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Ferromagnetic Relaxation Caused by Interaction with Thermally Excited Magnons*

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The contribution of three-magnon processes to the relaxation rate of spin waves is investigated. Relaxation occurs through the confluence of two magnons (with the generation of a third magnon), and through the splitting of a magnon into two magnons. The relaxation rate due to the confluence process is approximately proportional to the wave number, whereas that due to the splitting process is approximately independent of the wave number. The latter contribution vanishes at frequencies higher than $\frac{2}{3}(\gamma 4\pi M)$ (γ=gyromagnetic ratio, M=saturation magnetization), and increases with decreasing frequency. The implications of the theory with respect to the observation of spin-wave instability in a rf magnetic field parallel to the dc field are discussed.

1. INTRODUCTION

IT was pointed out in a previous paper[1] that the observation of spin-wave instability in an rf magnetic field applied parallel to the dc field provides a new, convenient method for the experimental study of relaxation mechanisms in ferromagnetic materials. In this experiment (henceforth called the "parallel pumping" experiment) the wave number of the potentially unstable spin waves can be changed by varying the dc magnetic field. One thus obtains information about the variation of the spin-wave relaxation time with wave number. This information should be helpful in identifying the various relaxation mechanisms that may be operative in any given case.

In this paper the spin-wave relaxation processes arising from the dipolar interaction will be investigated. The object of the paper is to determine their contribution to the relaxation rates observed in the parallel pumping experiment. Similar problems have previously been treated in the literature,[2–6] but never without the help of certain simplifying assumptions that are not justifiable under the conditions applicable in many experiments. It is hoped that the results of this investigation will help in the interpretation of experimental data obtained from the parallel pumping experiment.

The theoretical discussion will be confined to three-particle processes. It can be shown that the relaxation rate caused by these processes is proportional to the temperature in the high-temperature limit. Higher order processes (four particles etc.) produce relaxation rates which increase with a higher power of the temperature (T^2 etc.). The experimental results on yttrium iron garnet[7] indicate that the relaxation rate contains contributions which increase proportionally to the temperature. This suggests that three-particle processes account for an appreciable part of the observed relaxation rates.

The general theoretical method used in this paper is well known. The Hamiltonian containing Zeeman energy, exchange energy, and dipolar energy is expressed in terms of the amplitudes of the normal modes (spin waves). The amplitudes of the normal modes are quantum-mechanically interpreted as creation and annihilation operators. The Hamiltonian contains a term that is quadratic in the spin-wave amplitudes and also terms that are of higher order. The eigenstates of the quadratic part of the Hamiltonian can be characterized by the occupation numbers of the various normal modes. All higher order terms in the Hamiltonian lead to transitions between the eigenstates. Only that term in the Hamiltonian which is of third order in the amplitudes of the normal modes is considered explicitly, because only this term leads to relaxation rates proportional to the temperature in the high-temperature limit.

2. GENERAL THEORY OF THE THREE-MAGNON PROCESS

It was first pointed out by Akhiezer[2] that the dipolar energy contains a contribution which is of third order in the spin-wave amplitudes. It, therefore, leads to transitions in which one magnon is absorbed and two are emitted (splitting of a magnon), and to transitions in which two magnons are absorbed and one is emitted (confluence of two magnons). In these transitions Zeeman energy is converted into dipolar and exchange energy and vice versa. The three-magnon process has also been discussed by Kasuya[3] and by Sparks and Kittel,[6] who found that the relaxation rate increases proportionally to the wave number. Akhiezer,[2] Kaganov and Tsukernik,[4] and Akhiezer, Bar'yakhtar, and Peletminskii,[5] have calculated the thermal average of the relaxation rates due to this process.

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[5] A. I. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminskii, J. Exptl. Theoret. Phys. (U.S.S.R) **36**, 216 (1959) [translation: Soviet Phys.—JETP **36**(9), 146 (1959)].
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[7] E. G. Spencer and R. C. LeCraw, Bull. Am. Phys. Soc. **5**, 297 (1960); and private communication.

Following Holstein and Primakoff[8] and Akhiezer,[2] we begin by expressing the components of the magnetization vector in terms of creation and annihilation operators a^* and a:

$$\begin{aligned} M_x+iM_y&=(2g\mu_B M_0)^{\frac{1}{2}}a^\dagger(1-g\mu_B a^\dagger a/2M_0)^{\frac{1}{2}},\\ M_x-iM_y&=(2g\mu_B M_0)^{\frac{1}{2}}(1-g\mu_B a^\dagger a/2M_0)^{\frac{1}{2}}a,\\ M_z&=M_0-g\mu_B a^\dagger a. \end{aligned}\tag{1}$$

Here, g is the spectroscopic splitting factor ($g\approx 2$), μ_B the Bohr magneton, M_0 the saturation magnetization, and the superscript dagger denotes the Hermitian adjoint. M_x, M_y, and M_z as well as $a^\dagger$ and a are functions of $\mathbf{r}$ (position within the sample) and the operators $a(\mathbf{r})$ and $a^\dagger(\mathbf{r})$ obey the commutation relations

$$a(\mathbf{r})a^\dagger(\mathbf{r}')-a^\dagger(\mathbf{r}')a(\mathbf{r})=\delta(\mathbf{r}-\mathbf{r}').\tag{2}$$

We now introduce the Fourier components a_k of $a(\mathbf{r})$ by the relation

$$a(\mathbf{r})=V^{-\frac{1}{2}}\sum_k a_k e^{i\mathbf{k}\cdot\mathbf{r}}.\tag{3}$$

Here we have implied periodic boundary conditions and the components of $\mathbf{k}$ are integer multiples of $2\pi/L$, where $L^3=V$ is the periodicity volume. The assumption of periodic boundary conditions is merely a device for simplifying the theoretical problem and has no physical justification. Ideally one should use an expansion in terms of the correct normal modes which satisfy the physical boundary conditions, rather than in terms of plane waves. Since the normal modes are very involved, this approach is not very promising. The use of periodic boundary conditions is justified if the wavelength $2\pi/k$ is much smaller than the sample dimensions. Under these conditions the boundary conditions at the surface of the sample do not play an important role and the normal modes locally resemble plane waves. By using the plane-wave expansion (3) we thus restrict the validity of our results to modes whose wavelength is much smaller than the sample dimensions. It will be seen later that for the present purposes this is not a serious restriction.

