frac{G(y,z)}{G(y,z)_{(0)}}\cdot\left\langle 0\left|\tilde{\psi}_1{}^{*T}(x)\right.\right.$$
$$\left.\left.\times\exp\left(i\lambda\int_y^z dx'^{\mu}\epsilon_{\nu\mu}\tilde{J}_1{}^{\nu}\right)\tilde{\psi}_1(w)\right|0\right\rangle_{(0)}^{\text{linked}}. \quad (16)$$

The meaning of the term "linked" is that in expanding the matrix element only connected contractions are to be considered. The problem of evaluating this linked matrix element can be converted to an integral equation with a known kernel. In the case of the Thirring model this integral equation can be solved analytically. Once this matrix element is evaluated, say by solving the integral equation, $G(x,y)$ can be recovered by constructing the current operator with a limiting process on the field operators ψ and ψ^{*T}. The very interesting problem of thus constructing the current operator will next be considered.

The problem is that of constructing the renormalized current operator in terms of the renormalized field operators. One considers the operator

$$J^{\mu}(x,y)=i\psi^{*T}(x)\sigma_y\begin{bmatrix}\sigma_x\\ i\sigma_y\end{bmatrix}\psi(y) \quad (17)$$

and asks how the current operator $J^{\mu}(y)$ may be obtained from $J^{\mu}(x,y)$ as some limit $x\rightarrow y$. It is easiest to consider again a factorization such as Eq. (9) of a matrix element containing $J^{\mu}(x,y)$. In the course of such a factorization the operators $\tilde{\psi}^{*T}(x)$ and $\tilde{\psi}(y)$ appear in one factor, and $\exp(i\lambda\int_x^y dx'^{\mu}\epsilon_{\mu\nu}\tilde{J}_2{}^{\nu})$ appears in the other. As x approaches y in the first factor one must subtract out the singularity $\langle 0|J_1{}^{\mu}(x,y)|0\rangle_{(0)}$, i.e., the vacuum charge. In the second factor the exponential must disappear entirely, as there would be no such factor if the original matrix element contained $J^{\mu}(y)$. The following expression suffices:

$$J_1{}^{\mu}(y)=\lim_{\substack{x\rightarrow y,\\ \text{spacelike}}}\left\{\frac{J_1{}^{\mu}(x,y)-\langle 0|J_1{}^{\mu}(x,y)|0\rangle}{G(x,y)/G(x,y)_{(0)}}\right.$$
$$\left.-\langle 0|J_1{}^{\mu}(x,y)|0\rangle_{(0)}(+i\lambda)(y-x)^{\alpha}\epsilon_{\alpha\beta}J_2{}^{\beta}(y)\right\}. \quad (18)$$

In the first term the vacuum current is removed. The denominator divides out the vacuum expectation value of the exponential, that as pointed out above must be eliminated. It corresponds to multiplication by z_2, now equal to zero. The second term subtracts the contribution of the term $\sim(y-x)$ in the expansion of the exponential that unfortunately contributes since $G(x,y)_{(0)}\sim 1/(x-y)$. The limit, however, is now well defined. Since $J_2(y)$ appears on the right side of the equation; when this equation and a similar one for $J_2(y)$ are solved together the final expression for $J_1(y)$ would involve both $J_1(x,y)$ and $J_2(x,y)$. This clearly indicates the problems that may arise in considering operator products such as the current.

In particular, Eq. (18) suggests that even if the renormalization constants were finite $J_1{}^{\mu}$ and ψ_2 would not commute on a space-like surface. Thus in quantum electrodynamics, a much more complex theory, the fact that $[A^{\mu},J^{\mu}]\neq 0$ on a space-like surface (the so-called photon mass term) is not in itself inconsistent with finite renormalization. Commutators of an object such as J^{μ} cannot be deduced from commutators of the ψ's directly.

Finally we note that to define differential equations of motion for the renormalized field operator[2] a suitable definition of the formal product $\psi_1(x)J_2{}^{\mu}(x)$ must first be found. This is being further investigated.

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[2] J. G. Valatin, Proc. Roy. Soc. (London) **A226**, 254 (1954); R. Haag and G. Luzzatto, Nuovo cimento **13**, 415 (1959).