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Coulomb Scattering of Polarized Electrons*

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An approximate analytical expression for the spin-dependent cross section for Coulomb scattering, valid to order $(\alpha Z)^4$, is developed. The development is based on a modification of the Sommerfeld-Maue wave function. The $(\alpha Z)^2$ and $(\alpha Z)^3$ terms in the resulting cross section are identical with the corresponding terms found by the second Born approximation. The $(\alpha Z)^4$ term is verified by analytically summing the corresponding term in the Mott series. Graphs, comparing the approximate analytical expressions for both the cross section and asymmetry function with exact numerical results, are included.

I. INTRODUCTION

MOTT,[1,2] in his fundamental papers, gives the Coulomb scattering wave function as an infinite series. The asymptotic form of this function determines the scattering cross section exactly. Unfortunately, the resulting series for the cross section has not been summed analytically, and numerical techniques must be employed to obtain exact results. However, Mott derives an approximate formula for the cross section in which only the dominant term in a series in αZ is given.

McKinley and Feshbach[3] expand the summands of the Mott series in powers of αZ, obtain the αZ correction to the Mott formula in closed form, and thus find a rather simple expression for the cross section, containing terms to order $(\alpha Z)^3$. Numerical results for the $(\alpha Z)^4$ and $(\alpha Z)^5$ terms in the cross section are given, and numerical values for the cross section, useful for all but the heaviest elements, are computed.

An analytical result identical with the $(\alpha Z)^3$ cross section of McKinley and Feshbach was obtained in the second Born approximation by Dalitz.[4] Lewis[5] extended the Born approximation calculations to include scattering from a number of potentials. Mitter and Urban[6] considered the problem of Coulomb scattering in the third Born approximation; their results, which are given in integral form, do not seem to be in complete agreement with those presented in Sec. III.

More recently, Doggett and Spencer[7] have summed the Mott series numerically, and computed values for the spin-independent part of the cross section for a number of elements and a wide range of energies.

A general expression for the cross section for scattering of electrons from a state of definite polarization to another such state is given by Tolhoek.[8] With the aid of Tolhoek's work the spin dependent cross section can be obtained as an infinite series which is amenable to numerical calculation or to approximate analytical treatment.

In particular, exact numerical calculations for the scattering asymmetry function are presented by Sherman,[9] and numerical values of the cross section for scattering of longitudinally polarized electrons have been given by Tassie.[10] Analytical results for the spin dependent cross section, corresponding to the second Born approximation, are derived by Gürsey.[11]

Using a modification of the Sommerfeld-Maue wave function, valid to order $(\alpha Z)^2$, Johnson and Mullin[12] recently derived an approximate analytical expression for the spin-dependent cross section which agrees with that obtained by Gürsey.

In Sec. II a further modification of the Sommerfeld-

* Supported in part by the U. S. Atomic Energy Commission.
[1] N. F. Mott, Proc. Roy. Soc. (London) **A124**, 425 (1929).
[2] N. F. Mott, Proc. Roy. Soc. (London) **A135**, 429 (1932).
[3] W. A. McKinley and H. Feshbach, Phys. Rev. **74**, 1759 (1948).
[4] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).
[5] R. R. Lewis, Phys. Rev. **102**, 537 (1956).
[6] H. Mitter and P. Urban, Acta Phys. Austriaca **7**, 311 (1953).
[7] J. A. Doggett and L. V. Spencer, Phys. Rev. **103**, 1597 (1956).
[8] H. A. Tolhoek, Revs. Modern Phys. **28**, 277 (1956).
[9] Noah Sherman, Phys. Rev. **103**, 1601 (1956).
[10] L. J. Tassie, Phys. Rev. **107**, 1452 (1957).
[11] F. Gürsey, Phys. Rev. **107**, 1734 (1957).
[12] W. R. Johnson and C. J. Mullin, Phys. Rev. **119**, 1270 (1960).

Maue wave function valid to order $(\alpha Z)^3$ is presented. It is possible to obtain from this wave function an analytical expression for the spin-dependent cross section valid to order $(\alpha Z)^4$.

A general expression for the spin-dependent cross section, equivalent to that of Tolhoek, but more suitable for our analysis, is presented in Sec. III. This general expression together with the approximate wave function of Sec. II leads to a closed form expression for the spin-dependent cross section. The analytical results are compared graphically with the numerical results of Doggett and Spencer, and those of Sherman.

The validity of the cross section obtained by use of the modified Sommerfeld-Maue wave function has been verified by an independent calculation. In this calculation, which is presented in the Appendix, those terms in the Mott series which contribute to the $(\alpha Z)^4$ part of the cross section are summed in analytical form. These results are found to be identical with those derived by means of the modified Sommerfeld-Maue wave function.