It follows readily from Eqs. (2) and (3) that the operators a_k and $a_k^\dagger$ satisfy the commutation relation

$$a_k a_{k'}^\dagger-a_{k'}^\dagger a_k=\delta_{k-k'},\tag{4}$$

where $\delta_{k-k'}$ is the Kronecker symbol.

We now express the energy of the sample in terms of the operators a_k and $a_k^\dagger$. Taking into account Zeeman, exchange, and dipolar energy, one obtains for the Hamiltonian

$$\mathcal{H}=\mathcal{H}^{(2)}+\mathcal{H}^{(3)},\tag{5}$$

where

$$\mathcal{H}^{(2)}=\sum_k\{A_k a_k^\dagger a_k+\tfrac{1}{2}B_k[\exp(-2i\phi_k)a_k a_{-k}+\text{c.c.}]\}.\tag{6}$$

$$\mathcal{H}^{(3)}=\sum_{kk'k''}(\Phi_{kk'k''}a_k a_{k'}a_{-k''}^\dagger+\text{c.c.}).\tag{7}$$

[8] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).

Here c.c. denotes the complex conjugate of the expression preceding it and

$$\begin{aligned} A_k&=g\mu_B(H+Dk^2+2\pi M\sin^2\theta_k),\\ B_k&=g\mu_B 2\pi M\sin^2\theta_k,\\ \Phi_{kk'k''}&=C\sin 2\theta_k\exp(-i\phi_k)\delta_{k+k'+k''},\\ C&=-\pi g\mu_B(2g\mu_B M/V)^{\frac{1}{2}}. \end{aligned}\tag{8}$$

θ_k and ϕ_k are the polar angles that characterize the direction of the wave vector $\mathbf{k}$, and D is a phenomenological constant characterizing the strength of the exchange coupling. In the Hamiltonian (5) we have neglected all terms that are higher than third order in the wave amplitudes a_k and $a_k^\dagger$.

Following Holstein and Primakoff,[8] we now diagonalize the quadratic part of the Hamiltonian. This is achieved by a transformation to new operators $\tilde{a}_k$ and $\tilde{a}_k^\dagger$ defined by

$$\tilde{a}_k=\lambda_k a_k+\mu_k a_{-k}^\dagger,\tag{9}$$

where

$$\begin{aligned} \lambda_k&=\cosh\tfrac{1}{2}\psi_k, \qquad \mu_k=\sinh\tfrac{1}{2}\psi_k\exp 2i\phi_k,\\ \cosh\psi_k&=A_k/\hbar\omega_k, \qquad \sinh\psi_k=B_k/\hbar\omega_k, \end{aligned}\tag{10}$$

$$\begin{aligned} \hbar\omega_k&=(A_k^2-B_k^2)^{\frac{1}{2}}\\ &=g\mu_B[(H+Dk^2)(H+Dk^2+4\pi M\sin^2\theta_k)]^{\frac{1}{2}}. \end{aligned}\tag{11}$$

It is readily shown from Eqs. (9), (10), and (4) that the transformed operators $\tilde{a}_k$ obey the same commutation relations as the untransformed operators a_k, and that the quadratic part of the Hamiltonian assumes the very simple form

$$\mathcal{H}^{(2)}=\sum_k\hbar\omega_k\tilde{a}_k^\dagger\tilde{a}_k+\text{const.}\tag{12}$$

Thus, the equations of motion as derived from the quadratic part of the Hamiltonian are separated and the "spin-wave amplitudes" $\tilde{a}_k$ are the normal coordinates of the problem. In terms of these variables the third-order term of the Hamiltonian becomes

$$\mathcal{H}^{(3)}=\sum_{kk'k''}(\tilde{\Phi}_{kk'k''}\tilde{a}_k\tilde{a}_{k'}\tilde{a}_{-k''}^\dagger+\text{c.c.}),\tag{13}$$

where

$$\begin{aligned} \tilde{\Phi}_{kk'k''}&=(\Phi_{kk'k''}\lambda_k-\Phi_{kk'k''}^*\mu_k^*)(\lambda_{k'}\lambda_{k''}+\mu_{k'}^*\mu_{k''})\\ &\quad+(\Phi_{k''k'k}\mu_{k''}-\Phi_{k''k'k}^*\lambda_{k''})\mu_k^*\lambda_{k'}\\ &=C\delta_{k+k'+k''}[\sin 2\theta_k e^{-i\phi_k}(\lambda_k-|\mu_k|)\\ &\quad\times(\lambda_{k'}\lambda_{k''}+\mu_{k'}^*\mu_{k''})-\sin 2\theta_{k''}e^{i\phi_{k''}}\\ &\quad\times(\lambda_{k''}-|\mu_{k''}|)\mu_k^*\lambda_{k'}]. \end{aligned}\tag{14}$$

The eigenstates of the quadratic part of the Hamiltonian (12) are characterized by the occupation numbers n_k of the various spin waves ($n_k=0, 1,2,\cdots$). The cubic part of the Hamiltonian leads to transitions between these eigenstates. The transition probability is, according to the well-known time-dependent perturbation theory,

$$\lambda_{if}=(2\pi/\hbar)|\mathcal{H}^{(3)}|_{if}^2\delta(E_i-E_f).\tag{15}$$

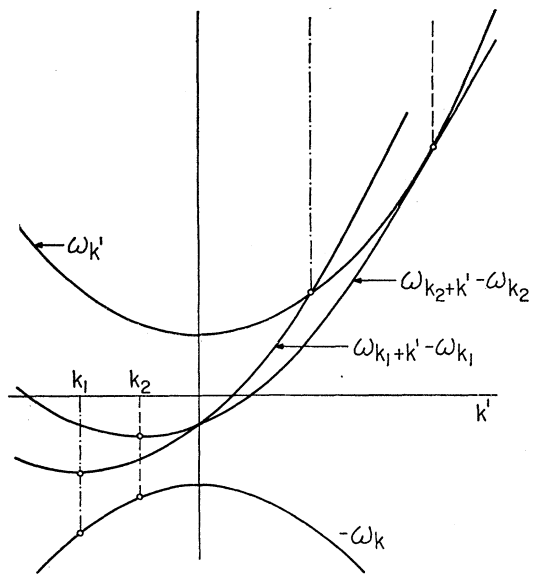

FIG. 1. Graphical solution of the interference condition for the confluence of a magnon k with another magnon k'. Note that the solution k' becomes very large as $k\to 0$.

