Perturbation Theory for an Infinite Medium of Fermions. III. Derivation of the Landau Theory of Fermi Liquids*

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(Received October 13, 1960)

The methods employed in the previous papers of this series for the ground-state energy of a normal Fermion system are extended to finite temperatures. The basic assumption is that the density matrix of the actual system in thermodynamic equilibrium can be derived by adiabatic transformation from the density matrix of a system of noninteracting particles at the same temperature, volume, and chemical potential. However, the spectrum of the noninteracting system must be dependent on the intensive parameters. The self-consistency of the scheme is exhibited by demonstrating that the effective potential characterizing this system can be related to the self-energy operator of the actual system. Two equivalent versions of the theory are derived. In one of these the basic equations are precisely those of the Landau theory of Fermi liquids with the difference, however, that all quantities are given explicitly from first principles.

I. INTRODUCTION

IN the previous papers of this series,[1] we have been concerned with the ground-state energy of a normal Fermion system. The attempt in the present work is to apply analogous ideas and methods to similar systems at finite temperature. We again choose to study a representative Hamiltonian, that of II, Eq. (1).

The previous developments have been based on the Green's function formalism to which has been adjoined a suitable adiabatic hypothesis permitting power series construction of the relevant propagators. It is, of course, well known that the boundary conditions on the propagators can be given independently of an adiabatic decoupling hypothesis and a general formulation thereby developed.[2] We shall, nevertheless, confine ourselves once more to the previous framework, in order to exhibit that what emerges is a completely explicit form of Landau's semiphenomenological theory of Fermi liquids.[3] As such, the theory should possess, in contrast to the zero-temperature case, a real domain of application, for example, to liquid He^3 and to normal metals. Several alternative versions of this theory have now been derived.[4] Aside from the virtue of conciseness, our version has perhaps its main advantage in showing from the start, in its fundamental assumption, that what must result, if that assumption can be justified, is the microscopic form of Landau's theory.

We assume, in effect, that the density matrix in the grand canonical representation is the adiabatic transform, in the sense of scattering theory, of the density matrix of a system of noninteracting particles (the model system), characterized, as usual, by states of definite momentum. The spectrum of single-particle energies must be allowed to vary, however, with the intrinsic parameters, temperature and chemical potential. We then develop in compact form (Sec. II) a kind of linked cluster expansion for the Gibbs potential expressed in terms of the exact propagator and of the so-far undetermined energy spectrum of the model system, the latter occurring only within factors expressing occupation number densities for the model system. As constructed, the actual and model systems yield the same average number of particles; with the help of this condition it is shown how the model energies may be evaluated from the properties of the actual system, in particular from its self-energy operator. An ambiguity in this evaluation is remarked and resolved. The form of the theory finally resulting, if expanded *completely* in a power series in the two-body interaction, would yield the expansions of Bloch and De Dominicis.[5]

In order to demonstrate the equivalence to the Landau theory, a transformed version of the theory is obtained in Sec. III. In contrast to the version of the previous section, it is the natural generalization of the expansions of the previous paper to which it reduces at zero temperature. The equations for the various thermodynamic quantities, such as average number, internal energy, and entropy, are, moreover, shown to be in one-to-one correspondence in form with the equations of Landau's theory.

II. FIRST FORM OF THE THEORY: DETERMINATION OF THE SELF-CONSISTENT POTENTIAL

Our object is to calculate the Gibbs potential, $A(\beta,\Omega,\alpha)$,

$$A(\beta,\Omega,\alpha)=\beta^{-1}\ln \mathrm{Tr}\exp[-\beta H+\alpha N], \qquad (1)$$

* Supported in part by the U. S. Atomic Energy Commission.

[1] A. Klein and R. E. Prange, Phys. Rev. **112**, 994 (1958); A. Klein, preceding paper [Phys. Rev. **121**, 950 (1961)]; hereinafter referred to as I and II. See, also, *Lectures at the International Spring School of Theoretical Physics, University of Naples*, **1960** [Nuovo cimento (to be published)].

[2] See, for example, P. C. Martin and J. S. Schwinger, Phys. Rev. **115**, 1342 (1959), from which most of the extensive Russian literature on this subject can be traced. See, also, V. L. Bonch-Bruevich and Sh. M. Kogan, Ann. Phys. **9**, 125 (1960).