II. THE MODIFIED SOMMERFELD-MAUE SCATTERING WAVE FUNCTION

We seek a solution of the Dirac equation

$$(H_0-W-\alpha Z/r)\psi(\mathbf{r})=0, \tag{1}$$

where

$$H_0=\frac{1}{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta m,$$

which behaves asymptotically like a plane wave and a spherical outgoing wave. Such a wave function is described, accurately to order αZ, by the Sommerfeld-Maue wave function[13–15]

$$\psi_{\mathrm{SM}}=Ne^{i\mathbf{p}_1\cdot\mathbf{r}}\left(1-\frac{i}{2W}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right)\times{}_1F_1(i\nu;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})u(\mathbf{p}_1), \tag{2}$$

with

$$N=\Gamma(1-i\nu)e^{\nu\pi/2}.$$

In the above formulas $\mathbf{p}_1$ represents the incident momentum vector, W the particle energy, and $\nu=\alpha ZW/p$. ${}_1F_1(a;b;x)$ is the confluent hypergeometric function. From the asymptotic form of the hypergeometric function it is easily seen that

$$\psi_{\mathrm{SM}}\underset{r\to\infty}{\longrightarrow}\exp[i\mathbf{p}_1\cdot\mathbf{r}-i\nu\ln 2pr\sin^2(\theta/2)]u(\mathbf{p}_1)+[H_0(\mathbf{p}_2)+W]T_{\mathrm{SM}}u(\mathbf{p}_1)\frac{\exp(ip_1r+i\nu\ln 2pr)}{r}, \tag{3}$$

where

$$T_{\mathrm{SM}}=\frac{\alpha Z}{4p_1^2\sin^2(\theta/2)}\frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)}\exp[i\nu\ln\sin^2(\theta/2)]. \tag{4}$$

Here $\mathbf{p}_2=p\mathbf{r}/r$, $\cos\theta=\mathbf{p}_1\cdot\mathbf{p}_2/p^2$, and $H_0(\mathbf{p}_2)=\boldsymbol{\alpha}\cdot\mathbf{p}_2+\beta m$.

Comparing Eq. (3) with the asymptotic form of the exact Coulomb wave function given by Mott,[16] one sees immediately that ψ_{SM} represents the incident plane wave exactly. It is therefore necessary to modify the spherical outgoing wave only.

Writing $\psi=\psi_{\mathrm{SM}}+\chi$, Eq. (1) becomes an inhomogeneous equation for χ:

$$(H_0-W-\alpha Z/r)\chi=-R(\mathbf{r}), \tag{5}$$

with

$$R(\mathbf{r})=N\frac{i\alpha Z}{2Wr}e^{i\mathbf{p}_1\cdot\mathbf{r}}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\,{}_1F_1(i\nu;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})u(\mathbf{p}_1). \tag{6}$$

The solution to Eq. (5) subject to the boundary condition that χ represents, asymptotically, a spherical outgoing wave is

$$\chi(\mathbf{r})=\int G(\mathbf{r},\mathbf{r}')R(\mathbf{r}')d\mathbf{r}', \tag{7}$$

where $G(\mathbf{r},\mathbf{r}')$ is the Green's function[17] for the operator $(H_0-W-\alpha Z/r)$. Since we want χ to be correct to third order in αZ, it is sufficient to replace $G(\mathbf{r},\mathbf{r}')$ by $G_1(\mathbf{r},\mathbf{r}')$, a Green's function accurate to first order in αZ. Such a Green's function has been constructed by Meixner.[18] Using the asymptotic form of $G_1(\mathbf{r},\mathbf{r}')$ given by Meixner, we find

$$\chi(\mathbf{r})\underset{r\to\infty}{\longrightarrow}[H_0(\mathbf{p}_2)+W]\tilde{T}u(\mathbf{p}_1)\frac{e^{ipr+i\nu\ln 2pr}}{r}, \tag{8}$$

where

$$\tilde{T}=-i\frac{N^2\alpha Z}{8\pi W}\int d\mathbf{r}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{r}\times\left[\left(1+\frac{i}{2W}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\right){}_1F_1(i\nu;1;ip_2r+i\mathbf{p}_2\cdot\mathbf{r})\right]\times[\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\,{}_1F_1(i\nu;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})], \tag{9}$$

with $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2$.

To calculate $\tilde{T}$ to third order in αZ, we represent the hypergeometric functions in Taylor series with respect to their first arguments and make use of the relations

$$\begin{aligned}\boldsymbol{\nabla}\,{}_1F_1(i\nu;1;ip_2r+i\mathbf{p}_2\cdot\mathbf{r})&=(p/r)\boldsymbol{\nabla}_{\mathbf{p}_2}\,{}_1F_1(i\nu;1;ip_2r+i\mathbf{p}_2\cdot\mathbf{r}),\\ \boldsymbol{\nabla}\,{}_1F_1(i\nu;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})&=-(p/r)\boldsymbol{\nabla}_{\mathbf{p}_1}\,{}_1F_1(i\nu;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r}).\end{aligned} \tag{10}$$

[13] A. Sommerfeld and A. W. Maue, Ann. Physik **22**, 629 (1935).
[14] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954).
[15] A. Sommerfeld, *Atombau und Spektrallinien* (F. Vieweg und Sohn, Braunschweig, 1939), Vol. II, p. 408.
[16] N. F. Mott, reference 2, p. 440.
[17] The Green's function used here is the negative of that used in reference 12.
[18] J. Meixner, Ann. Physik **29**, 97 (1937).

