tionary with respect to variations of $\mathcal{V}(\mathbf{p},\beta,\alpha)$ with N_m a function of $\mathcal{V}$.

(2) The power series for $\Phi[N_m]$ in terms of time-ordered diagrams is precisely of the Goldstone form as follows by comparison of (38) with the corresponding functional at zero temperature II, Eq. (12). The present form is, therefore, the natural analog of the expansions given in II.

(3) The form (38) goes over simply into the Goldstone expansion or into its generalization for anisotropic situations as $\beta\rightarrow\infty$. It is to be recalled that the appropriate limit is

$$\lim_{\beta\rightarrow\infty,\ \mu\ \text{fixed}} A(\beta,\Omega,_\alpha)=E-\mu N. \tag{40}$$

(4) By direct calculation or with the help of the variational principle, it can be shown that the internal energy is given by

$$E=(\partial/\partial\beta)(\beta A)_{\Omega,\alpha}=\sum\nolimits_{\mathbf{p}} p^2N_m(\mathbf{p},\beta,\alpha)+\Phi[N_m]. \tag{41}$$

From this we find

$$\begin{aligned}\epsilon(\mathbf{p},\beta,\alpha)&=p^2+\mathcal{V}(\mathbf{p},\beta,\alpha)\\&=\delta E/\delta N_m(\mathbf{p},\beta,\alpha).\end{aligned} \tag{42}$$

(5) The entropy is given by the even simpler expression

$$\begin{aligned}S&=\beta^2(\partial A/\partial\beta)_{\Omega,\mu}\\&=-\sum\nolimits_p\{(1-N_m)\ln(1-N_m)+N_m\ln N_m\}.\end{aligned} \tag{43}$$

Equations (41) and (43), together with the equations for $N_m(\mathbf{p},\beta,\alpha)$, are the fundamental equations of the Landau theory. All known consequences of this theory can, therefore, be derived from our equations.

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Cyclotron Radiation from Relativistic Particles with an Arbitrary Velocity Distribution*

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A formula is derived which represents the spectral and angular behavior of cyclotron radiation emitted by a relativistic particle moving in a constant magnetic field and having arbitrary velocity components parallel and perpendicular to the field. This formula is used to describe the emission of an assembly of particles having a Maxwellian distribution function. All broadening mechanisms of practical importance—collisions, relativistic mass variability and Doppler effects—are included and discussed.

1. INTRODUCTION

IT was shown in a previous article[1] (hereinafter called I) that the influences of "collisions" of the emitting particles with disturbers can be accounted for by a formula of the Lorentz type, even if the particle energies have relativistic values. It turned out that this result holds whenever the particle completes a large number of revolutions in the magnetic field between two interactions. Although the formulas were derived for a set of particles with equal velocities they are readily generalized to an arbitrary velocity distribution, provided there are *no* motions along the magnetic field. The result then is, briefly, that the lines that make up the spectrum are broadened, not due to Doppler shifts but due to the relativistic dependence of the resonance frequency on the particle energy.[2–4]

When there *are* motions along the field, the transformation becomes quite elaborate and the result, as in many other relativistic instances, not beforehand foreseeable. It seems therefore desirable to have an independent derivation which, in our case, includes right from the beginning the motion along the field. This is carried out in Sec. 2. The result—the spectral and angular behavior of the radiation from particles with given velocities along and across the field, but no velocity spread—is then compared with the result from an appropriate transformation of our previous formula (Sec. 3).

Practical applications are discussed in Sec. 4, starting from a formula for a completely arbitrary distribution function of particle energies along and across the field. We then specialize to a Maxwellian distribution which is the one of main interest for laboratory experiments.

* Supported by the Office of Naval Research.

[1] L. Oster, Phys. Rev. **119**, 1444 (1960).

[2] B. A. Trubnikov, Doklady Akad. Nauk SSSR **118**, 913 [translation: Soviet Phys.-Doklady **3**, 136 (1958)].

[3] B. A. Trubnikov and V. S. Kudryatsev, *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958* (United Nations, Geneva, 1958), Paper A-5, P/2213, Vol. 31, p. 93.

[4] D. B. Beard, Phys. Fluids **2**, 379 (1959); **3**, 45 (1960).

2. LINE CONTOUR IN THE CASE OF PARTICLE MOTIONS ALONG AND ACROSS THE MAGNETIC FIELD

In this section, we discuss in close analogy to the procedure developed in I the contour of cyclotron lines emitted by particles which have velocities along *and* across the magnetic lines of force. All particles move together in both directions. Hence, the broadening of the otherwise infinitely narrow lines at multiples of the resonance frequency is solely due to collisions that break up the emitted wave train into finite sections, of lengths corresponding to durations Δt with an average time T_0 between collisions.

