Symmetry of the ND^{-1} Solutions for Coupled Scattering Amplitudes

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We prove that the ND^{-1} matrix solutions for coupled scattering amplitudes are symmetric provided the given discontinuity of the scattering matrix across the unphysical cut is symmetric.

A USEFUL method developed by Omnes[1] and Chew and Mandelstam[2] to construct single-channel partial-wave amplitudes satisfying the requirements of analyticity and unitarity has been extended by Mandelstam[3] and Bjorken[4] to the multiple-channel case. The method consists in writing the scattering matrix G as a product of a matrix N which contains the unphysical (dynamical) singularities of G and the inverse of a matrix D which contains the physical (unitarity) singularities of G. A well-known requirement from time-reversal invariance is that G be a symmetric matrix. We prove that G, written in the form

$$G=ND^{-1}, \tag{1}$$

satisfies this condition, provided only that the given discontinuity of G across the unphysical cuts is a symmetric matrix.

By assumption, the matrices N and D satisfy dispersion relations

$$N=\frac{1}{\pi}\int_{-\infty}^{s_L}\frac{\mathrm{Im}N(s')}{s'-s}ds', \tag{2}$$

$$D=1+\frac{1}{\pi}\int^{+\infty}\frac{\mathrm{Im}D(s')}{s'-s}ds', \tag{3}$$

where s_L and s_R are the end points of the unphysical $(-\infty<s<s_L)$ and the physical $(s_R<s<\infty)$ cuts. For simplicity we restrict the unphysical singularities to the real axis. From Eqs. (1)–(3) it follows that

$$\begin{aligned}\mathrm{Im}N&=(\mathrm{Im}G)D \quad \text{when} \quad s<s_L\\ &=0 \quad \text{when} \quad s_L<s,\\ \mathrm{Im}D&=\mathrm{Im}(G^{-1})N \quad \text{when} \quad s_R<s\\ &=0 \quad \text{when} \quad s<s_R.\end{aligned} \tag{4}$$

The unitarity of the S matrix requires that

$$\mathrm{Im}G^{-1}=-\rho \quad \text{when} \quad s_R<s, \tag{5}$$

where ρ is the diagonal density of states matrix, and is therefore known. On the other hand, $\mathrm{Im}G$ along the unphysical cut depends on the form of the interaction. The only assumption we make is that $\mathrm{Im}G$ is a symmetric matrix.

Consider now

$$G-G^T=ND^{-1}-(D^{-1})^TN^T, \tag{6}$$

where the superscript T indicates the transposed matrix. Multiplying Eq. (6) on the right-hand side by D and on the left-hand side by D^T, taking the imaginary part, and substituting Eqs. (4) and (5), we obtain

$$\begin{aligned}&\mathrm{Im}[D^T(G-G^T)D]\\ &\quad=D^T(\mathrm{Im}G)D-D^T(\mathrm{Im}G)^TD \quad \text{when} \quad s<s_L\\ &\quad=-N^T\rho^TN+N^T\rho N \quad \text{when} \quad s_R<s.\end{aligned} \tag{7}$$

Since ρ is diagonal and $\mathrm{Im}G=(\mathrm{Im}G)^T$, the right-hand side of Eq. (7) is everywhere zero. It follows that the function $D^T(G-G^T)D$ is analytic everywhere in the s plane, and vanishes at ∞, Eqs. (2)–(3). Hence it is identically zero and we obtain

$$G=G^T. \tag{8}$$

If D has zeros, consideration of the matrix $(\det D)G$ leads to the same conclusion, Eq. (8).

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