$\tilde{T}$ may be written then as $\tilde{T}=T^{(2)}+T_1^{(3)}+T_2^{(3)}$, where

$$T^{(2)}=-(\alpha Z)^2\frac{N^2}{8\pi}\int_0^\infty d\mu\; \boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_1}\lim_{a\to 0}\frac{d}{da}A(\mathbf{p}_1,\mathbf{q},a,\mu), \quad (11a)$$

$$T_1^{(3)}=-(\alpha Z)^3\frac{iN^2}{16\pi}\frac{W}{p}\times\int_0^\infty d\mu\; \boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_1}\lim_{a\to 0}\frac{d^2}{da^2}A(\mathbf{p}_1,\mathbf{q},a,\mu), \quad (11b)$$

$$T_2^{(3)}=-(\alpha Z)^3\frac{iN^2}{8\pi}\frac{W}{p}\times\left[\int_0^\infty d\mu\; \boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_1}\lim_{a\to 0,\, b\to 0}\frac{d^2}{da\,db}B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q},a,b,\mu)\right.$$
$$\left.+i\frac{p}{2W}\int_0^\infty d\nu\int_\nu^\infty d\mu\; \boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_2}\boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_1}\lim_{a\to 0,\, b\to 0}\frac{d^2}{da\,db}\times B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q},a,b,\mu)\right]. \quad (11c)$$

The functions A and B occurring in Eq. (11) are spatial integrals involving the hypergeometric functions which have been evaluated by Sommerfeld.[19] In particular:

$$A=\int\frac{d\mathbf{r}}{r}e^{i\mathbf{q}\cdot\mathbf{r}-\mu r}\,{}_1F_1(a;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})=\frac{4\pi}{q^2+\mu^2}\left(\frac{q^2+\mu^2}{q^2+\mu^2-2\mathbf{p}_1\cdot\mathbf{q}-2ip_1\mu}\right)^a, \quad (12a)$$

$$B=\int\frac{d\mathbf{r}}{r}e^{i\mathbf{q}\cdot\mathbf{r}-\mu r}\,{}_1F_1(a;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})\times{}_1F_1(b;1;ip_2r+i\mathbf{p}_2\cdot\mathbf{r})$$
$$=\frac{4\pi}{q^2+\mu^2}\left(\frac{q^2+\mu^2}{q^2+\mu^2-2\mathbf{p}_1\cdot\mathbf{q}-2ip_1\mu}\right)^a\times\left(\frac{q^2+\mu^2}{q^2+\mu^2+2\mathbf{p}_2\cdot\mathbf{q}-2ip_2\mu}\right)^b{}_2F_1(a,b;1;y), \quad (12b)$$

with

$$y=2\frac{(\mu^2+q^2)(p_1p_2+\mathbf{p}_1\cdot\mathbf{p}_2)-2(\mathbf{p}_2\cdot\mathbf{q}-ip_2\mu)(\mathbf{p}_1\cdot\mathbf{q}+ip_1\mu)}{(q^2+\mu^2+2\mathbf{p}_2\cdot\mathbf{q}-2ip_2\mu)(q^2+\mu^2-2\mathbf{p}_1\cdot\mathbf{q}-2ip_1\mu)}.$$

Making use of the Dirac equation, and the fact that $\tilde{T}$ occurs in Eq. (8) bracketed by the operator $H_0(\mathbf{p}_2)+W$ and the plane wave spinor $u(\mathbf{p}_1)$, we find

$$\boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_1}\lim_{a\to 0}\frac{dA}{da}=\frac{8\pi i(W-\beta m)}{p(q^2+\mu^2)(\mu-2ip)}, \quad (13a)$$

$$\boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_1}\lim_{a\to 0}\frac{d^2A}{da^2}=\frac{16\pi i(W-\beta m)}{p(q^2+\mu^2)(\mu-2ip)}\ln\frac{q^2+\mu^2}{\mu(\mu-2ip)}, \quad (13b)$$

$$\boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_1}\lim_{a\to 0,\, b\to 0}\frac{d^2B}{da\,db}=\frac{8\pi i(W-\beta m)}{p(q^2+\mu^2)(\mu-2ip)}\ln\frac{q^2+\mu^2}{\mu(\mu-2ip)}, \quad (13c)$$

$$\boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_2}\boldsymbol{\alpha}\cdot\nabla_{\mathbf{p}_1}\lim_{a\to 0,\, b\to 0}\frac{d^2B}{da\,db}=\frac{16\pi m(W\beta-m)}{p^2}\left[\frac{2}{\mu(\mu^2+q^2)(\mu-2ip)}+\frac{1}{q^2(\mu-2ip)^2}\ln\frac{\mu^2+q^2}{\mu^2}\right]. \quad (13d)$$

Combining the results of Eqs. (11) and (13), we obtain the following relations:

$$T^{(2)}=-i(\alpha Z)^2\frac{N^2}{pq^2}(W-\beta m)I_1, \quad (14a)$$

$$T_1^{(3)}=(\alpha Z)^3\frac{N^2}{p^2q^2}W(W-\beta m)I_2, \quad (14b)$$

$$T_2^{(3)}=T_1^{(3)}+(\alpha Z)^3\frac{N^2}{p^2q^2}m(W\beta-m)I_3. \quad (14c)$$