[3] For a recent review see A. A. Abrikosov and I. M. Khalatnikov, *Reports on Progress in Physics* (The Physical Society, London, 1959), Vol. XXII, p. 329.

[4] J. M. Luttinger and J. C. Ward, Phys. Rev. **119**, 1417 (1960); J. M. Luttinger, Phys. Rev. **119**, 1153 (1960); R. Balian, C. Bloch, and C. De Dominicis, *Lectures at the International Spring School of Theoretical Physics, University of Naples*, **1960** [Nuovo cimento (to be published)]; R. Balian and C. De Dominicis, Nuclear Phys. **16**, 502 (1960); Compt. rend. **250**, 3285, 4111 (1960). Only in the work of these latter authors is an explicit derivation of the Landau equations given.

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where β is the reciprocal temperature and $\alpha=\beta\mu$. As is well known, the complete scheme of thermo-dynamics follows from the identifications

$$A(\beta,\Omega,\alpha)=-P\Omega, \tag{2}$$

where P is the pressure, and

$$\begin{aligned} -N&=\beta(\partial A/\partial\alpha)_{\beta,\Omega},\\ -P&=(\partial A/\partial\Omega)_{\beta,\alpha}\\ E-\mu N&=(\partial/\partial\beta)(\beta A)_{\Omega,\mu}=A+\beta(\partial A/\partial\beta)_{\Omega,\mu}. \end{aligned} \tag{3}$$

Since (units: Boltzmann's constant$=k=1$)

$$E=\beta^{-1}S-P\Omega+\mu N, \tag{4}$$

where S is the entropy, we have

$$\beta^2(\partial A/\partial\beta)_{\Omega,\mu}=S. \tag{5}$$

To obtain a linked cluster expansion for A, we compute the derivative

$$[\partial A(\beta,\Omega,\alpha)/\partial\lambda]_{\beta,\Omega,\alpha}=\langle H_1\rangle, \tag{6}$$

where $\langle\,\rangle$ indicates a grand canonical average. The integral of (6) is

$$A(\beta,\Omega,\alpha)=A_0(\beta,\Omega,\alpha)+\int_0^\lambda d\lambda'\langle H_1\rangle_{\lambda'}, \tag{7}$$

where $A_0(\beta,\Omega,\alpha)$ is the well-known value of the unperturbed thermodynamic potential,

$$A_0(\beta,\Omega,\alpha)=-\frac{1}{\beta}\frac{\Omega}{(2\pi)^3}\int d\mathbf{p}\,\ln[1+\exp(-\beta p^2+\alpha)]. \tag{8}$$

A suitable expansion of the second term of (7) will again be sought with the help of the Green's function formalism. The relevant set of propagators are defined characteristically by the equations[2]

$$\begin{aligned} &G_{\beta,\alpha}(x;x')=i\,\mathrm{Tr}\{T(\psi(x)\psi^\dagger(x'))\rho(\beta,\Omega,\alpha)\},\\ &G_{\beta,\alpha}(x_1x_2;x_1'x_2')\\ &\quad=i^2\,\mathrm{Tr}\{T(\psi(x_1)\psi(x_2)\psi^\dagger(x_2')\psi^\dagger(x_1'))\rho(\beta,\Omega,\alpha)\}, \end{aligned} \tag{9}$$

etc., where $\rho(\beta,\Omega,\alpha)$ is the density matrix

$$\rho(\beta,\Omega,\alpha)=\exp[-\beta H+\alpha N]/\mathrm{Tr}\exp[-\beta H+\alpha N]. \tag{10}$$

Equation (7) may now be rewritten in complete *formal* analogy with Eqs. I (9), (17), and (29) and then Eqs. II (3) and (5) as

$$A-A_0=-\tfrac{1}{2}\Omega\int_0^\lambda\frac{d\lambda'}{\lambda'}\,\mathrm{tr}M_{\beta,\alpha}(\mathbf{p},p_0)G_{\beta,\alpha}(\mathbf{p},p_0), \tag{11}$$

where the operation "tr" has the same meaning as in II, namely $i(2\pi)^{-4}\int d\mathbf{p}\int dp_0$. To make any real progress, we require definite forms for the Green's function and associated self-energy operators which occur in (11). In this paper we shall obtain such by means of an adiabatic hypothesis which generalizes to finite temperature the discussion of II. The restriction to normal systems will again be manifest in this assumption. Moreover, the formalism thereby attained in the next section will reduce concisely to the previous zero-temperature results.