Here E_i and E_f are the initial and final energy, respectively, and $|\mathcal{H}^{(3)}|_{if}$ is the matrix element of the perturbation energy with respect to the initial and final states. Using well-known properties of the creation and annihilation operators, one thus obtains for the rate of change of the average occupation number $\langle n_k\rangle$ of the mode k

$$\langle \dot{n}_k\rangle=\frac{2\pi}{\hbar^2}\sum_{k'k''}\{A_{kk'k''}[\langle n_k+1\rangle\langle n_{k'}+1\rangle\langle n_{k''}\rangle$$
$$-\langle n_k\rangle\langle n_{k'}\rangle\langle n_{k''}+1\rangle]\delta(\omega_k+\omega_{k'}-\omega_{k''})$$
$$+\tfrac{1}{2}A_{k''k'k}[\langle n_k+1\rangle\langle n_{k'}\rangle\langle n_{k''}\rangle$$
$$-\langle n_k\rangle\langle n_{k'}+1\rangle\langle n_{k''}+1\rangle]\delta(\omega_k-\omega_{k'}-\omega_{k''})\}. \quad (16)$$

Here

$$A_{kk'k''}=|\tilde{\Phi}_{kk'-k''}+\tilde{\Phi}_{k'k-k''}|^2. \quad (17)$$

The factor $\frac{1}{2}$ in Eq. (16) compensates for the fact that the final states are counted twice in the summation over k' and k''.

The factor of $-\langle n_k\rangle$ on the right-hand side of the rate equation (16) is the inverse of the relaxation time for the mode under consideration. For convenience it may be separated into two contributions, one arising from processes in which the magnon k combines with another magnon k' to form k'' (confluence), the other arising from processes in which k splits into two magnons k' and k'' (splitting). The two relevant relaxation rates are, from Eq. (16),

$$\tau_k^{-1}|_{\text{confl}}=2\pi\hbar^{-2}\sum_{k'k''}A_{kk'k''}(\langle n_{k'}\rangle-\langle n_{k''}\rangle)$$
$$\times\delta(\omega_k+\omega_{k'}-\omega_{k''}), \quad (18)$$
$$\tau_k^{-1}|_{\text{spl}}=\pi\hbar^{-2}\sum_{k'k''}A_{k''k'k}(1+\langle n_{k'}\rangle+\langle n_{k''}\rangle)$$
$$\times\delta(\omega_k-\omega_{k'}-\omega_{k''}). \quad (19)$$

For the confluence process, $\omega_{k''}>\omega_{k'}$. Therefore $\langle n_{k''}\rangle<\langle n_{k'}\rangle$, so that all the factors under the summation signs are necessarily positive. It is obvious that the relaxation due to confluence will tend to zero as the temperature tends to zero. On the other hand, the relaxation due to splitting will remain finite at absolute zero, provided that the interference conditions can be satisfied.

In thermal equilibrium the average occupation numbers are given by

$$\langle n_k\rangle=[\exp(\alpha\omega_k)-1]^{-1}, \quad (20)$$

where $\alpha=\hbar/k_BT$, k_B is Boltzmann's constant, and T the absolute temperature. It follows from Eq. (20) that under the side condition appropriate for the confluence process $(\omega_k+\omega_{k'}-\omega_{k''}=0)$,

$$\langle n_{k'}\rangle-\langle n_{k''}\rangle=\frac{1}{2}\frac{\sinh(\alpha\omega_k/2)}{\sinh(\alpha\omega_{k'}/2)\sinh(\alpha\omega_{k''}/2)}. \quad (21a)$$

Similarly, under the side condition appropriate for the splitting process $(\omega_k-\omega_{k'}-\omega_{k''}=0)$,

$$1+\langle n_{k'}\rangle+\langle n_{k''}\rangle=\frac{1}{2}\frac{\sinh(\alpha\omega_k/2)}{\sinh(\alpha\omega_{k'}/2)\sinh(\alpha\omega_{k''}/2)}. \quad (21b)$$

These relations are only valid subject to the side conditions with which the two expressions on the left of

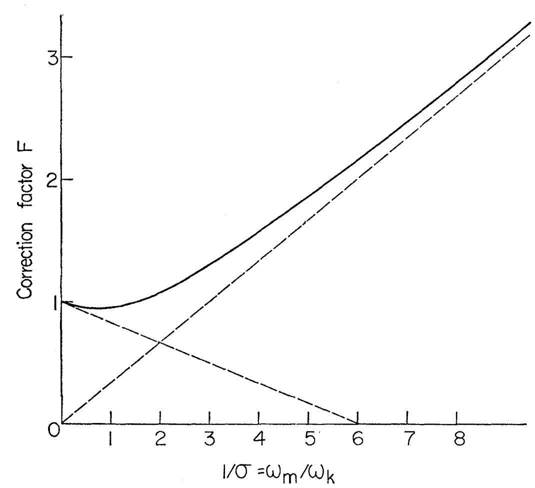

FIG. 2. Correction factor relating the theoretical result of Sparks and Kittel[6] to that derived in this paper plotted versus $1/\omega$ on a reduced scale.

Eqs. (21a, b) occur in the sums representing the relaxation rates. It should be noticed that as $\alpha\rightarrow\infty$ (i.e., $T\rightarrow 0$), the right-hand side of Eq. (21a) approaches zero, whereas the right-hand side of Eq. (21b) approaches unity.