The integrals I_1, I_2, and I_3 occurring in Eqs. (14) can be evaluated explicitly. Setting $x=\sin(\theta/2)$, we find

$$I_1=\int_0^\infty d\mu\frac{q^2}{(\mu^2+q^2)(\mu-2ip)}=\frac{x^2}{1-x^2}\left\{i\frac{\pi(1-x)}{2x}-\ln x\right\}, \quad (15a)$$

$$I_2=\int_0^\infty d\mu\frac{q^2}{(\mu^2+q^2)(\mu-2ip)}\ln\frac{\mu^2+q^2}{\mu(\mu-2ip)}=i\pi\frac{x^2}{1-x^2}\left\{\frac{\ln 2}{x}-\ln(1+x)-\frac{(1-x)}{2x}\ln\frac{1+x}{x}\right\}$$
$$+\frac{1}{2}\frac{x^2}{1-x^2}\left\{\frac{1+x}{2x}\mathcal{L}_2(1-x^2)-\ln^2x-\frac{2}{x}\mathcal{L}_2(1-x)-\frac{\pi^2}{4}\frac{(1-x)}{x}\right\}, \quad (15b)$$

$$I_3=\int_0^\infty d\nu\int_\nu^\infty d\mu\left[\frac{2q^2}{\mu(\mu-2ip)(\mu^2+q^2)}+\frac{1}{(\mu-2ip)^2}\ln\frac{\mu^2+q^2}{\mu^2}\right]$$
$$=\int_0^\infty\frac{d\nu}{(\nu-2ip)}\ln\frac{\nu^2+q^2}{\nu^2}$$
$$=\{i\pi\ln(1+x)-\tfrac{1}{2}\mathcal{L}_2(1-x^2)+\tfrac{1}{12}\pi^2\}. \quad (15c)$$

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[19] A. Sommerfeld, reference 15, p. 503. It should be noted that A can be obtained from B by setting $b=0$.

In Eqs. (15), $\mathcal{L}_2(x)$ denotes Euler's dilogarithm[20,21] defined by

$$\mathcal{L}_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}=-\int_0^x\frac{\ln(1-\eta)}{\eta}d\eta. \quad (16)$$

From Eqs. (3) and (8) it is clear that the third order Coulomb scattering wave function may be represented asymptotically by Eq. (3) with T_{SM} replaced by

$$T=T_{\mathrm{SM}}+T^{(2)}+T_1^{(3)}+T_2^{(3)}. \quad (17)$$

One should note that the only matrices which occur in T are I and $\beta=\gamma_4$.

III. THE FOURTH-ORDER COULOMB SCATTERING CROSS SECTION

The differential cross section for scattering from a state of momentum $\mathbf{p}_1$, spin direction $\boldsymbol{\zeta}_1$ (in the electron's rest system), to a state of momentum $\mathbf{p}_2$, spin direction $\boldsymbol{\zeta}_2$ (again in the electron's rest system), is given by

$$\begin{aligned}\sigma(\mathbf{p}_1,\boldsymbol{\zeta}_1;\,\mathbf{p}_2,\boldsymbol{\zeta}_2)&=d\sigma/d\Omega=\tfrac{1}{4}\,\mathrm{Tr}[T^{\dagger}(1-is_2\gamma_5)\\&\quad\times(m-ip_2)\gamma_4T(1-is_1\gamma_5)(m-ip_1)\gamma_4],\end{aligned} \quad (18)$$

where T is determined from the asymptotic form of the scattering wave function as shown in Sec. II. In Eq. (18) we have adopted the notation: $q=q_\mu\gamma_\mu$, $p_1=(\mathbf{p}_1,iW_1)$, and

$$s_1=\left(\boldsymbol{\zeta}_1+\frac{\mathbf{p}_1\cdot\boldsymbol{\zeta}_1}{m(W_1+m)},\ i\frac{\mathbf{p}_1\cdot\boldsymbol{\zeta}_1}{m}\right).$$

Making use of the fact that T may be written in the form $T=a+\gamma_4 b$ and carrying out the trace in Eq. (18), we find

$$\begin{aligned}\sigma(\mathbf{p}_1,\boldsymbol{\zeta}_1;\,\mathbf{p}_2,\boldsymbol{\zeta}_2)&=\tfrac{1}{2}I(\theta)(1+\boldsymbol{\zeta}_1\cdot\boldsymbol{\zeta}_2)-\tfrac{1}{2}D(\theta)(\mathbf{n}\cdot\boldsymbol{\zeta}_1+\mathbf{n}\cdot\boldsymbol{\zeta}_2)\\&\quad+\tfrac{1}{2}F(\theta)\mathbf{n}\cdot(\boldsymbol{\zeta}_2\times\boldsymbol{\zeta}_1)-\tfrac{1}{2}G(\theta)(\mathbf{n}\times\boldsymbol{\zeta}_1)\cdot(\mathbf{n}\times\boldsymbol{\zeta}_2),\end{aligned} \quad (19)$$

where $\mathbf{n}=(\mathbf{p}_2\times\mathbf{p}_1)/(|\mathbf{p}_2\times\mathbf{p}_1|)$ is the unit normal to the plane of scattering. The quantities $I(\theta)$, $D(\theta)$, $F(\theta)$, and $G(\theta)$ occurring in Eq. (19) are functions of θ, independent of spin, given by

$$\begin{aligned}I(\theta)&=4\{[W^2-p^2\sin^2(\theta/2)]|a|^2\\&\quad+[W^2-p^2\cos^2(\theta/2)]|b|^2+2Wm\,\mathrm{Re}(ab^*)\},\end{aligned} \quad (20a)$$

$$D(\theta)=4p^2\sin\theta\,\mathrm{Im}(ab^*), \quad (20b)$$

$$\begin{aligned}F(\theta)&=2\sin\theta[(p^2+(W-m)^2\cos\theta)(|a|^2-|b|^2)\\&\quad-2(W-m)^2\cos\theta\,\mathrm{Re}(ab^*)],\end{aligned} \quad (20c)$$

$$G(\theta)=2(W-m)^2\sin^2\theta[|a|^2+|b|^2-2\,\mathrm{Re}(ab^*)]. \quad (20d)$$