We now assume that there exists a model Hamiltonian

$$\begin{aligned} H_m(\beta,\alpha;\lambda)&=\textstyle\sum_p a^\dagger(\mathbf{p})a(\mathbf{p})\{p^2+\mathcal{U}(\mathbf{p},\beta,\alpha;\lambda)\}\\ &=\textstyle\sum_p a^\dagger(\mathbf{p})a(\mathbf{p})\epsilon(\mathbf{p},\beta,\alpha;\lambda)\\ &=H_0+V(\beta,\alpha;\lambda), \end{aligned} \tag{12}$$

with the real potential $\mathcal{U}$ ultimately to be determined, and a model density matrix $\rho_m(\beta,\alpha;\lambda)$,

$$\rho_m(\beta,\alpha;\lambda)=\exp[-\beta H_m(\beta,\alpha;\lambda)+\alpha N]/\mathrm{Tr}\exp[-\beta H_m+\alpha N], \tag{13}$$

such that the actual density matrix $\rho(\beta,\alpha;\lambda)$, Eq. (10), is the adiabatic transform of $\rho_m(\beta,\alpha;\lambda)$ for every λ,

$$\rho(\beta,\alpha;\lambda)=\frac{U_{\beta,\alpha}(0,-\infty;\lambda)\rho_m(\beta,\alpha;\lambda)U_{\beta,\Omega}(\infty,0;\lambda)}{\mathrm{Tr}U_{\beta,\alpha}(\infty,-\infty;\lambda)\rho_m(\beta,\alpha;\lambda)}. \tag{14}$$

Here $U_{\beta,\alpha}(t,t';\lambda)$ is the time development operator in the interaction representation for the interaction $\lambda H_1-V(\beta,\alpha;\lambda)$. With the help of (14) and the definition,

$$a(\mathbf{p}t)=U_{\beta,\alpha}(0,t)a_m(\mathbf{p},t)U_{\beta,\alpha}(t,0), \tag{15}$$

of the interaction representation operators $a_m(\mathbf{p}t)$,

$$a_m(\mathbf{p}t)=\exp[-i\epsilon(\mathbf{p},\beta,\alpha;\lambda)t]a(\mathbf{p}), \tag{16}$$

we may write for the one-particle Green's function, for instance

$$\begin{aligned} \overline{G_{\beta,\alpha}}(\mathbf{p},t-t')&=i\,\mathrm{Tr}\{T(a(\mathbf{p}t)a^\dagger(\mathbf{p}t'))\rho(\beta,\alpha)\}\\ &=i\frac{\mathrm{Tr}\{T(a_m(\mathbf{p}t)a_m{}^\dagger(\mathbf{p}t')U_{\beta,\alpha}(\infty,-\infty))\rho_m(\beta,\alpha)\}}{\mathrm{Tr}\{U_{\beta,\alpha}(\infty,-\infty)\rho_m(\beta,\alpha)\}}. \end{aligned} \tag{17}$$

The problems associated with determining the appropriate form of the self-consistent potential $\mathcal{U}(\mathbf{p},\beta,\alpha;\lambda)$ will be deferred until the end of this section. Assuming its existence, we first discuss the evaluation of (11). The Green's function (17), which is needed for this purpose, can be analyzed either by the methods of I or by means of Wick's theorem. The simplest expression of the result is (henceforth suppressing the subscripts β,α)

$$\begin{aligned} G(\mathbf{p},p_0;\lambda)&=\frac{1-N_m(\mathbf{p},\lambda)}{-p_0+p^2-M(p,p_0;\lambda)-i\eta}\\ &\quad+\frac{N_m(\mathbf{p},\lambda)}{-p_0+p^2-M(\mathbf{p},p_0;\lambda)+i\eta}=P[-p_0+p^2-M]^{-1}\\ &\quad+i\pi[1-2N_m(\mathbf{p},\lambda)]\delta(-p_0+p^2-M(\mathbf{p},p_0;\lambda)), \end{aligned} \tag{18}$$