Consider now the "high-temperature limit" in which the energy of all magnons that are involved in the relaxation process is small compared to k_BT. If the frequency ω_k of the relaxing spin wave is in the X-band region, the condition $\alpha\omega_k\ll 1$ is satisfied for all temperatures larger than approx. 1°K. For the splitting process the frequencies of the interacting spin waves must necessarily be smaller than ω_k. Thus, the high-temperature approximation is valid down to temperatures of approximately 1°K in this case. For the confluence process the frequency of the interacting spin waves can be larger than ω_k. Thus, the high-temperature approximation breaks down at somewhat higher temperatures in this case. It appears, however, that the high-temperature approximation is valid in most cases that are conveniently accessible to experimentation. We shall, therefore, concentrate on this case and replace the right-hand sides of Eqs. (21a,b) by $k_BT\omega_k/\hbar\omega_{k'}\omega_{k''}$.

3. RELAXATION DUE TO CONFLUENCE OF TWO MAGNONS

According to Eqs. (18) and (19), the relaxation rates are represented by double sums over the wave numbers k' and k''. Because of the two conservation laws for the "pseudo-momentum" and for the energy, however, the summation is actually extended only over a two dimensional surface in k' space. Before evaluating the relaxation rates explicitly we shall discuss the qualitative features of this surface by a graphical method.

For the case of confluence of magnons $\mathbf{k}$ and $\mathbf{k}'$, the

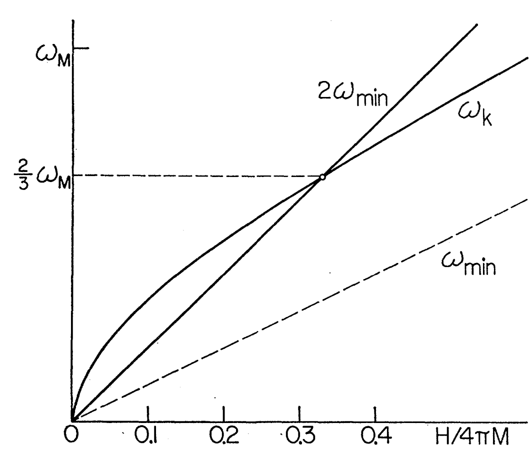

FIG. 3. Frequency ω_k of long wavelength spin waves propagating perpendicular to the dc field and two times the minimum spin-wave frequency (waves propagating parallel to the dc field) as a function of the internal magnetic field in reduced units. The splitting process is allowed only if $\omega_k>2\omega_{\min}$.

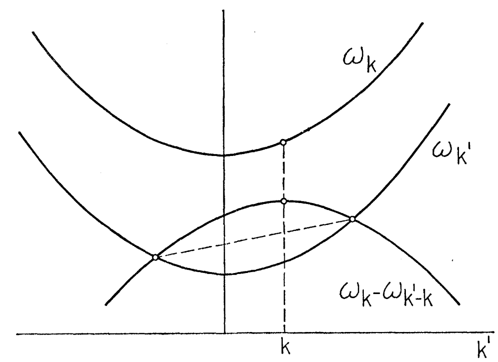

FIG. 4. Graphical solution of the interference condition for the splitting of a magnon $\mathbf{k}$ into two magnons $\mathbf{k}'$ and $\mathbf{k}-\mathbf{k}'$.

conservation laws are

$$\mathbf{k}+\mathbf{k}'=\mathbf{k}'', \qquad \omega_k+\omega_{k'}=\omega_{k''}. \tag{22a}$$

These two relations may be combined to give the "interference condition"

$$\omega_{k'}=\omega_{k+k'}-\omega_k. \tag{22b}$$

In Fig. 1 both sides of this equation are plotted as functions of k' for two fixed values of k (k_1 and k_2). The drawing represents a two-dimensional cut through a four-dimensional figure. Thus the surface in k' space, along which the interference condition is satisfied, is reduced to isolated points. The extension of this construction to the two-dimensional k' space can easily be envisaged. The parabolas shown in Fig. 1 must then be replaced by paraboloids of revolution which intersect along certain lines. The projection of these lines on the plane of the drawing is also indicated in the figure. It may be seen from Fig. 1 that with decreasing wave number k of the magnon whose relaxation rate is being calculated the wave number k' of the interacting magnon becomes larger. As k approaches zero the solution of the interference condition k' approaches infinity provided that the dispersion relation can be represented by a parabola. (Of course, this condition breaks down at high wave numbers.) It is plausible that this behavior will lead to a wave-number dependence of the relaxation rate. For details of this calculation the reader is referred to an unpublished paper of the author[9] and to a forthcoming paper by Sparks, Loudon, and Kittel.[10] It is shown in these papers that the relaxation rate vanishes proportionally to k for sufficiently small wave numbers. One obtains for spin waves propagating perpendicularly to the dc magnetic field ($\theta_k=\pi/2$)

$$\tau_k^{-1}|_{\mathrm{confl}}=\tfrac{1}{2}\pi k_BT\gamma^2MD^{-1}\omega_k^{-1}kF, \tag{23}$$

where

$$F=\gamma(H+\tfrac{1}{3}4\pi M)\omega_k^{-1}. \tag{23a}$$

[9] E. Schlömann, Raytheon Technical Memo T-233, July, 1960 (unpublished).

[10] M. Sparks, R. Loudon, and C. Kittel (to be published).