Since the calculation duplicates most of the corresponding derivations carried out in I, we do not reproduce the details here, but indicate briefly the differences.

The calculation goes as follows: One first writes down the solutions of the Liénard-Wiechert potentials for the radiation field, i.e., the electric and magnetic field vectors $\mathbf{E}$ and $\mathbf{H}$, from which the emission follows via Poynting's vector. One then Fourier-analyzes the components of $\mathbf{E}$ ($\mathbf{H}$ gives no new information), which yields the spectral distribution in the case that the wave train persists over a time interval Δt. Finally, the resulting integrals must be averaged according to the statistical distribution of Δt's.

Using the same system of reference as in I, we have now for the velocity vector $\mathbf{v}$:

$$\mathbf{v}=(v_0\cos\phi, v_0\sin\phi, v_1), \tag{1}$$

with the corresponding expressions for $\mathbf{n}^*$ and s, as defined in I.

Phase angle ϕ and time are connected, as in I, by the relation

$$\phi=\omega_0 t. \tag{2}$$

The relativistic β's are introduced, as usual, by the relation

$$\beta_0=v_0/c,\quad \beta_1=v_1/c, \tag{3}$$

for the perpendicular and parallel component of the velocity vector. The cyclotron frequency may be written in terms of the rest mass m_0 of the electron:

$$\omega_0=\omega_{00}(1-\beta_0^2)^{\frac{1}{2}},\quad \omega_{00}\equiv eH_0/m_0c. \tag{4}$$

Actual time t^* and retarded time t that enter the Fourier analysis are now connected by the relation

$$dt^*=sdt=(1-B_0\cos\phi-\beta_1\cdot\sin\theta)dt, \tag{5}$$

which leads to

$$t^*=t-B_0\sin\phi/\omega_0-t\beta_1\sin\theta. \tag{6}$$

Here, B_0 is defined by

$$B_0=\beta_0\cos\theta. \tag{7}$$

Introducing the abbreviations

$$2\phi_0\equiv\omega_0\Delta t,\quad \Omega\equiv\omega/\omega_0, \tag{8}$$

and, in addition,

$$\lambda=1-\beta_1\sin\theta, \tag{9}$$

we find for the Fourier components of the electric field vector

$$E_x(\Omega)=2\frac{ev_0}{\pi c^2}\sin\theta(\sin\theta-\beta_1)I_1(\Omega), \tag{10}$$

$$E_y(\Omega)=2\frac{ev_0}{\pi c^2}I_2(\Omega), \tag{11}$$

$$E_z(\Omega)=-2\frac{ev_0}{\pi c^2}\cos\theta(\sin\theta-\beta_1)I_1(\Omega). \tag{12}$$

The integrals I_1 and I_2 are given by the relations, corresponding to I, Eqs. (36) and (37),

$$I_1(\Omega)=-\frac{\cos\phi_0\sin[\Omega(\lambda\phi_0-B_0\sin\phi_0)]}{\lambda-B_0\cos\phi} +\frac{\Omega}{\lambda}\int_0^{\phi_0}\cos\phi\cos[\Omega(\lambda\phi-B_0\sin\phi)]d\phi, \tag{13}$$

and

$$I_2(\Omega)=-\frac{\sin\phi_0\cos[\Omega(\lambda\phi_0-B_0\sin\phi_0)]}{\lambda-B_0\cos\phi} -\Omega\int_0^{\phi_0}\sin\phi\sin[\Omega(\lambda\phi-B_0\sin\phi)]d\phi. \tag{14}$$

Equations (13) and (14) show the effects of the inclusion of a motion parallel to the magnetic field: while the general form of the Fourier integrals is exactly the same as in the case $v_1=0$, the velocity v_1 enters in the combinations $\lambda=1-\beta_1\sin\theta$ as a factor multiplying ϕ whenever it appears in a linear form.

The next step consists in applying the expansion analysis of I, Sec. 11, to the integrals (13) and (14). It is easy to see, going through the mathematical details, that the appearance of the quantity λ changes the conclusion reached in I, Sec. II, only to a very minor extent, that is, the definition of Δ_n now includes the quantity λ:

$$\Delta_n(\lambda\Omega)=-\int_{-\phi_0}^{+\phi_0}\sin(\lambda\Omega\phi)\sin(n\phi)d\phi, \tag{15}$$

or, approximately,

$$\Delta_n\approx-\sin[(\lambda\Omega-n)\phi_0]/(\lambda\Omega-n), \tag{16}$$

with the new condition for the validity of the approximation

$$|\lambda\Omega-n|\ll n. \tag{17}$$

The physical meaning of Eq. (17) is the same as the one of Eq. (I, 105): The representation of the spectrum with the help of functions of the type of Eq. (16) is restricted to frequencies close to the resonance. How-

ever, the frequencies close to resonance as observed from a particle moving with velocity v_1 along the magnetic field are Doppler shifted by an amount λ with respect to frequencies as seen emitted from a particle with $v_1=0$. As we will discuss in more detail in the next section, the transformation of Ω between the two systems leads to the substitution of $\lambda\Omega$ for Ω.