where $N_m(\mathbf{p},\lambda)$ is the momentum distribution of the

model system,

$$N_m(\mathbf{p},\lambda)=\{1+\exp[\beta\epsilon(\mathbf{p},\beta,\alpha;\lambda)-\alpha]\}^{-1}, \quad (19)$$

and $M[\mathbf{p},p_0;\lambda]$ is *exactly* the same functional of G or of $G^{(0)}$ as at zero temperature with $G^{(0)}$ here given by (18) in the limit $M=0$. It should be clear that the essential effect of the adiabatic hypothesis chosen has been a clean separation of statistics and dynamics.

We are finally prepared to evaluate (11). We by-pass any discussion of the complete power series for A, since this would in any event require the value of $\mathcal{U}(\mathbf{p},\lambda)$. *A posteriori*, it could be established that what results is the suitable form of the expansion of Bloch and De Dominicis.[5] Instead, we pursue without pause the methods of II, since it now follows from the above that diagrammatically $(A-A_0)$ has the same properties essential for this purpose as does the expression for the ground-state energy. We write

$$M[G]=\sum_1^\infty \lambda^n M^{(n)}[G], \quad (20)$$

and employ the formula [II, Eq. (8)]

$$(\partial/\partial\lambda)\,\mathrm{tr}\{M^{(n)}[G]G\}=2n\,\mathrm{tr}M^{(n)}[G](\partial G/\partial\lambda), \quad (21)$$

which utilizes the structure of the diagrams in terms of equivalent lines. Inserting then in (11) the form (18), evaluated at λ' and proceeding as in II, Eq. (9)–(12), we find ultimately that

$$\begin{aligned}(\Delta A/\Omega)&=(A-A_0)/\Omega\\&=-\sum_1^\infty(\lambda^n/2n)\,\mathrm{tr}M^{(n)}G+\mathrm{tr}MG+\mathrm{tr}\ln|G^{(0)}G^{-1}|\\&\quad-\pi i\,\mathrm{tr}\int_0^M dM(\lambda')[1-2N_m(\lambda')]\\&\qquad\times\delta(-p_0+p^2-M(\mathbf{p},p_0;\lambda')). \quad (22)\end{aligned}$$

In (22), the absolute value signs in the logarithmic term indicate the result of a principal value integral.

We can now show that Eq. (22) for $(\Delta A/\Omega)$ is stationary with respect to variations of $M(\mathbf{p},p_0)$, where $M^{(n)}=M^{(n)}[G]$ and $G^{-1}=-p_0+p^2-M$. It is also stationary with respect to variations of $G(\mathbf{p},p_0)$ if we understand that $M=G^{(0)-1}-G^{-1}$. The proofs are the same as in II.

Having postponed its consideration to this point, we must finally study the all-important question of the value (and indeed existence) of $\mathcal{U}(\mathbf{p},\beta,\alpha;\lambda)$. For this determination we shall utilize several equivalent forms of $\bar{N}$, the average number of particles. We have first of all from the definition of the Green's function, Eq. (17),

$$\bar{N}=\mathrm{tr}G. \quad (23)$$

More essential for our purposes is the observation that also

$$\bar{N}=\mathrm{tr}G^{(0)}=\textstyle\sum_{\mathbf{p}} N_m(\mathbf{p},\beta,\alpha), \quad (24)$$

where we have reverted for reasons to be clarified below to a discrete spectrum (finite system). We content ourselves with a few remarks concerning the validity of Eq. (24). This formula is the exact analog of Eq. (55) of II, which was established on general grounds. Next we remark that diagram for diagram the difference between (23) and (24) is the same as the difference between the corresponding expressions at zero temperature. There the vanishing of the difference could have been established also by showing the cancellation in pairs of the time-ordered diagrams representative of its series expansion. A similar cancellation obtains here.