Apart from the last factor F, our result (23) is identical to that given by Sparks and Kittel,[6] who have calculated the transition probabilities using the untransformed perturbation energy (7) instead of the transformed perturbation energy (13). This procedure is valid if the spin wave k travels in the direction of the dc field or for a general direction of propagation if $H \gg 4\pi M$. In order to compare our theoretical result (23) with the experimental data obtained from the parallel pumping experiment, the correction factor F should be evaluated for that value of H for which the wave number of the unstable spin waves approaches zero (H_c in reference 1). For convenience we introduce the abbreviation

$$\sigma = \omega_k / \omega_M. \tag{24}$$

Replacing H by[11]

$$H_c = 2\pi M[(1+4\sigma^2)^{\frac{1}{2}} - 1], \tag{25}$$

the correction factor F becomes

$$F = (1+\tfrac{1}{4}\sigma^{-2})^{\frac{1}{2}} - \tfrac{1}{6}\sigma^{-1}. \tag{23b}$$

In Fig. 2, F is plotted versus σ^{-1}. It may be seen that the correction factor differs by less than ten percent from unity for $\sigma > 0.5$. In the case of yttrium iron garnet this implies that the correction is smaller than ten percent for pump frequencies larger than 5 kMc/sec. At lower pump frequencies, however, the correction factor is quite large ($F \approx 3$ for yttrium iron garnet at a pump frequency of 1 kMc/sec).

It should be stressed that the result (23) is applicable only if k is sufficiently small (and T sufficiently large). It can be shown that the next term in an expansion of τ_k^{-1} in powers of k is proportional to k^3, if the high-temperature approximation is applicable.

4. RELAXATION DUE TO SPLITTING

The conservation laws for the splitting process are

$$\mathbf{k} = \mathbf{k}' + \mathbf{k}'', \quad \omega_k = \omega_{k'} + \omega_{k''}. \tag{26a}$$

Thus the interference condition is

$$\omega_{k'} = \omega_k - \omega_{k-k'}. \tag{26b}$$

For a given value of k this equation does not always have a solution k'. Consider in particular the case that $\mathbf{k}$ approaches zero in such a way that $\theta_k = \pi/2$. A solution k' of Eq. (26b) then exists only if ω_k is at least twice as large as the lowest spin-wave frequency at the given dc magnetic field; i.e., only if

$$\omega_k \gg 2\omega_{\min}. \tag{27}$$

In Fig. 3 both sides of this inequality are plotted versus the internal dc field H. It may be seen that the condition (27) is satisfied only if

$$\omega_k \leqslant \tfrac{2}{3}\omega_M, \tag{28}$$

where $\omega_M = \gamma 4\pi M$.

The graphical solution of the interference condition for the present case is sketched in Fig. 4. Again the two sides of Eq. (26b) are plotted as functions of k' for a given value of k. The two parabolas, whose intersection determines the solution, now open toward different sides. As a consequence the solutions k' of the interference condition remain finite as k approaches zero. We may thus expect that the relaxation rate due to this process will contain a contribution which is independent of the wave number (in contradistinction to the relaxation rate due to confluence). This expectation is verified by the detailed calculation presented below.

We shall evaluate the relaxation rate due to confluence only in the limit in which $k \to 0$ and $\theta_k = \pi/2$. Under these conditions $\omega_{k'} = \omega_{k''} = \frac{1}{2}\omega_k$. Replacing the summation over k' by an integration, the relaxation rate is, according to Eq. (19),

$$\tau_k^{-1}|_{\mathrm{spl}} = \tfrac{1}{2} V k_B T \pi^{-2} \hbar^{-3} \omega_k^{-1} \int d^3k' \times A_{-k'k'k}\,\delta(\omega_k - 2\omega_{k'}). \tag{29}$$

The integral is evaluated in the Appendix, where it is shown that

$$\tau_k^{-1}|_{\mathrm{spl}} = \tau_s^{-1} f(\sigma), \tag{30}$$

where

$$\tau_s^{-1} = \tfrac{1}{16} k_B T (4\pi M)^{\frac{1}{2}} D^{-\frac{3}{2}} \gamma, \tag{31}$$

and $f(\sigma)$ is a universal function of $\sigma = \omega_k/\omega_M$. An integral representation of this function is given in the Appendix. The integral can be evaluated exactly in terms of elliptic integrals, but for the present purposes it has been found more expedient to calculate an upper and a lower bound [$f_1(\sigma)$ and $f_2(\sigma)$] which bracket the function $f(\sigma)$. The two bounds together with the esti-

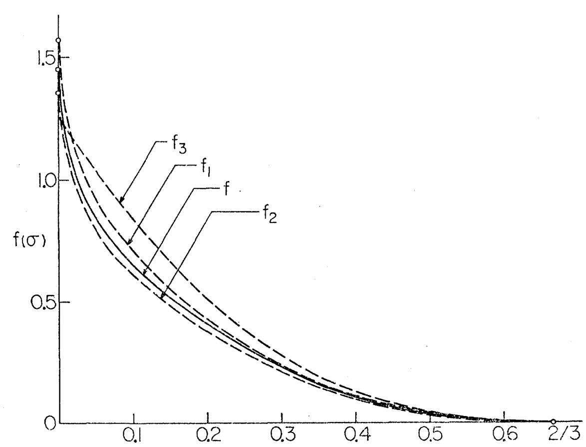

FIG. 5. Frequency dependence of the relaxation rate due to splitting. The various functions are defined in the text.

[11] Equation (28) of reference 1. Note that in reference 1, ω denotes the pump frequency, i.e., two times the spin-wave frequency.

mated behavior of $f(\sigma)$ are shown in Fig. 5. $f(\sigma)$ increases with decreasing σ (i.e., decreasing frequency), and vanishes for $\sigma>\frac{2}{3}$. For $\frac{2}{3}-\sigma\ll 1$ the function $f(\sigma)$ is approximately

$$f_3(\sigma)=3.47(\tfrac{2}{3}-\sigma)^{\frac{5}{2}}. \qquad (32)$$

This function is also shown in Fig. 5. It may be seen that it approximates $f(\sigma)$ quite well.

5. DISCUSSION

In the present section we shall briefly discuss the implication of our theoretical results with respect to the parallel pumping experiment. A theoretical analysis pertaining to a similar situation has recently been published by Kaganov and Tsukernik.[12] These authors have calculated the absorption in a rf magnetic field applied parallel to the dc field using the assumption that the spin waves are in thermal equilibrium. This assumption restricts the validity of their results to the range of relatively small rf magnetic field strengths. The effects predicted by Kaganov and Tsukernik[12] are very small and have not yet been observed experimentally.