Using Eqs. (19) and (20) together with the expression for T given in Eq. (17) we may compute the spin-dependent cross section to order $(\alpha Z)^4$.

If we average over initial spins and sum over final spins in Eq. (19), we see that $\sigma(\mathbf{p}_1;\,\mathbf{p}_2)=I(\theta)$ is the cross section for scattering of unpolarized electrons. Summing over final spins only, we find $S(\theta)=-D(\theta)/I(\theta)$, the so-called asymmetry function. $F(\theta)$ and $G(\theta)$ are functions which are important when both initial and final states are polarized. Such a case arises, for example, in the intermediate stage of a triple scattering.

From Eqs. (4), (14), and (17) we may write

$$a=\frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)}[a^{(1)}+a^{(2)}+a^{(3)}], \quad (21a)$$

$$b=\frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)}[b^{(2)}+b^{(3)}]; \quad (21b)$$

where

$$a^{(1)}=\frac{\alpha Z}{q^2}, \quad (22a)$$

$$a^{(2)}=i\frac{(\alpha Z)^2}{q^2}\frac{W}{p}[\ln x^2-I_1], \quad (22b)$$

$$a^{(3)}=\frac{(\alpha Z)^3}{q^2}\frac{W^2}{p^2}\left[-2\ln^2x-i\pi I_1+2I_2-\frac{m^2}{W^2}I_3\right], \quad (22c)$$

$$b^{(2)}=i\frac{(\alpha Z)^2}{q^2}\frac{m}{p}I_1, \quad (22d)$$

$$b^{(3)}=\frac{(\alpha Z)^3}{q^2}\frac{Wm}{p^2}[i\pi I_1-2I_2+I_3]. \quad (22e)$$

The expressions for $a^{(1)}$ and $b^{(2)}$ agree with the corresponding expressions from the Born approximation[22] except for a phase factor. The quantity $a_{\mathrm{Born}}^{(2)}$, however, has a divergent imaginary part of the form $a_{\mathrm{Born}}^{(2)}=2i\nu\ln(2p/\lambda)a^{(1)}+a^{(2)}$, where λ is a screening parameter which is allowed to vanish after the cross section is computed. Since to order $(\alpha Z)^2$ one may write $a_{\mathrm{Born}}^{(1)}+a_{\mathrm{Born}}^{(2)}=(2p/\lambda)^{2i\nu}(a^{(1)}+a^{(2)})$, it is clear that to second order in αZ the Born cross section will be identical to that computed from Eqs. (22). The value of $\mathrm{Im}b^{(3)}$ may be deduced from the Born five-denominator integral without great difficulty. It can be shown that

$$\mathrm{Im}(b_{\mathrm{Born}}^{(2)}+b_{\mathrm{Born}}^{(3)})=\mathrm{Im}[(2p/\lambda)^{2i\nu}(b^{(2)}+b^{(3)})],$$

and that the expression for $D(\theta)$ computed from the third Born approximation is identical with that given below.

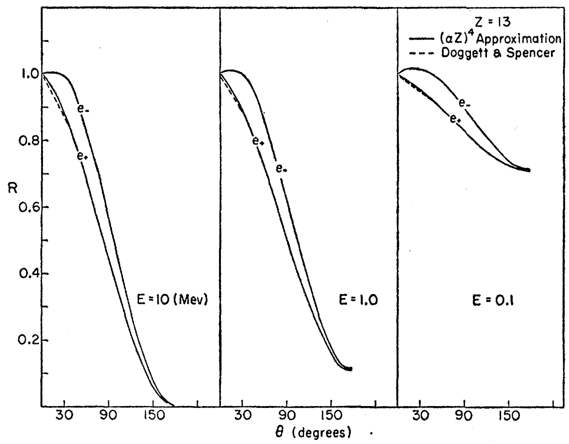
The calculation based on the Sommerfeld-Maue wave function has the obvious merit that divergent expressions are never encountered.

Writing $I(\theta)=\sigma_R[I^{(0)}(\theta)+I^{(1)}(\theta)+I^{(2)}(\theta)]$ and $D(\theta)$

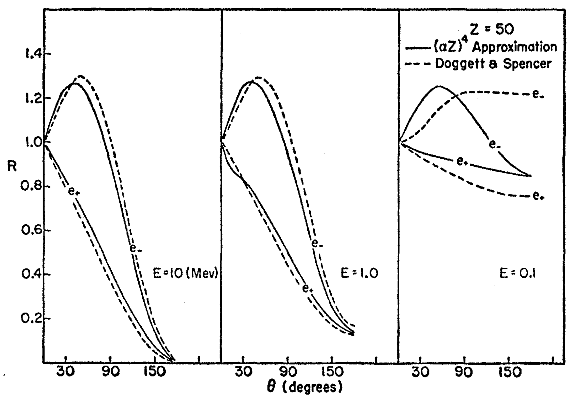
[20] *Leonhardi Euleri Opera Omnia*, edited by C. Boehm (B. G. Teubner, Basil, 1935), Ser. 1, Vol. 16, Sec. 2, p. 117.