We now require that the result (24) be consistent with the value of $\bar{N}$ computed from the Gibb's potential according to the recipe of Eq. (3),

$$-\bar{N}=\beta(\partial A/\partial\alpha)_{\beta,\Omega}. \quad (25)$$

According to the variational principle described after Eq. (22), the only contributions to (25) which need be taken into account are those from A_0 and from the quantity $N_m[M(\beta,\alpha;\lambda');\beta,\alpha;\lambda']$ in the last term of (22). The evaluation of (25) thus reads explicitly

$$\begin{aligned}\bar{N}&=\sum_{\mathbf{p}} N_0(\mathbf{p},\beta,\alpha)\\&\quad+2\pi\beta\sum_{\mathbf{p}}\int\frac{dp_0}{(2\pi)}\int_0^\lambda d\lambda'(\partial/\partial\lambda')M(\mathbf{p},p_0,\lambda')\\&\quad\times[\partial N_m(\lambda')/\partial\alpha]_{M(\lambda')}\delta(-p_0+p^2-M(\mathbf{p},p_0;\lambda')), \quad (26)\end{aligned}$$

where

$$N_0(\mathbf{p},\beta,\alpha)=\{1+\exp[\beta p^2-\alpha]\}^{-1} \quad (27)$$

is the occupation number for a set of free particles.

Further progress is contingent upon a recognition of the meaning to be assigned to the δ function condition in (26). It has its origin as part of the Green's function (18). We must now understand that Eq. (22) is simply a highly condensed expression for a power series which is evaluated explicitly by expanding G in a power series in M and the latter in turn in an *explicit* power series in λ. Next, in every order the integrals over p_0, well-defined by the boundary conditions imposed on G, are carried out (by contour methods, for example), ignoring the contributions which may arise from the coincidence of poles which are *generally* different. After these integrations we may set $\eta\to 0$ and go to the continuous limit for the sums over momenta, as no singularities should remain, at least when terms are suitably grouped according to their origin.

We may now ask: What is the contribution to be assigned on this basis to the last term of (26) or of the corresponding term of A from which it emanates? The answer again follows by analogy with the corresponding expansion at zero temperature where the proof of the Hugenholtz-Van Hove theorem given in II (Sec. III) demonstrated that this term is correctly evaluated in the perturbation sense by assigning a root $\epsilon(\mathbf{p},\beta,\alpha;\lambda)$

to the equation

$$p_0=p^2-M(\mathbf{p},p_0;\beta,\alpha;\lambda) \tag{28}$$

according to the Lagrange formula

$$p_0=\epsilon(\mathbf{p})=p^2-\sum_1^\infty \frac{(-1)^{n-1}}{n!}\left[\left(\frac{\partial}{\partial p_0}\right)^{n-1}M^n(\mathbf{p},p_0)\right]_{p_0=p^2}$$
$$=p^2-M(\mathbf{p},\epsilon(\mathbf{p})). \tag{29}$$

If, provisionally, we accept this evaluation, and use the result in (29), we find after a short calculation, analogous to that of II Eqs. (22)–(27), that

$$\sum_{\mathbf{p}} N_m(\mathbf{p},\beta,\alpha;\lambda)=\sum_{\mathbf{p}} N_0(\mathbf{p},\beta,\alpha)$$
$$+\beta\sum_{\mathbf{p}}\int_0^\lambda d\lambda'[(d/d\lambda')M(\mathbf{p},\epsilon(\mathbf{p},\beta,\alpha;\lambda'))]$$
$$\times[\partial N_m(\lambda')/\partial\alpha]_{M(\lambda')}. \tag{30}$$

A solution of this equation is of the form (19), if we make the identification

$$\mathcal{U}(\mathbf{p},\beta,\alpha;\lambda)=-M[\mathbf{p},\epsilon(\mathbf{p},\beta,\alpha;\lambda);\beta,\alpha;\lambda]. \tag{31}$$

It remains only to consider the consistency of this identification. We have, after all, required the reality of $\mathcal{U}$. In the limit $\Omega\to\infty$, which we must ultimately consider, $M(\mathbf{p},\epsilon(p))$ will, in general, be complex. In this connection we consider simultaneously the contribution to A which corresponds to the last term of (22). This can be put in the form [compare (30)]

$$-\tfrac{1}{2}\sum_{\mathbf{p}}\int_0^\lambda d\lambda'[(d/d\lambda')M(\mathbf{p},\epsilon(\mathbf{p},\lambda'))][1-2N_m(\lambda')]. \tag{32}$$