The parallel pumping experiment described in reference 1 is carried out at high power levels. Under these conditions certain spin waves become excited. The assumption of thermal equilibrium is, therefore, not applicable for these (unstable) spin waves, although it is still applicable for almost all other spin waves.

According to the theory described by Schlömann *et al.*,[1,13] the unstable spin waves have $\theta_k=\pi/2$ and $k\neq 0$ if $H<H_c$, where H_c is given by Eq. (25). Under these conditions the assumption of periodic boundary conditions can be justified, because the wavelength is much smaller than the sample dimensions, so that the boundary conditions cannot play an important role. If $H>H_c$, however, the simple theory described in references 1 and 13 is not accurate enough because the wavelength of the unstable spin waves is now comparable with the sample dimensions. In this situation it is imperative to formulate and solve the instability problem subject to the actual boundary conditions. Since this problem has not yet been solved, we shall restrict our discussion to the case in which $H<H_c$.

The three-magnon process contributes to the relaxation rate at $k\approx 0$ only if $\sigma<\frac{2}{3}$. This relaxation rate can be determined experimentally from the critical field at $H=H_c$ (i.e., from the minimum of the critical field with respect to the dc field). If $H<H_c$, the wave number of the unstable spin waves is given by

$$H+Dk^2=H_c. \qquad (33)$$

According to references 1 and 13 ,the critical rf field is related to the relaxation rate by

$$H_{1\mathrm{crit}}=2\sigma\gamma^{-1}\tau_k^{-1}. \qquad (34)$$

Here we have taken into account that the relaxation rates τ_k^{-1} as calculated in this paper refer to energy relaxation. Therefore τ_k^{-1} equals twice the amplitude relaxation rate (usually denoted by η_k). If τ_k^{-1} varies linearly with k (as was shown in Sec. 3), the critical field will, according to Eqs. (34) and (33), vary linearly with $(H_c-H)^{\frac{1}{2}}$:

$$H_{1\mathrm{crit}}=\alpha+\beta(H_c-H)^{\frac{1}{2}}. \qquad (35)$$

Here α contains contributions from all those relaxation processes, for which the relaxation rate does not reduce to zero in the limit as $k\rightarrow 0$. Thus the splitting process contributes to α, whereas the confluence process contributes only to β.

The experimental data are very often presented in the form of an equivalent linewidth ΔH_k of the kth spin wave, which is related to the relaxation rate τ_k^{-1} by

$$\gamma\Delta H_k=\tau_k^{-1}. \qquad (36)$$

The k-independent contribution to the equivalent linewidth that arises from the splitting process is, according to Eqs. (30), (31), and (36),

$$\Delta H_{k\mathrm{spl}}=\tfrac{1}{2}\alpha_{\mathrm{spl}}\sigma^{-1}=\tfrac{1}{16}k_BT(4\pi M)^{\frac{1}{2}}D^{-\frac{3}{2}}f(\sigma), \qquad (37)$$

where $f(\sigma)$ is given in Fig. 5.

The exchange constant D is related to the Landau-Lifshitz exchange constant A by

$$D=2A/M. \qquad (38)$$

Numerical values of A and D can be inferred from the specific heat at low temperatures. According to Kunzler *et al.*,[14] for yttrium iron garnet $A=4.3\times 10^{-7}$ erg/cm; hence $D=4.4\times 10^{-9}$ oe cm². We thus find that at room temperature ($4\pi M=1800$ oe) the factor of $f(\sigma)$ on the right of Eq. (37) is approximately 0.37 oe.

It should be kept in mind that other relaxation processes also contribute to α. For instance, the presence of minute impurities of rare earth ions will lead to a k-independent relaxation rate through a mechanism discussed in detail by Kittel and his collaborators.[15] It can be shown that relaxation processes arising from the magnetoelastic energy and involving either two magnons and one phonon, or one magnon and two phonons, also contribute to α.

The second term in Eq. (35) arises from those processes for which the relaxation rate varies linearly with wave number. According to Eqs. (35), (34), (33), and (23), the contribution of the three-magnon process is

$$\beta_{\mathrm{confl}}=\tfrac{1}{4}k_BTD^{-\frac{3}{2}}F(\sigma), \qquad (39)$$

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where $F(\sigma)$ is given by Eq. (23b) and Fig. 2. It is conceivable that relaxation processes other than the three-magnon process also contribute to β.

Equation (35) may be regarded as the beginning of a power series in $(H_c-H)^{\frac{1}{2}}$. Higher order terms arise if the relaxation rate contains contributions that vary as k^2, k^3, etc., and/or if the contributions that vary as k^0 and k^1 depend explicitly on the magnetic field. The relaxation rate due to confluence of two magnons in fact depends explicitly on H [Eq. (23a)]. This has not been taken into consideration in Eqs. (35) and (39) because its effect on the critical field is equivalent to the effect of those contributions to the relaxation rate that vary as k^3. Since the latter have been neglected it is more consistent to neglect the former also. The measured values of the critical field can usually be fitted quite well by a relation of the type of Eq. (35) with α and β independent of H.

In reference 1 the experimental data were tentatively explained on the basis of a linear dependence of the relaxation rate on k^2. It has now been found that the data agree much better with Eq. (35), indicating that the relaxation rate varies in fact linearly with k. The theoretical value of β_{confl} [Eq. (39)] agrees in order of magnitude with the experimental value obtained at room temperature on yttrium iron garnet with a pump frequency of 9.42 kMc/sec. LeCraw and Spencer[16] have found that for yttrium iron garnet the relaxation rate at $k=0$ increases sharply at frequencies below 3 kMc/sec. This is exactly the behavior expected on the basis of the present theory. Further experimental evidence in support of the theory has been presented by Green and Schlömann.[17]

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APPENDIX. RELAXATION DUE TO SPLITTING

In the limit as $k\rightarrow 0$, k'' approaches $-k'$. Thus, according to Eqs. (17) and (14) for $\theta_k=\pi/2$,

$$\begin{aligned}A_{-k'k'k}&=4|\Phi_{-k'k'-k}|^2\\&=4C^2\sin^2 2\theta_{k'}(\lambda_{k'}-|\mu_{k'}|)^2|\lambda_{k'}\lambda_k+\mu_{k'}{}^*\mu_k|^2.\end{aligned}\tag{A1}$$