Expressing the integrals, Eqs. (13) and (14), in terms of Bessel functions $J_r(\Omega B)$, we obtain[5] (n designates the number of the harmonic):

$$-I_1^{(n)}(\Omega B_0)=\frac{n}{\lambda B_0}J_n(\Omega B_0)\Delta_n, \tag{18}$$

and

$$I_2^{(n)}(\Omega B_0)=\Omega\frac{d[J_n(\Omega B_0)]}{d(\Omega B_0)}\Delta_n. \tag{19}$$

Writing again n instead of Ω in the amplitude functions, we obtain the final result:

$$\begin{aligned}\bar{S}_n(\omega)d\omega do &= \frac{e^2\beta_0^2\omega_0^2}{2\pi^3c}n^2\left\{\frac{(\sin\theta-\beta_1)^2}{\beta_0^2\cos^2\theta}J_n^2(n\beta_0\cos\theta)\right.\\ &\left.+\left[\left(\frac{dJ_n(x)}{dx}\right)^2\right]_{x=n\beta_0\cos\theta}\right\}\{[\omega(1-\beta_1\sin\theta)\\ &-n\omega_{00}(1-\beta_0^2-\beta_1^2)^{\frac{1}{2}}]^2+T_0^{-2}\}^{-1}d\omega do. \end{aligned}\tag{20}$$

Equation (20) gives the emission of a particle with a velocity between v_0 and v_0+dv_0 perpendicular to the magnetic field and v_1 and v_1+dv_1 along the field, into a solid angle element do in unit distance and in the direction given by the angle θ, and in a frequency interval $d\omega$ near the nth resonance.

Equation (20) must be compared with our previous result [I, Eq. (123)] for a particle stationary with respect to the magnetic field ($\beta_1=0$), which read

$$\begin{aligned}\bar{S}_n'(\omega')d\omega' do' &= \frac{e^2\beta_0'^2\omega_0'^2}{2\pi^3c}n^2\left\{\frac{\tan^2\theta'}{\beta_0'^2}J_n^2(n\beta_0'\cos\theta')\right.\\ &\left.+\left(\frac{dJ_n(x')}{dx'}\right)_{x'=n\beta_0'\cos\theta'}\right\}\\ &\times\{[\omega'-n\omega_{00}(1-\beta_0'^2)^{\frac{1}{2}}]^2+T_0'^{-2}\}d\omega' do'. \end{aligned}\tag{21}$$

It might be worthwhile to summarize here the basic features of Eq. (21) and—*mutatis mutandis*—Eq. (20). First, we find that the spectral and angular characteristics are independent of each other [except for the contribution of the Doppler effect in Eq. (20) which, of course, depends on θ]. Second, the influence of collisions which are described by the well-known Lorentz term, the second bracket in Eqs. (20) and (21), does not affect the relative intensity of the various harmonics, given by the remaining factors. As a matter of fact, the bracket with the Bessel functions that determines essentially the relative intensities (and the angular behavior) is a feature common to all equations describing the emission from charged particles on periodic orbits. This result can be recovered even in treatments which make no use of relativity, such as that by Schott[6] who is usually credited with the first derivation. A thorough discussion from a modern point of view can be found in Landau and Lifshitz's well-known text.[6]

The comparison of Eqs. (20) and (21) reveals that the finite velocity along the field, given by β_1, enters the final formula in at first glance surprisingly few terms. In order to understand better the physical processes involved and, also, to obtain an independent check on the correctness of our result, we discuss in the next section its transformation behavior. To be more precise, we will investigate the transformation of the reference frame K' in which Eq. (21) holds to a frame K in which Eq. (20) is valid.