In the continuous limit, (32) will also be complex because of M. Insofar as the contribution to A is concerned, however, we are assured that the imaginary part *must* cancel against other contributions. In this sense we have made an artificial decomposition of A into constituent pieces. We can, therefore, consistently replace M by its real part, ReM, in (32) if compensating replacements are made in other parts of A. This means, however, that our original requirement on $\mathcal{U}$ can be satisfied by substituting for (31) the identification

$$\mathcal{U}(\mathbf{p},\beta,\alpha,\lambda)=-\mathrm{Re}M[\mathbf{p},\epsilon(\mathbf{p},\beta,\alpha;\lambda);\beta,\alpha;\lambda]. \tag{33}$$

This ambiguity in the identification of $\mathcal{U}$ is perhaps not surprising, since it lacks direct physical significance. The possibility of choosing a real $\mathcal{U}$, however, commends itself by its simplicity.

III. DERIVATION OF THE LANDAU THEORY

The theory developed above will in this section be given a new form which will render more apparent the fact that it embodies nothing more or less than the Landau theory of Fermi liquids, in a completely explicit form, however.

We start with Eq. (3) in the form,

$$\bar{N}=-(\partial A/\partial\mu)_{\beta,\Omega}, \tag{34}$$

with $\alpha=\beta\mu$, and integrate, using Eq. (24) for $\bar{N}$,

$$\bar{N}=\textstyle\sum_{\mathbf{p}} N_m(\mathbf{p},\beta,\alpha). \tag{35}$$

We find straightforwardly

$$A(\beta,\Omega,\alpha)$$
$$=-\sum_{\mathbf{p}}\int_{-\infty}^{\mu}d\mu'\frac{\exp[-\beta\epsilon(\mathbf{p},\beta,\alpha')+\beta\mu']}{1+\exp[-\beta\epsilon(\mathbf{p},\beta,\alpha')+\beta\mu']}$$
$$=-\sum_{\mathbf{p}}\{\beta^{-1}\ln[1+\exp(-\beta\epsilon+\alpha)]+\mathcal{U}(\mathbf{p},\beta,\alpha)N_m(\mathbf{p},\beta,\alpha)\}$$
$$+\sum_{\mathbf{p}}\int_{-\infty}^{\mu}d\mu'(\partial N_m/\partial\mu')\mathcal{U}(\mathbf{p},\beta,\alpha'). \tag{36}$$

If we can find a functional $\Phi[N_m]$, whose sole dependence on μ is through the $N_m(\mathbf{p},\beta,\alpha)$, with the property, furthermore, that

$$\mathcal{U}(\mathbf{p},\beta,\alpha)=\delta\Phi[N_m]/\delta N_m(\mathbf{p},\beta,\alpha), \tag{37}$$

then the last term of (36) becomes simply $\Phi[N_m]$, and $A(\beta,\Omega,\alpha)$ may itself be rewritten

$$A(\beta,\Omega,\alpha)=\textstyle\sum_{\mathbf{p}}\{\beta^{-1}\ln[1-N_m(\mathbf{p},\beta,\alpha)]$$
$$-\mathcal{U}(\mathbf{p},\beta,\alpha)N_m(\mathbf{p},\beta,\alpha)\}+\Phi[N_m]. \tag{38}$$

Examination of the developments of Sec. II is sufficient to present a ready candidate for the role of $\Phi[N_m]$, namely,

$$\Phi[N_m]=-\sum_1^\infty(\lambda^n/2n)\,\mathrm{tr}M^{(n)}G+\mathrm{tr}MG$$
$$+\mathrm{tr}\ln|G^{(0)}G^{-1}|-\pi i\,\mathrm{tr}[1-2N_m(\mathbf{p},\beta,\alpha)]$$
$$\times\int_0^{M(\lambda)}dM(\lambda')\delta(-p_0+p^2-M(\lambda')). \tag{39}$$

The functional, Eq. (39), is closely related to that in Eq. (22), this relationship having suggested its choice, but differs in that in the last term N_m appears at value λ rather than λ'. It shares with (22), however, its stationary property with respect to variations of M and of G, which property simplifies the proof of (37). Indeed, the proof of the latter now follows by the same methods as lead from (26) to (31).