Here we have used the fact that $\lambda_{-k'}=\lambda_{k'}$ and $\mu_{-k'}=\mu_{k'}$. With the help of Eqs. (10) and (11), it can easily be shown that

$$(\lambda_{k'}-|\mu_{k'}|)^2=\exp(-\psi_{k'})=\gamma(H+Dk'^2)\omega_{k'}{}^{-1},\tag{A2}$$

$$\begin{aligned}&|\lambda_{k'}\lambda_k+\mu_{k'}{}^*\mu_k|^2\\&=\tfrac{1}{2}[1+\cosh\psi_{k'}\cosh\psi_k+\sinh\psi_{k'}\sinh\psi_k\cos 2(\phi_k-\phi_{k'})]\\&=\tfrac{1}{2}(\omega_k\omega_{k'})^{-1}[\omega_k\omega_{k'}+\gamma^2(H+2\pi M)\\&\quad\times(H+Dk'^2+2\pi M\sin^2\theta_{k'})\\&\qquad+\gamma^2(2\pi M)^2\sin^2\theta_{k'}\cos 2(\phi_k-\phi_{k'})].\end{aligned}\tag{A3}$$

Combining Eqs. (A1), (A2), and (A3), and using the fact that $\omega_{k'}=\frac{1}{2}\omega_k$, one obtains

$$\begin{aligned}A_{-k'k'k}&=4C^2\omega_k{}^{-3}\sin^2 2\theta_{k'}\gamma(H+Dk'^2)\\&\quad\times[\omega_k{}^2+2\gamma^2(H+2\pi M)(H+Dk'^2+2\pi M\sin^2\theta_{k'})\\&\qquad+2\gamma^2(2\pi M)^2\sin^2\theta_{k'}\cos 2(\phi_k-\phi_{k'})].\end{aligned}\tag{A4}$$

Consider now the delta function that occurs under the integration sign in Eq. (29). For convenience it may be rewritten as

$$\begin{aligned}\delta(\omega_k-2\omega_{k'})&=2\omega_k\delta(\omega_k{}^2-4\omega_{k'}{}^2)\\&=2\omega_k\delta\{\omega_k{}^2-4\gamma^2[(H+Dk'^2)\\&\qquad\times(H+Dk'^2+4\pi M\sin^2\theta_{k'})]\}.\end{aligned}\tag{A5}$$

The surface in k' space over which the argument of the delta function vanishes is given by

$$\begin{aligned}Dk'^2=&-(H+2\pi M\sin^2\theta_{k'})\\&+[(2\pi M)^2\sin^4\theta_{k'}+(\omega_k/2\gamma)^2]^{\frac{1}{2}}.\end{aligned}\tag{A6}$$

Since H and ω_k are related by

$$\omega_k{}^2=\gamma^2 H(H+4\pi M),\tag{A7}$$

one finds, after elimination of H,

$$Dk'^2/2\pi M=\cos^2\theta_{k'}+(\sin^4\theta_{k'}+\sigma^2)^{\frac{1}{2}}-(1+4\sigma^2)^{\frac{1}{2}},\tag{A8}$$

where $\sigma=\omega_k/\omega_M$. In Fig. 6 we have plotted $K'=k'\times(D/2\pi M)^{\frac{1}{2}}$ versus $\theta_{k'}$ in polar coordinates for various values of σ. Each curve has the form of a "figure eight." Only one loop of the curve is shown for the cases $\sigma^2=0.4$, 0.25, 0.03, and 0.01. As σ approaches $\frac{2}{3}$ and zero, the curve contracts into the origin. The angle θ_0 at which the two arms of the eight cross is always less than 30 degrees. The maximum value of K' is obtained when $\theta_{k'}=0$ and $\sigma^2=\frac{1}{12}$.

Equations (A4) and (A5) are now inserted into Eq. (29) and the integral is evaluated in polar coordinates. The integration over $\phi_{k'}$ can immediately be carried

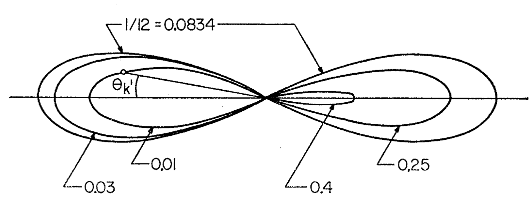

FIG. 6. Solutions of the interference condition for splitting in the limit as $k\rightarrow 0$, $\theta_k=\pi/2$. The numbers in the figure refer to σ^2, where $\sigma=\omega_k/\omega_M$.

[16] R. C. LeCraw and E. G. Spencer, Bull. Am. Phys. Soc. **5**, 297 (1960).

[17] J. J. Green and E. Schlömann, J. Appl. Phys. **32**, 168S (1961).

out, since this variable does not occur in the argument of the delta function. We may thus replace $A_{-k'k'k}$ by the average value $\bar{A}_{-k'k'k}$ taken with respect to $\phi_{k'}$; in other words, the expression given under Eq. (A4) with the last ϕ-dependent term omitted. For convenience we introduce the following abbreviations:

$$\begin{aligned} x&=\cos^2\theta_{k'},\\ \bar{A}_{-k'k'k}&=\sin^2\theta_{k'}\cos^2\theta_{k'}(a-b\cos^2\theta_{k'})=x(1-x)(a-bx),\\ a&=16C^2\omega_k^{-3}\gamma(H+Dk'^2)[\omega_k^2+2\gamma^2(H+2\pi M)\\ &\qquad\times(H+2\pi M+Dk'^2)], \qquad (A9)\\ b&=16C^2\omega_k^{-3}\gamma^3(H+Dk'^2)4\pi M(H+2\pi M),\\ c&=4\gamma^2 4\pi M(H+Dk'^2),\\ d&=4\gamma^2(H+Dk'^2)(H+Dk'^2+4\pi M)-\omega_k^2. \end{aligned}$$