[21] K. Mitchell [Phil. Mag. **40**, 351 (1949)] gives nine place tables for $\mathcal{L}_2(x)$, $-1\leqq x\leqq 1$.

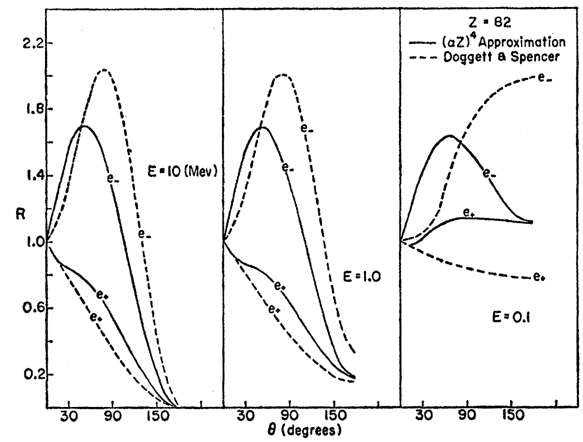
[22] See for example Dalitz, reference 4.



(a)



(b)



(c)

FIG. 1. Comparison between $(\alpha Z)^4$ approximation and exact calculation of Doggett and Spencer for scattering of electrons and positrons of various kinetic energies. $R=I(\theta)/\sigma_R$, the ratio of the scattering cross section to Rutherford scattering.

$=\sigma_R[D^{(1)}(\theta)+D^{(2)}(\theta)]$, where $\sigma_R=4(\alpha Z)^2W^2/q^4$ is the Rutherford cross section, we find

$$I^{(0)}(\theta)=(1-\beta^2x^2), \tag{24a}$$

$$I^{(1)}(\theta)=\pi\alpha Z\beta x(1-x), \tag{24b}$$

$$I^{(2)}(\theta)=(\alpha Z)^2x\Big\{[\mathcal{L}_2(1-x^2)-4\mathcal{L}_2(1-x)+2x\ln^2x+\tfrac{1}{2}\pi^2(1-x)+\tfrac{1}{6}\pi^2x]+\beta^2x\Big[\mathcal{L}_2(1-x^2)+\frac{x^2\ln^2x}{1-x^2}+\frac{\pi^2}{4}\frac{1-x}{1+x}-\frac{\pi^2}{6}\Big]\Big\}, \tag{24c}$$

$$D^{(1)}(\theta)=2\alpha Z\beta(1-\beta^2)^{\frac{1}{2}}\frac{x^3}{(1-x^2)^{\frac{1}{2}}}\ln x, \tag{24d}$$

$$D^{(2)}(\theta)=2\pi(\alpha Z)^2(1-\beta^2)^{\frac{1}{2}}\frac{x^3}{(1-x^2)^{\frac{1}{2}}}\Big[\ln x+\frac{\ln 4}{x}-\frac{1}{x}\Big(1+\frac{1}{x}\Big)\ln(1+x)\Big]. \tag{24e}$$

In Eqs. (24), $x=\sin(\theta/2)$ and $\beta=p/W$. The corresponding formulas for $F(\theta)$ and $G(\theta)$ may be written down directly from Eqs. (20) and (22).

In the extreme relativistic limit, $p=W$, one finds

$$D_{\mathrm{ER}}(\theta)=0, \tag{25a}$$

$$F_{\mathrm{ER}}(\theta)=\sin\theta I_{\mathrm{ER}}(\theta), \tag{25b}$$

$$G_{\mathrm{ER}}(\theta)=(1-\cos\theta)I_{\mathrm{ER}}(\theta), \tag{25c}$$

where

$$I_{\mathrm{ER}}(\theta)=\sigma_R\Big\{(1-x^2)+\pi\alpha Zx(1-x)+(\alpha Z)^2\times x\Big[(1+x)\mathcal{L}_2(1-x^2)-4\mathcal{L}_2(1-x)+2x\ln^2x+\frac{x^3\ln^2x}{1-x^2}+\frac{\pi^2}{2}(1-x)+\frac{\pi^2x(1-x)}{4(1+x)}\Big]\Big\}. \tag{26}$$

The relations given in Eq. (25) are true to all orders of αZ, whereas the expression for $I_{\mathrm{ER}}(\theta)$ in Eq. (26) is valid to fourth order only.