This completes the formulation. In what follows, we shall summarize some of its salient properties, largely without detailed proof:

(1) $A(\beta,\Omega,\alpha)$ as given by (38) is stationary with respect to variations of $N_m(p,\beta,\alpha)$, where $\mathcal{U}(\mathbf{p},\beta,\alpha)$ is to be expressed in terms of the latter, and equally sta-

tionary with respect to variations of $\mathcal{U}(\mathbf{p},\beta,\alpha)$ with N_m a function of $\mathcal{U}$.

(2) The power series for $\Phi[N_m]$ in terms of time-ordered diagrams is precisely of the Goldstone form as follows by comparison of (38) with the corresponding functional at zero temperature II, Eq. (12). The present form is, therefore, the natural analog of the expansions given in II.

(3) The form (38) goes over simply into the Goldstone expansion or into its generalization for anisotropic situations as $\beta \to \infty$. It is to be recalled that the appropriate limit is

$$\lim_{\beta\to\infty,\ \mu \text{ fixed}} A(\beta,\Omega,_\alpha) = E-\mu N. \tag{40}$$

(4) By direct calculation or with the help of the variational principle, it can be shown that the internal energy is given by

$$E=(\partial/\partial\beta)(\beta A)_{\Omega,\alpha}=\sum_{\mathbf{p}} p^2 N_m(\mathbf{p},\beta,\alpha)+\Phi[N_m]. \tag{41}$$

From this we find

$$\begin{aligned}\epsilon(\mathbf{p},\beta,\alpha) &= p^2+\mathcal{U}(\mathbf{p},\beta,\alpha)\\ &= \delta E/\delta N_m(\mathbf{p},\beta,\alpha).\end{aligned} \tag{42}$$

(5) The entropy is given by the even simpler expression

$$\begin{aligned}S &= \beta^2(\partial A/\partial\beta)_{\Omega,\mu}\\ &= -\sum_p \{(1-N_m)\ln(1-N_m)+N_m \ln N_m\}.\end{aligned} \tag{43}$$

Equations (41) and (43), together with the equations for $N_m(\mathbf{p},\beta,\alpha)$, are the fundamental equations of the Landau theory. All known consequences of this theory can, therefore, be derived from our equations.

ACKNOWLEDGMENTS

The author is indebted to Dr. R. Balian, Dr. C. Bloch, Dr. C. De Dominicis, and Dr. J. M. Luttinger for informing him of their results prior to publication and for illuminating discussions of these results.

PHYSICAL REVIEW VOLUME 121, NUMBER 4 FEBRUARY 15, 1961

Cyclotron Radiation from Relativistic Particles with an Arbitrary Velocity Distribution*

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A formula is derived which represents the spectral and angular behavior of cyclotron radiation emitted by a relativistic particle moving in a constant magnetic field and having arbitrary velocity components parallel and perpendicular to the field. This formula is used to describe the emission of an assembly of particles having a Maxwellian distribution function. All broadening mechanisms of practical importance—collisions, relativistic mass variability and Doppler effects—are included and discussed.

1. INTRODUCTION

IT was shown in a previous article[1] (hereinafter called I) that the influences of "collisions" of the emitting particles with disturbers can be accounted for by a formula of the Lorentz type, even if the particle energies have relativistic values. It turned out that this result holds whenever the particle completes a large number of revolutions in the magnetic field between two interactions. Although the formulas were derived for a set of particles with equal velocities they are readily generalized to an arbitrary velocity distribution, provided there are *no* motions along the magnetic field. The result then is, briefly, that the lines that make up the spectrum are broadened, not due to Doppler shifts but due to the relativistic dependence of the resonance frequency on the particle energy.[2–4]

When there *are* motions along the field, the transformation becomes quite elaborate and the result, as in many other relativistic instances, not beforehand foreseeable. It seems therefore desirable to have an independent derivation which, in our case, includes right from the beginning the motion along the field. This is carried out in Sec. 2. The result—the spectral and angular behavior of the radiation from particles with given velocities along and across the field, but no velocity spread—is then compared with the result from an appropriate transformation of our previous formula (Sec. 3).

Practical applications are discussed in Sec. 4, starting from a formula for a completely arbitrary distribution function of particle energies along and across the field. We then specialize to a Maxwellian distribution which is the one of main interest for laboratory experiments.

* Supported by the Office of Naval Research.
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