3. COMPARISON OF THE FORMULAS FOR $\beta_1=0$ AND FOR FINITE β_1

Equation (21) was derived for a particle whose velocity relative to the velocity of light is

$$\boldsymbol{\beta}'=(\beta_0'\cos\phi',\beta_0'\sin\phi',0). \tag{22}$$

The first thing to remember is that the transformation which leads to a value β_1 for the z component of the velocity, yields instead of Eq. (22)

$$\boldsymbol{\beta}=[\beta_0'(1-\beta_1^2)^{\frac{1}{2}}\cos\phi',\beta_0'(1-\beta_1^2)^{\frac{1}{2}}\sin\phi',\beta_1], \tag{23}$$

while the phase angle

$$\phi'=\omega_0't'=\omega_0t=\phi \tag{24}$$

is invariant.

Hence, in order to end up in K with a transverse velocity given by Eq. (1), we must start in K' with

$$\beta_0'=\beta_0/(1-\beta_1^2)^{\frac{1}{2}}. \tag{25}$$

Equation (24) is due to the fact that the (particle's) time and the orbital frequency transform as follows:

$$t'=t(1-\beta_1^2)^{\frac{1}{2}}; \tag{26}$$

in particular,

$$T_0'=T_0(1-\beta_1^2)^{\frac{1}{2}}, \tag{27}$$

[5] Equations (18) and (19) are written in a slightly more careful way than in I, namely, making a distinction between the number n of the harmonic and Ω, the frequency in units of the relativistic gyrofrequency. This is necessary for the correct performance of transformations of the reference frame, as discussed in the next section.

[6] A. G. Schott, *Electromagnetic Radiation* (University Press, Cambridge, England, 1912), p. 109; Landau and Lifshitz, *Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951).

and

$$\omega_0' = \omega_0/(1-\beta_1^2)^{\frac{1}{2}}. \quad (28)$$

All this is rather trivial.

From the same principles one finds for the observer's angles θ' and θ, respectively,[7]

$$\sin\theta' = (\sin\theta - \beta_1)/\lambda, \quad \cos\theta' = (1-\beta_1^2)^{\frac{1}{2}}\cos\theta/\lambda, \quad (30)$$

with λ given by Eq. (9).

The combination B_0 [see Eq. (7)] then transforms as follows:

$$B_0' = B_0/\lambda = \beta_0 \cos\theta/\lambda. \quad (31)$$

As a first application, we consider Poynting's vector for the "white light," i.e., before the Fourier analysis is carried out. Obviously, the integral over the whole sphere should not depend on the reference frame. This condition leads to

$$\int S'(\theta')do' = \int S(\theta)do = 2\pi \int S(\theta)\cos\theta d\theta. \quad (32)$$

do' and do are the elements of solid angle in K' and K, respectively. Poynting's vector is given in K by the relation

$$|S| = \frac{c}{4\pi}E^2 = \frac{e^2\beta_0^2\omega_0^2}{4\pi c}(\lambda - B_0\cos\phi)^{-6} \times \{(\sin\theta-\beta_1)^2\sin^2\phi + (B_0 - \lambda\cos\phi)^2\}. \quad (33)$$

The corresponding formula valid in K' [I, Eq. (20)] is obtained from Eq. (33) by putting $\beta_1=0$ and substituting the primed quantities for the unprimed:

$$|S'| = \frac{e^2\beta_0'^2\omega_0'^2}{4\pi c}(1-B_0'\cos\phi)^{-6} \times \{\sin^2\theta'\sin^2\phi + (B_0' - \cos\phi)^2\}. \quad (34)$$

In transforming Eq. (34) into Eq. (35) we have to consider the transformation formulas for the electromagnetic field components,[8] that is, we have to transform first $\mathbf{E}'(\theta')$ into $\mathbf{E}(\theta')$ from the relations

$$E_x(\theta') = [E_x'(\theta') + \beta_1 H_y'(\theta')](1-\beta_1^2)^{-\frac{1}{2}}, \quad (35)$$

$$E_y(\theta') = (E_y' - \beta_1 H_x')(1-\beta_1^2)^{-\frac{1}{2}}, \quad (36)$$

$$E_z(\theta') = E_z'(\theta'). \quad (37)$$

Finally, we have to express E_x, E_y, and E_z in terms of θ instead of θ': this takes care of the fact that the moving observer sees in his reference frame the radiation emerge from a different angle, than the observer at rest sees it in his frame. The detailed calculation leads to the result

$$\mathbf{E}'(\theta) = \lambda(1-\beta_1^2)^{-\frac{1}{2}}\mathbf{E}(\theta). \quad (38)$$

This verifies Eq. (32), since the element of solid angle is transformed according to the relation[9]

$$do' = \cos\theta' d\theta' d\phi' = \frac{1-\beta_1^2}{\lambda^2}\cos\theta d\theta d\phi = \frac{1-\beta_1^2}{\lambda^2}do. \quad (39)$$