Figures 1(a), 1(b), and 1(c) compare the results of the fourth-order calculation with the exact numerical results of Doggett and Spencer. One can see the increase in error as αZ increases and similarly as β decreases. The

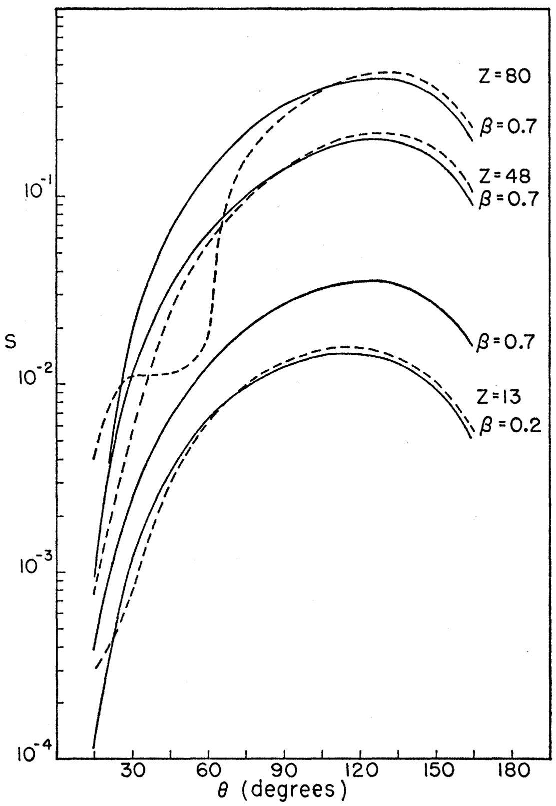


FIG. 2. Comparison between $(\alpha Z)^4$ approximation and exact calculation of Sherman for the asymmetry function, $S=-D(\theta)/I(\theta)$.

discrepancies between the fourth-order results and the exact results for positron scattering from aluminum, apparent in Fig. 1(a), are remarkable since an error of $(\alpha Z)^3 \sim 0.1\%$ is to be expected here. The function $S(\theta)$ illustrated in Fig. 2 is seen to be substantially in agreement with the exact numerical calculations of Sherman. The agreement is not expected to be as good as that for $I(\theta)$, however, since only the first two terms in the series for $D(\theta)$ are available from this calculation.

Note added in proof. Since submission of this article for publication our attention has been called to the work of B. Nagel [Kl. Tek. Högskol. Handl. No. 157 (1960)] in which some of the results derived in our article have been obtained by somewhat different methods.

APPENDIX

The expressions for $I(\theta)$, $D(\theta)$, $F(\theta)$, and $G(\theta)$ introduced in Eq. (19) can be written in terms of two functions $\mathcal{F}(\theta)$ and $\mathcal{G}(\theta)$ introduced by Mott:

$$I(\theta)=\nu'^2|\mathcal{F}|^2\csc^2(\theta/2)+|\mathcal{G}|^2\sec^2(\theta/2), \tag{1a}$$

$$D(\theta)=4\nu'\csc\theta\,\mathrm{Re}(\mathcal{F}^*\mathcal{G}), \tag{1b}$$

$$F(\theta)=-2\nu'^2|\mathcal{F}|^2\cot(\theta/2)+2|\mathcal{G}|^2\tan(\theta/2)+4\nu'\cot\theta\,\mathrm{Im}(\mathcal{F}^*\mathcal{G}), \tag{1c}$$

$$G(\theta)=2\nu'^2|\mathcal{F}|^2\cot^2(\theta/2)+2|\mathcal{G}|^2\tan^2(\theta/2)+4\nu'\,\mathrm{Im}(\mathcal{F}^*\mathcal{G}), \tag{1d}$$

where $\nu'=(m/p)\alpha Z=(1-\beta^2)^{\frac{1}{2}}\nu$. Following McKinley and Feshbach,[3] $\mathcal{F}$ and $\mathcal{G}$ are written as $\mathcal{F}=\mathcal{F}_0+\mathcal{F}_1$ and $\mathcal{G}=\mathcal{G}_0+\mathcal{G}_1$, where

$$\mathcal{F}_0=\frac{i}{2p}\frac{\Gamma(1-i\nu)}{\Gamma(1+i\nu)}\exp(i\nu\ln x^2), \tag{2a}$$

$$\mathcal{G}_0=-i\nu\frac{1-x^2}{x^2}\mathcal{F}_0, \tag{2b}$$

$$\mathcal{F}_1=\frac{i}{2p}\sum_{k=0}^{\infty}[kD_k+(k+1)D_{k+1}](-1)^kP_k(\cos\theta), \tag{2c}$$

$$\mathcal{G}_1=\frac{i}{2p}\sum_{k=0}^{\infty}[k^2D_k-(k+1)^2D_{k+1}](-1)^kP_k(\cos\theta), \tag{2d}$$

with

$$D_k=(-1)^k\frac{\Gamma(k-i\nu)}{\Gamma(k+1+i\nu)}\times\left\{1-e^{-i\pi(\gamma_k-k)}\frac{\Gamma(\gamma_k-i\nu)}{\Gamma(k-i\nu)}\frac{\Gamma(k+1+i\nu)}{\Gamma(\gamma_k+1+i\nu)}\right\}, \tag{3}$$

and $\gamma_k=[k^2-(\alpha Z)^2]^{\frac{1}{2}}$. D_k is expanded in a series in αZ to third order to give

$$D_k=(-1)^{k+1}\left\{(\alpha Z)^2\left(\frac{i\pi}{2k^2}+\frac{1}{2k^3}\right)+\frac{(\alpha Z)^3}{\beta}\left(\frac{\pi\psi_1(k)}{k^2}+\frac{\pi}{2k^3}+i\frac{\psi_2(k)}{k^2}-i\frac{\psi_1(k)}{k^3}-\frac{i}{k^4}\right)\right\}, \tag{4}$$

where

$$\psi_1(x)=\frac{d}{dx}\ln\Gamma(x)\quad\text{and}\quad\psi_2(x)=\frac{d}{dx}\psi_1(x).$$

