Effect of Electron Exchange on the Dispersion Relation of Plasmons

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A calculation of the influence of electron exchange on the dispersion relation of a high-density electron gas at 0°K is described. The result is compared with those obtained by various authors using different methods.

THE influence of electron exchange on the dispersion relation of collective oscillations of a high-density electron gas has been considered by a variety of authors.[1] They all concluded that exchange contributes a term $\sim K^2$ (K is the wave vector of a plasmon). The magnitude of this term, however, varies considerably with author. Recently, Kanazawa, Misawa, and Fujita[2] calculated the exchange correction again using a Green's function approach and found the correction to be in agreement with the one obtained by Silin as well as by Nozières and Pines.[3] We wish to report here a calculation of the same effect using the transport theory recently developed by one of us.[4] The exchange contribution we obtained is again the same as the one given by Kanazawa *et al.* and by Nozières and Pines. In what follows we will sketch the derivation.

Starting with the Fourier transform $\alpha(\mathbf{K},\mathbf{k},\omega)$ of the perturbed electron distribution function, the pertinent equation of motion is given by[4]

$$\left(-\omega+\frac{\hbar}{m}\mathbf{K}\cdot\mathbf{k}+\frac{\hbar}{2m}K^2\right)\alpha(\mathbf{K},\mathbf{k},\omega)$$
$$=\omega_p{}^2\frac{m}{\hbar}\frac{1}{K^2}[F_0(\mathbf{k}+\mathbf{K})-F_0(\mathbf{k})]\int\alpha(\mathbf{K},\mathbf{k}',\omega)d^3k'$$
$$-\tfrac{1}{2}\omega_p{}^2\frac{m}{\hbar}\int d^3k'(\mathbf{k}-\mathbf{k}')^{-2}\{[F_0(\mathbf{k}+\mathbf{K})-F_0(\mathbf{k})]$$
$$\times\alpha(\mathbf{K},\mathbf{k}',\omega)-[F_0(\mathbf{k}'+\mathbf{K})-F_0(\mathbf{k}')]\alpha(\mathbf{K},\mathbf{k},\omega)\}. \quad (1)$$

Here $\mathbf{k}$ is the wave vector of the electron and F_0 is the Fermi-Dirac distribution. The last term on the right-hand side of Eq. (1) is due to exchange. The factor $\frac{1}{2}$ in front of it arises from the followlng consideration. In the derivation leading to Eq. (1), it was shown[4] that particles with a symmetric (antisymmetric) wave function in configuration space are described by an equation in which the exchange term appears with a plus (minus) sign. For electrons, the two-particle wave function including spin must be antisymmetric. But this means that the configuration part of the wave function is antisymmetric in $\frac{3}{4}$ of all cases (triplet) and symmetric in $\frac{1}{4}$ of all cases (singlet), so that the average in spin space leads to a factor $\frac{3}{4}(-1)+\frac{1}{4}=-\frac{1}{2}$ for the exchange term in Eq. (1). Since the exchange contribution we wish to calculate is known to be small,[1] we are applying a simple perturbation scheme to Eq. (1).

Let

$$\alpha=\alpha_0+\alpha_1, \quad (2)$$

$$\omega=\omega_0+\omega_1, \quad (3)$$

where $\alpha_1\ll\alpha_0$ and $\omega_1\ll\omega_0$. We then have, with the abbreviations

$$D=-\omega_0+\frac{\hbar}{m}\mathbf{K}\cdot\mathbf{k}+\frac{\hbar}{2m}K^2, \quad (4)$$

$$\Delta F_0=F_0(\mathbf{k}+\mathbf{K})-F_0(\mathbf{k}), \quad (5)$$

the following set of equations:

$$D\alpha_0(\mathbf{K},\mathbf{k},\omega_0)=\omega_p{}^2\frac{m}{\hbar}\frac{\Delta F_0}{K^2}\int\alpha_0(\mathbf{K},\mathbf{k}',\omega_0)d^3k', \quad (6)$$

$$D\alpha_1(\mathbf{K},\mathbf{k},\omega_1)-\omega_1\alpha_0(\mathbf{K},\mathbf{k},\omega_0)$$
$$=\omega_p{}^2\frac{m}{\hbar}\frac{\Delta F_0}{K^2}\int\alpha_1(\mathbf{K},\mathbf{k}',\omega_1)d^3k'$$
$$-\tfrac{1}{2}\omega_p{}^2\frac{m}{\hbar}\int d^3k'|\mathbf{k}-\mathbf{k}'|^{-2}\{\Delta F_0\alpha_0(\mathbf{K},\mathbf{k}',\omega_0)$$
$$-\Delta F_0{}'\alpha_0(\mathbf{K},\mathbf{k},\omega_0)\}. \quad (7)$$

Equation (6) leads immediately to the well-known dispersion relation:

$$1=\omega_p{}^2\int d^3k\frac{F_0(\mathbf{k})}{[\omega_0-(\hbar/m)\mathbf{K}\cdot\mathbf{k}]^2-[(\hbar/2m)K^2]^2}. \quad (8)$$

Observing Eqs. (6), (7), and (8) the exchange correction ω_1 is then determined by

$$-\omega_1\int D^{-1}\alpha_0(\mathbf{K},\mathbf{k},\omega_0)d^3k$$
$$=-\tfrac{1}{2}\omega_p{}^2\frac{m}{\hbar}\int d^3kd^3k'\frac{\Delta F_0\alpha_0{}'-\Delta F_0{}'\alpha_0}{D|\mathbf{k}-\mathbf{k}'|^2}, \quad (9)$$