Equation (37) is also in accordance with the general formula for the transformation of wave amplitudes.[10]

The corresponding calculation for the Fourier components needs some more reflection. First, we remember from Parseval's theorem that

$$\pi\int_0^\infty S(\omega)d\omega = \int_{\Delta t} S(t)dt. \quad (40)$$

This holds, of course, for the primed quantities as well. Taking the angular dependence into account at the same time, we may write for the emission per unit time

$$\int \frac{S'(\theta',\omega')}{\Delta t'}d\omega' do' = \int \frac{S(\theta,\omega)}{\Delta t}d\omega do. \quad (41)$$

The difficulties arise from the definition of unit time: As a matter of fact, the time intervals Δt and $\Delta t'$ do *not* transform according to Eq. (26). This can be seen in the following way: by means of the Fourier analysis, the finite wave train emerging from the origin during a time $\Delta t'$ as measured in K' is expressed in terms of infinite wave trains with various frequencies ω'. The wave trains are then measured during the time $\Delta t'$ *simultaneously* at each point of the sphere. Going over to the reference frame K, we have the corresponding condition of measuring the wave trains with different frequencies ω *simultaneously* over the sphere in K. Carrying out the transformation, we have to assure the respective simultaneity.

Let us consider a wave train in K belonging to a given frequency ω and travelling in a direction θ. At a point A on the sphere, we integrate over the intensities during the time interval $\Delta t = t_1 - t_0$. This, however, corresponds to an integration of the wave train over the distance ΔR in direction θ, ΔR and Δt being connected by the relation

$$\Delta R = c\Delta t. \quad (42)$$

Equation (42) holds of course in the same form in K' as well. Thus, the integration interval Δt transforms as ΔR. For the latter,[10] the transformation rule reads

$$\Delta R' \propto \Delta t' = (1-\beta_1^2)^{\frac{1}{2}}\Delta t/\lambda. \quad (43)$$

From Eq. (43) our previous formula, Eq. (26), follows at once for $\lambda = 1$ (orbital plane).

The remaining tasks are now easily accomplished. From the relativistic Doppler effect[9] we obtain at once the transformation rules for frequency and frequency

[7] W. Pauli, *Relativitätstheorie* (B. G. Teubner, Leipzig, 1921), Sec. 6.

[8] W. Pauli, reference 7, Sec. 28.

[9] W. Pauli, reference 7, Sec. 6.

[10] W. Pauli, reference 7, Sec. 32.

interval:

$$\omega' = \lambda\omega/(1-\beta_1^2)^{\frac{1}{2}}, \quad d\omega' = \lambda d\omega/(1-\beta_1^2)^{\frac{1}{2}}. \tag{44}$$

Ω, the frequency in units of the cyclotron frequency, transforms as follows:

$$\Omega' = \omega'/\omega_0' = \lambda\Omega. \tag{45}$$

Thus, the combination

$$\Omega' B_0' = \Omega B_0 \tag{46}$$

is invariant.

Also invariant is the combination of differentials and Δt in Eq. (41):

$$d\omega' do' / \Delta t' = d\omega do / \Delta t, \tag{47}$$

which is found from Eqs. (39), (43), and (44). What we have to verify therefore is the invariance of $S(\omega)$.

First, we have for the integrals I_1 and I_2

$$I_1'(\Omega') = \lambda^2 I_1(\Omega), \tag{48}$$

and

$$I_2'(\Omega') = \lambda I_2(\Omega). \tag{49}$$

This is easily shown from Eqs. (13) and (14). If we use the representation with the help of Bessel functions [Eqs. (18) and (19)], we must consider the fact that according to Eq. (16)

$$\Delta_n'(\Omega') = \Delta_n(\Omega), \tag{50}$$

and that n is of course, unchanged, while Ω transforms according to Eq. (48). Allowing for the transformation of the field components [see Eqs. (35)–(37)] yields, after some algebraic manipulations, the result that I_1 and I_2 must each be multiplied with a factor

$$(1+\beta_1 \sin\theta')/(1-\beta_1^2)^{\frac{1}{2}}. \tag{51}$$

That transforms into

$$(1-\beta_1^2)^{\frac{1}{2}}/\lambda \tag{52}$$

in K. From here it is obvious that the combination

$$\beta_0'^2 \frac{(1+\beta_1 \sin\theta')^2}{1-\beta_1^2} [(\sin^2\theta') I_1'^2(\Omega') + I_2'^2(\Omega')] \tag{53}$$

transforms into

$$\beta_0^2 [(\sin\theta - \beta_1)^2 I_1^2(\Omega) + I_2^2(\Omega)], \tag{54}$$

which proves our point.