Since only the second order part of $\mathcal{F}_1$ and the third order part of $\mathcal{G}_1$ are necessary for the fourth order cross section, we write

$$\mathcal{F}_1(\theta)=(\alpha Z)^2\mathcal{A}(\theta), \tag{5}$$

$$\mathcal{G}_1(\theta)=(\alpha Z)^2\mathcal{E}(\theta)+\frac{(\alpha Z)^3}{\beta}\mathcal{H}(\theta). \tag{6}$$

It is apparent from McKinley and Feshbach[3] that

$$\mathcal{E}(\theta)=\frac{1}{4p}\left[\pi\frac{1-x}{x}+i\ln x^2\right], \tag{7}$$

$$\mathrm{Re}\,\mathcal{A}(\theta)=-\frac{\pi}{2p}\ln(1+x), \tag{8}$$

$$\mathrm{Im}\mathcal{K}(\theta)=\frac{\pi}{2p}\left[\gamma\frac{(1-x)}{x}+\frac{1}{x}\ln x-\frac{1}{x}\ln\left(\frac{1+x}{4}\right)-\ln(1+x)\right]. \quad (9)$$

From Eqs. (2), (3), and (4) it is found that

$$\mathrm{Im}\mathcal{a}(\theta)=\frac{1}{4p}\sum_{k=1}^{\infty}\frac{1}{k^2}(P_{k-1}-P_k), \quad (10)$$

$$\mathrm{Re}\mathcal{K}(\theta)=\frac{1}{2p}\sum_{k=1}^{\infty}\left(\psi_2(k)-\frac{\psi_1(k)}{k}-\frac{1}{k^2}\right)(P_k+P_{k-1}). \quad (11)$$

Using the relations:

$$\sum_{k=1}^{\infty}\frac{1}{k^2}P_{k-1}=\tfrac{3}{2}\mathcal{L}_2(x^2)-2\mathcal{L}_2(x)+\ln x\ln(1-x^2)+\tfrac{1}{6}\pi^2, \quad (12a)$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2}P_k=\tfrac{1}{2}\mathcal{L}_2(x^2)-2\mathcal{L}_2(x)-\ln x\ln(1-x^2)+\tfrac{1}{6}\pi^2, \quad (12b)$$

$$\sum_{k=1}^{\infty}\psi_2(k)P_{k-1}=\frac{1}{2x}\left\{2\mathcal{L}_2(x)-\tfrac{1}{2}\mathcal{L}_2(x^2)+\ln x\ln\frac{1-x}{1+x}\right\}, \quad (12c)$$

$$\sum_{k=1}^{\infty}\psi_2(k)P_k=\sum_{k=1}^{\infty}\psi_2(k)P_{k-1}+\tfrac{1}{2}\mathcal{L}_2(x^2)-2\mathcal{L}_2(x)-\ln x\ln(1-x^2), \quad (12d)$$

$$\sum_{k=1}^{\infty}\frac{1}{k}\psi_1(k)P_{k-1}=-\gamma\ln\frac{1+x}{x}+\tfrac{1}{2}\ln^2 x-\ln x\ln(1+x)-\ln(1+x)\ln(1-x)+\ln 2\ln(1-x)-\tfrac{1}{2}\ln^2 2-\mathcal{L}_2(x^2)+2\mathcal{L}_2(x)-\mathcal{L}_2(\tfrac{1}{2}+\tfrac{1}{2}x), \quad (12e)$$

$$\sum_{k=1}^{\infty}\frac{1}{k}\psi_1(k)P_k=\gamma\ln x(1+x)+\ln x\ln(1+x)+\tfrac{1}{2}\ln^2 x+\tfrac{1}{2}\ln^2 2-\ln 2\ln(1-x)+\ln(1+x)\ln(1-x)-\tfrac{1}{6}\pi^2+\mathcal{L}_2(\tfrac{1}{2}+\tfrac{1}{2}x), \quad (12f)$$

where γ denotes Euler's constant, one can immediately show that

$$\mathrm{Im}\mathcal{a}(\theta)=\frac{1}{4p}[\mathcal{L}_2(x^2)+\ln x^2\ln(1-x^2)] \quad (13)$$

and

$$\mathrm{Re}\mathcal{K}(\theta)=\frac{1}{2p}\left\{\frac{1}{x}\left[2\mathcal{L}_2(x)-\tfrac{1}{2}\mathcal{L}_2(x^2)+\ln x\ln\frac{1-x}{1+x}\right]-\tfrac{1}{2}\mathcal{L}_2(x^2)-\tfrac{1}{6}\pi^2-\gamma\ln x^2-\ln^2 x-\ln x\ln(1-x^2)\right\}. \quad (14)$$

With the aid of Eqs. (7), (8), (9), (13), and (14) one finds expressions for $I(\theta)$, $D(\theta)$, $F(\theta)$, and $G(\theta)$ which are identical with those given in Sec. III.