[1] D. Pines, Revs. Modern Phys. **28**, 184 (1956); P. A. Wolff, Phys. Rev. **92**, 18 (1953); R. A. Ferrell, Phys. Rev. **107**, 450 (1957); D. F. Dubois, Ann. Phys. **8**, 24 (1959).
[2] H. Kanazawa, S. Misawa, and E. Fujita, Progr. Theoret. Phys. (Kyoto) **23**, 426 (1960).
[3] P. Nozières and D. Pines, Phys. Rev. **111**, 442 (1958); V. P. Silin, J. Exptl. Theoret. Phys. U.S.S.R. **37**, 273 (1959) [translation: Soviet Phys.-JETP **37** (10), 192 (1960)].
[4] O. von Roos, Phys. Rev. **119**, 1174 (1960).

which may be written, with the help of Eq. (6):

$$\omega_1\int D^{-2}\Delta F_0 d^3k=\tfrac{1}{2}\omega_p{}^2\frac{m}{\hbar}\int d^3kd^3k' \times\frac{\Delta F_0\Delta F_0'}{|\mathbf{k}-\mathbf{k}'|^2}\left(\frac{1}{DD'}-\frac{1}{D^2}\right). \quad (10)$$

In the long-wavelength limit, an expansion of the terms in Eq. (10) in powers of K is allowed. Keeping only the lowest order terms, Eq. (10) goes over into

$$\omega_1=\frac{\omega_p{}^2}{8K^2\omega_0}\int d^3kd^3k'[\mathbf{K}\cdot(\mathbf{k}-\mathbf{k}')]^2|\mathbf{k}-\mathbf{k}'|^{-2} \times\mathbf{K}\cdot\nabla_kF_0(\mathbf{k})\mathbf{K}\cdot\nabla_{k'}F_0(\mathbf{k}'). \quad (11)$$

The integral in Eq. (11) is easily evaluated at 0°K temperature and the result is:

$$\omega_1=-\frac{3}{40}\frac{K^2}{k_F{}^2}\frac{\omega_p{}^2}{\omega_0}, \quad (12)$$

where k_F is the Fermi momentum. But since ω_0 is given by the unperturbed plasma frequency ω_p plus small correction terms of order K^2, etc., we see finally that

$$\begin{aligned}\omega^2&=(\omega_0+\omega_1)^2=\omega_0{}^2+2\omega_0\omega_1\\&=\omega_0{}^2-(3/20)(K^2/k_F{}^2)\omega_p{}^2,\end{aligned} \quad (13)$$

identical with the result of Kanazawa *et al.*[2]

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Analytic Properties of Single-Particle Propagators for Many-Fermion Systems*

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Certain general properties of single-particle propagators for a system of interacting fermions are derived. In addition, the properties of the proper self-energy part $G_k(\zeta)$ which were used in previous work on the ground-state energy and on the Fermi surface are established. In particular, the fact that to all orders of perturbation theory in the interaction, Im $G_k(x-i0^+)$ behaves like $C_k(x-\mu)^2$ ($C_k>0$) for x very near μ, is proved.

1. GENERAL DISCUSSION OF THE PROPAGATOR

In some recent work[1] on the theory of a system of interacting fermions, certain analytical properties of the so-called "single-particle propagator" were made use of. No proof of those properties was given at that time. It is the purpose of this brief note to establish these properties. For simplicity we shall restrict ourselves to the case of spinless fermions interacting among themselves, but not moving in an external potential. The resulting simplification is mainly notational, and there is no difficulty in extending our results to the more complicated cases.

The single-particle propagator as used in LW was defined as the sum (with appropriate coefficients) of all connected diagrams having a single line entering and leaving. For the purposes of general discussion it is often convenient to have an explicit closed expression for it. As is well known in field theory, such an expression is given as follows.[2] Consider the quantity

$$S_k'(u,u')\equiv\langle T[a_k{}^\dagger(u)a_k(u')]\rangle. \quad (1)$$

In (1) the quantity a_k is the destruction operator for a particle of momentum k,

$$a_k(u')=e^{u'H}a_ke^{-u'H},\quad a_k{}^\dagger(u)=e^{uH}a_k{}^\dagger e^{-uH}; \quad (2)$$

H is the total Hamiltonian of the system and the angular bracket represents the average of the enclosed quantity with respect to the grand canonical distribution

$$\langle A\rangle\equiv\mathrm{Tr}\,(e^{\beta(\Omega-H-\mu N)}A),\quad \beta=1/kT. \quad (3)$$

The operation T is the usual Wick chronological operator meaning

$$\begin{aligned}T[a_k{}^\dagger(u)a_k(u')]&=a_k{}^\dagger(u)a_k(u'),\quad u>u'\\&=-a_k(u')a_k{}^\dagger(u),\quad u<u'.\end{aligned} \quad (4)$$

Equation (1) provides an expression for the propagators in the "temperature" variables u,u', which are constrained to vary between zero and β. From (1) we see that $S_k'(u,u')$ is a function of $u-u'\equiv v$ only:

$$S_k'(v)=\mathrm{Tr}\,e^{\beta(\Omega-H-\mu N)}\begin{cases}e^{vH}a_k{}^\dagger e^{-vH}a_k, & \beta>v>0\\ -a_ke^{vH}a_k{}^\dagger e^{-vH}, & -\beta<v<0.\end{cases} \quad (5)$$

Using (5), we see at once that the quantity

$$S_k'(v)e^{-(i\pi/\beta+\mu)v}$$

is a periodic function of v of period β in the interval

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[1] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960). (We shall refer to this paper as LW.) J. M. Luttinger, Phys. Rev. **119**, 1153 (1960). We shall follow the notation of these papers as far as is practiced.

[2] The representation we shall use here is essentially the same as that of A. A. Abrikosov, L. P. Gorkov, and I. E. Dsyaloshinskii, Soviet Phys.-JETP **36** (9), 636 (1959), except for minor differences of notation and definition.