4. THE EMISSION FROM AN ASSEMBLY OF PARTICLES WITH A SPREAD IN VELOCITIES

We are now ready to make use of our principal result, Eq. (20), for the determination of the radiation pattern emitted by an actual laboratory plasma.

It might be worthwhile to state at this point once again the approximations made: First, it is obvious that self-absorption is not considered. This is legitimate for most cases of interest. It is even possible to account for this effect by going over from the emission to the absorption coefficient and the optical depth; this is a standard procedure, but restricted to legitimate applications of Kirchhoff's law.

The second approximation concerns the refractive behavior that affects not only the ray paths, but also the amount of local emission (source function): in our treatment, the refractive index is set equal to unity. This is also legitimate in most cases of not too dense plasmas. The calculation of refractive indices and, even more, their correct application to radiation formulas is still in a rather unsatisfactory state, so that it seems wise to omit them here.[11]

We also neglect the energy loss of the particle due to the emitted radiation. This again is no practical restriction, since the classical electrodynamic treatment breaks down, if the emitted quanta have energies comparable to the total energy of the particle.[12]

Finally, there is the assumption basic for the validity of the presented collision treatment and discussed in all detail in I, that the radiating particle accomplishes many revolutions between two disturbing collisions.

Our starting point is Eq. (20), which gave the emission of a particle as a function of its velocity along the magnetic field $\sim\beta_1$ and across the field $\sim\beta_0$, and other variable quantities. The emission of N_0 particles which have an arbitrary distribution function

$$f(\beta_0,\beta_1) d\beta_0 d\beta_1, \tag{55}$$

with the normalization

$$\iint_B f(\beta_0,\beta_1) d\beta_0 d\beta_1 = 1 \tag{56}$$

(B is the volume of the β-space that corresponds to the total volume in velocity space), is then simply

$$\langle S(\omega)\rangle_{\beta_0,\beta_1} d\omega do = N_0 \iint S(\omega,\beta_0,\beta_1) \times f(\beta_0,\beta_1) d\beta_0 d\beta_1 d\omega do. \tag{57}$$

We investigate the effects of a spread in velocities of the emitting particles in the case of a relativistic Maxwellian distribution in all three coordinates. Here, we make implicitly the assumption that the presence of the magnetic field does not change appreciably the behavior of the gas. This should be true for all laboratory plasmas. As a matter of fact, too strong magnetic fields again make the use of quantum mechanics instead of classical electrodynamics necessary[12]: The Larmor radius must be at least larger than the de Broglie wavelength of the electrons.

[11] D. B. Beard, Phys. Fluids **2**, 379 (1959), derives refractive indices for the relativistic plasma with magnetic fields, but restricts his discussion to first order effects in β_1. See also the article by K. C. Westfold, Astrophys. J. **130**, 241 (1959).

[12] A. Sokolov, Suppl. Nuovo cimento **3**, 743 (1956).

In terms of the momentum

$$\mathbf{p}=m_0\mathbf{v}/(1-\beta^2)^{\frac{1}{2}}, \quad (58)$$

with β defined again by the relation

$$\beta^2=\beta_x^2+\beta_y^2+\beta_z^2=\beta_0^2+\beta_1^2, \quad (59)$$

the equilibrium distribution function reads[13]

$$f(\mathbf{p})d\mathbf{p}=N_0f_0e^{-\epsilon/KT}d\mathbf{p}. \quad (60)$$

The relativistic energy ϵ is given by

$$\epsilon=m_0c^2(1+p^2/m_0c^2)^{\frac{1}{2}}. \quad (61)$$

K is the Boltzmann constant, T the temperature, N_0 the total number of electrons per unit volume. f_0, the normalization factor, is found from the integral

$$1/f_0=\int e^{-\epsilon/KT}d\mathbf{p} \quad (62)$$

to be

$$1/f_0=-i2\pi^2m_0cKTH_2^{(1)}\left(i\frac{m_0c^2}{KT}\right). \quad (63)$$

Here, $H_2^{(1)}(ir)$ is the Hankel function of the first kind and order two. We can check Eq. (63) by going over to the limit $\beta\ll1$. Then, the distribution function, Eq. (56), must represent the normal Maxwellian distribution. With the well-known nonrelativistic limit of ϵ,

$$\epsilon\approx m_0c^2(1+\tfrac{1}{2}\beta^2), \quad (64)$$

the exponential function in Eq. (60) becomes

$$\exp\left[\frac{m_0c^2}{KT}\right]\exp\left[-\frac{m_0v^2}{2KT}\right]. \quad (65)$$

Next, we expand the Hankel function for the nonrelativistic case

$$r\equiv m_0c^2/KT\gg1. \quad (66)$$

The calculation yields[14] for f_0:

$$1/f_0\approx m_0^3(2\pi KT/m_0)^{\frac{3}{2}}\exp[-m_0c^2/KT]. \quad (67)$$

Hence, in the nonrelativistic limit, Eq. (60) represents the normal Maxwellian distribution.