Since $d^3k'=4\pi k'^2dk'\ d(\cos\theta_{k'})$ (where the integration limits on $\cos\theta_{k'}$ are 0 and 1), and since $d(\cos\theta)=\frac{1}{2}x^{-\frac{1}{2}}dx$, one obtains from Eqs. (29), (A4), (A5), and (A9)

$$\tau_k^{-1}|_{\mathrm{spl}}=2Vk_BT\pi^{-1}\hbar^{-3}\int_0^\infty k'^2dk'\,F(k'), \qquad (A10)$$

where

$$\begin{aligned} F(k')&=\int_0^1 dx\,x^{\frac{1}{2}}(1-x)(a-bx)\delta(cx-d)\\ &=d^{\frac{1}{2}}c^{-7/2}(c-d)(ac-bd), \end{aligned} \qquad (A11)$$

provided that $0<d/c<1$. This condition is satisfied as long as $k'<k_0$, where

$$2\gamma(H+Dk_0^2)-\omega_k=0. \qquad (A12)$$

$F(k')$ vanishes if $k'>k_0$. Thus k_0 is effectively the upper integration limit in Eq. (A10).

For convenience, we introduce a new integration variable

$$z=(H+Dk'^2)/2\pi M. \qquad (A13)$$

The lower integration limit is then, according to Eq. (A7),

$$z_0=(1+4\sigma^2)^{\frac{1}{2}}-1, \qquad (A14)$$

and the upper limit, according to Eq. (A12),

$$z_1=\sigma, \qquad (A15)$$

where $\sigma=\omega_k/\omega_M$. Using Eq. (A7) the functions a, b, c, and d are now expressed in terms of z and σ:

$$\begin{aligned} a&=4C^2\sigma^{-3}z[2\sigma^2+(1+4\sigma^2)^{\frac{1}{2}}(1+z)],\\ b&=4C^2\sigma^{-3}(1+4\sigma^2)^{\frac{1}{2}}z,\\ c&=2\omega_M^2z,\\ d&=\omega_M^2[z(z+2)-\sigma^2]. \end{aligned} \qquad (A16)$$

Hence

$$\begin{aligned} c-d&=\omega_M^2(\sigma^2-z^2),\\ ac-bd&=4C^2\omega_M^2\sigma^{-3}z[4\sigma^2z+(1+4\sigma^2)^{\frac{1}{2}}(\sigma^2+z^2)]. \end{aligned} \qquad (A17)$$

According to Eqs. (A13) and (A7),

$$k'^2dk'=\frac{1}{2}\left(\frac{2\pi M}{D}\right)^{\frac{3}{2}}[z+1-(1+4\sigma^2)^{\frac{1}{2}}]^{\frac{1}{2}}dz. \qquad (A18)$$

Thus, from Eqs. (A10), (A11), (A16), (A17), (A18), and (8),

$$\tau_k^{-1}|_{\mathrm{spl}}=\tau_s^{-1}f(\sigma), \qquad (A19)$$

where

$$\tau_s^{-1}=\tfrac{1}{16}k_BT(4\pi M)^{\frac{1}{2}}D^{-\frac{3}{2}}\gamma, \qquad (A20)$$

and

$$f(\sigma)=\frac{1}{\sigma^3}\int_{z_0}^{z_1}g(z)dz, \qquad (A21)$$

$$\begin{aligned} g(z)=z^{-\frac{5}{2}}&[z+1-(1+4\sigma^2)^{\frac{1}{2}}]^{\frac{1}{2}}\times[2z-(\sigma^2-z^2)]^{\frac{1}{2}}\\ &\times[\sigma^2-z^2]\times[4\sigma^2z+(1+4\sigma^2)^{\frac{1}{2}}(\sigma^2+z^2)]. \end{aligned} \qquad (A22)$$

In order to obtain an estimate of $f(\sigma)$, we note that throughout the integration interval

$$(\sigma^2-z^2)/2z<(\sigma^2-z_0^2)/2z_0. \qquad (A23)$$

Thus,

$$2z\geqslant 2z-(\sigma^2-z^2)\geqslant 2z[1-\tfrac{1}{2}(\sigma^2-z_0^2)z_0^{-1}]. \qquad (A24)$$

We now define a function $g_1(z)$ analogous to Eqs. (A22) except that $2z-(\sigma^2-z^2)$ is replaced by $2z$. Then, according to Eq. (A24),

$$g_1(z)\geqslant g(z)\geqslant g_1(z)[1-\tfrac{1}{2}(\sigma^2-z_0^2)z_0^{-1}]^{\frac{1}{2}}. \qquad (A25)$$

The integral over $g_1(z)$ can readily be evaluated, with the result

$$\begin{aligned} f_1(\sigma)&=\sigma^{-3}\int_{z_0}^{z_1}g_1(z)dz\\ &=2^{\frac{1}{2}}\sigma^{-3}\{-(2/7)xy^7-\tfrac{4}{5}(1+6\sigma^2-x)y^5\\ &\quad+\tfrac{4}{3}(1+6\sigma^2-x^3)y^3+\sigma^3(8\sigma-x)y\\ &\quad+\sigma^4(8-7x)(x-1)^{-\frac{1}{2}}\tan^{-1}[y(x-1)^{-\frac{1}{2}}]\}, \end{aligned} \qquad (A26)$$

where

$$\begin{aligned} x&=(1+4\sigma^2)^{\frac{1}{2}},\\ y&=(\sigma-x+1)^{\frac{1}{2}}. \end{aligned}$$

According to Eq. (A25), since the integrand is always positive,

$$f_1(\sigma)\geqslant f(\sigma)\geqslant f_2(\sigma), \qquad (A27)$$

where

$$f_2(\sigma)=[1-\tfrac{1}{2}(\sigma^2-z_0^2)z_0^{-1}]^{\frac{1}{2}}f_1(\sigma). \qquad (A28)$$

The functions $f_1(\sigma)$ and $f_2(\sigma)$ are plotted in Fig. 5.