In order to maintain our previous terminology, we transform $d\mathbf{p}$ into $d\boldsymbol{\beta}$. The calculation gives, after some algebraic transformations,

$$f(\boldsymbol{\beta})d\boldsymbol{\beta}=N_0f_0\exp\left[-\frac{m_0c^2}{KT}[1+\beta^2/(1-\beta^2]\right]\frac{c^3m_0^3}{(1-\beta^2)^{\frac{5}{2}}}d\boldsymbol{\beta}, \quad (68)$$

with f_0 given by Eq. (63).

For our purpose, the appropriate reference frame in velocity space is cylindrical, that is,

$$d\boldsymbol{\beta}=2\pi\beta_0d\beta_0d\beta_1. \quad (69)$$

The limits of integrations are determined from Eq. (59) to be

$$\int d\boldsymbol{\beta}=2\pi\int_0^1\beta_0d\beta_0\int_{-(1-\beta_0^2)^{\frac{1}{2}}}^{+(1-\beta_0^2)^{\frac{1}{2}}}d\beta_1. \quad (70)$$

From Eqs. (20), (63), (68) and (70) we find for the radiation of N_0 particles in the nth harmonic:

$$\begin{aligned}\langle\bar{S}_n(\omega)\rangle_{\beta_0,\beta_1}d\omega do
&=N_0f_0\frac{e^2m_0^3c^2}{\pi^2}\frac{\beta_0^3\omega_{00}^2}{(1-\beta_0^2-\beta_1^2)^{\frac{3}{2}}}n^2\\
&\times\left\{\frac{(\sin\theta-\beta_1)^2}{\beta_0^2\cos^2\theta}J_n^2(n\beta_0\cos\theta)+\left(\frac{dJ_n(x)}{dx}\right)^2_{x=n\beta_0\cos\theta}\right\}\\
&\times\{[\omega(1-\beta_1\sin\theta)-n\omega_{00}(1-\beta_0^2-\beta_1^2)^{\frac{1}{2}}]^2+T_0^{-2}\}^{-1}\\
&\times\exp\left\{-\frac{m_0c^2}{KT}(1-\beta_0^2-\beta_1^2)^{-\frac{1}{2}}\right\}d\omega do\\
&\times\int_0^1d\beta_0\int_{-(1-\beta_1^2)^{\frac{1}{2}}}^{+(1-\beta_1^2)^{\frac{1}{2}}}d\beta_1. \quad (71)\end{aligned}$$

Before we comment on Eq. (71), let us briefly note the nonrelativistic limit. The expansion of the distribution function for nonrelativistic velocities (affecting the exponential function and f_0) has been given already in Eqs. (65) and (67). The expansion of the Bessel functions leads to the following result: Since the order is proportional to the number of the harmonic, we can neglect all higher harmonics, that is $n>1$, and find[15]

$$J_1(\beta_0\cos\theta)\approx\tfrac{1}{2}\beta_0\cos\theta, \quad (72)$$

and

$$\left(\frac{dJ_1}{dx}\right)_{x=\beta_0\cos\theta}\approx\tfrac{1}{2}. \quad (73)$$

The bracket with the Bessel functions in Eq. (71), thus, becomes

$$\{\ \}\approx\tfrac{1}{4}\{2-\cos^2\theta-2\beta_1\sin\theta\}, \quad (74)$$

while the Lorentz term, the second large bracket in Eq. (71), reads

$$\{\ \}^{-1}=\{[\omega(1-\beta_1\sin\theta)-\omega_{00}]^2+T_0^{-2}\}^{-1}. \quad (75)$$

The integration is extended over all velocity components $v_0=c\beta_0$ from 0 to ∞, and over all components $v_1=c\beta_1$

[13] W. Pauli, reference 7, Sec. 49.

[14] Jahnke-Emde, *Tables of Higher Functions* (B. G. Teubner, Leipzig, 1948), p. 136.

[15] Jahnke-Emde, reference 14, p. 127.

from $-\infty$ to $+\infty$. Hence, we obtain

$$\begin{aligned}\langle \bar{S}_1(\omega)\rangle d\omega do &\approx N_0(2\pi KT/m_0)^{-\frac{3}{2}}e^2v_0{}^3\omega_{00}{}^2c^{-3}(1-\tfrac{1}{2}\cos^2\theta-\beta_1\sin\theta)\\ &\times\{[\omega(1-\beta_1\sin\theta)-\omega_{00}]^2+T_0{}^{-2}\}^{-1}\\ &\times\exp[-m_0(v_0{}^2+v_1{}^2)/2KT]d\omega do\\ &\times\int_0^\infty dv_0\int_{-\infty}^{+\infty}dv_1. \qquad (76)\end{aligned}$$

This result has been previously demonstrated.[16]

Equation (71) may be rewritten for numerical purposes by going over from cylindrical coordinates in velocity space to spherical coordinates. For this purpose, one writes, for instance,

$$\beta_0{}^2+\beta_1{}^2=\rho^2, \qquad (77)$$

with

$$\beta_0=\rho\cos\delta,\quad \beta_1=\rho\sin\delta, \qquad (78)$$

and obtains the following limits of integration:

$$\int_0^1 d\rho,\quad \int_0^{2\pi}d\delta. \qquad (79)$$

The resulting form of Eq. (71) is straightforward and can be omitted here.

A detailed discussion of Eq. (71) cannot be made without elaborate numerical calculations. The main features, however, are clearly visible: the broadening of the originally sharp lines (the "natural line width" being replaced by the collision parameter, $T_0{}^{-2}$) is due to two, to a certain extent, independent factors, namely, the mass variability and Doppler shifts. This has been pointed out previously.[17] Both are equally important and enter Eq. (71) mainly through the Lorentz term. Transverse Doppler effect and mass variability are both connected in the square-root factor multiplying ω_0. In the nonrelativistic limit, ω_0 is a constant and we are left with the normal Doppler effect of first order in β_1.

The feature particular to cyclotron radiation is that the velocity spread affects also the *intensity* of the single lines, represented mainly by the term including the Bessel functions. The corresponding behavior in optics would result from temperature fluctuations of the emitting region. Again, both β_0 and β_1 must be taken into account.

A third effect which takes part in the broadening of the lines, namely the influence of collisions, is discussed in all detail in I. The practical problem is the derivation of an appropriate collision time T_0; in our treatment we have succeeded in separating this problem from the general derivation of the cyclotron radiation formulas. For practical applications, this has little importance, since the collision broadening is much less than the broadening by the purely relativistic effects.

For application in thermofusion devices, the distribution function, Eq. (60), might not quite be adequate,[18] since here the particles have (or are supposed to have) in general no appreciable velocity components in the direction of the magnetic field. Hence, at least for a certain time, $\beta_1\ll\beta_0$.[19] As a first order approximation, one might put $\beta_1=0$ in Eq. (71) and obtains the broadening of the single lines that is due only to the relativistic mass variability. A preliminary calculation showed that already for $\beta_0=0.9$ the first cyclotron lines are broadened to such an extent, that no single lines are observable.

[16] L. Oster, Phys. Rev. **116**, 479 (1959), Eq. (49). For a term to term comparison, one must first notice that in this reference the integration over β_0 has been carried out, resulting in an average value $\langle v_0{}^2\rangle$. Secondly, the condition "frequency close to resonance," i.e., $|\omega-\omega_0|\ll\omega$ and ω_0, implies that the Doppler shift in Eq. (76) can be applied to either ω or ω_0. Finally, the above-mentioned reference did not include the aberration represented by the term $(-\beta_1\sin\theta)$ in the expansion of the bracket with the Bessel function, Eq. (75).

[17] For instance by D. B. Beard, reference 4. Our formula, however, is valid without restrictions as to the total particle energy (as long as quantum effects can be neglected) as well as to the ratio between "longitudinal velocity" v_1 and "transverse velocity" v_0.

[18] J. Drummond and M. N. Rosenbluth, Phys. Fluids **3**, 45 (1960). See also references 3 and 4.

[19] Obviously, the distribution function of the electrons will become isotropic after an initial period due to colllsions with ions. The duration of this initial period characterized by the condition $\beta_1\ll\beta_0$ depends of course on the magnetic geometry, the ion type, and the injection mechanism as well as on the momentum transfer cross section. However, a final thermalization apparently cannot be avoided in any thermofusion device. The equilibrium distribution of cyclotron radiation, as given by Eq. (71), might therefore be of interest even if initially the electron distribution function is close to the case $\beta_1=0$. Most recently, this case has been re-examined by D. B. Beard and M. N. Rosenbluth in communications presented to the Second Annual Meeting of the Division of Plasma Physics of the American Physical Society, Gatlinburg, Tennessee, November, 1960 (unpublished), Papers C8 and C12, respectively.