Range of 2- to 60-Kev Recoil Atoms in Cu, Ag, and Au†

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The ranges of atoms recoiling from photoneutron reactions in Cu, Ag, and Au have been measured utilizing foil-sandwich irradiations followed by standard radiochemical detection techniques. Different energy spectra of recoil atoms were achieved by varying the bremsstrahlung energy and the angle of emission relative to the incident photons. The results have been compared with calculations of Holmes and Leibfried. The best fit to the data is achieved by assigning values of the multiplicative parameter of the screening radius, α, equal to 1.7 for Cu, 2.0 for Ag, and 3.0 for Au.

INTRODUCTION

THE range of atoms recoiling from photoneutron (γ,n) reactions has been measured by observing the fraction of the resultant radioactive atoms which recoil out of a target foil into an adjacent catcher foil. The preliminary work, which was reported previously,[1] has been extended by irradiating target foils with bremsstrahlung spectra of various peak energies between 11 Mev and 26 Mev.

A sandwich of foils was constructed in which target foils (Cu, Ag, or Au) were inserted between catcher foils (Al). The sandwich was irradiated with a bremsstrahlung spectrum produced by passing the electron beam from the General Atomic electron linear accelerator through a 0.010-in. platinum converter foil. Two orientations were used: In one the beam was perpendicular to the plane of the foils, and in the other it was at an angle of 15° to the plane of the foils.

The gamma rays from the converter foils are absorbed in the target foils to produce, among other reactions, photoneutron reactions. A compound nucleus in the foil receives a momentum from the absorption process which equals numerically (in Mev/c) the energy of the absorbed gamma ray (in Mev). Experimental evidence[2] and evaporation theory[3] indicate that the photoneutrons are emitted isotropically in the center-of-mass system. The energy spectrum of the photoneutrons has been measured in some cases and can be described by evaporation theory with a small admixture of direct-ejection particles. Thereby, the momentum distribution of the recoil nucleus is calculable.

The contribution to the recoil momentum from cascade gamma radiation which accompanies neutron emission is usually neglected. The principal effect of these gamma rays is to broaden slightly an already broad momentum distribution.

If the recoil atoms were emitted isotropically with a unique range R, a simple geometrical argument shows that the fraction, f, which recoil out of one surface of a foil of thickness t would be

$$f=R/4t \quad \text{for} \quad t\geq R.$$

In practice, there are two deviations from this simple picture: There is an appreciable spectrum of ranges and the initial recoil energy is correlated with its direction of motion via the momentum imparted by the absorbed gamma ray. These problems are discussed in the next section.

The measurement of the fraction of atoms which recoil out of the target foils was performed by conventional radiochemical counting techniques. Since the radioactive products of (γ,n) reactions in copper and silver are positron emitters, the 0.511-Mev annihilation radiation was counted. The 0.354- and 0.426-Mev gamma rays emitted by 5.6-day Au^{196} were used to determine the foil activity. Beta counting was avoided to minimize corrections for self-absorption and backscattering in the sample.

GAMMA-RAY ABSORPTION

The electron linear accelerator produces electrons which are monoenergetic within 5%. The average electron energy was determined by the current in a deflecting magnet.

Gamma rays were produced by passing the electron beam through a 0.010-in. platinum converter. The bremsstrahlung spectrum was assumed to be a Schiff spectrum as evaluated by Penfold and Leiss.[4] The experimental energy dependence of the cross section for the photoneutron reaction[5] was used to reduce the data. For each bremsstrahlung energy, E_0, the neutrons were assumed to be emitted isotropically with a

† This work was sponsored by Air Research and Development Command, U. S. Air Force.

[1] R. A. Schmitt and R. A. Sharp, Phys. Rev. Letters **1**, 445 (1958).

[2] K. Geller, J. Halpern, and P. F. Yergin, Phys. Rev. **95**, 695(A) (1954).

[3] V. F. Weisskopf and D. H. Ewing, Phys. Rev. **57**, 472 (1940).

[4] A. S. Penfold and J. F. Leiss, University of Illinois Report, 1958 (unpublished).

[5] R. Montalbetti, L. Katz, and J. Goldemberg, Phys. Rev. **91**, 659 (1953).

spectrum given by[6]

$$F(E_n)dE_n = KE_n \int_{E_t+E_n}^{E_0} dk\, \sigma_{\gamma n}(k) \frac{\phi(E_0,k)}{k} \times \left\{ \frac{\exp\{2[\alpha(k-E_t-E_n)]^{\frac{1}{2}}\}}{\int_0^{k-E_t} dx\, x \exp\{2[\alpha(k-E_t-x)]^{\frac{1}{2}}\}} + b\delta(k-E_t) \right\} dE_n,$$

where $F(E_n)$ is the neutron energy distribution function, E_n is the neutron energy, K is the normalization constant, k is the gamma energy, E_0 is the peak bremsstrahlung energy, E_t is the gamma energy at the photoneutron threshold, $\sigma_{\gamma n}(k)$ is the photoneutron cross section, $\phi(E_0,k)/k$ is the bremsstrahlung photon spectrum, α is a constant assumed to be $\alpha = A/20$ (Mev), A is the mass number of the product nucleus, and b is a constant to be specified (assumed values 0.15 for Cu, 0.10 for Ag, and 0.10 for Au).

The first term in the brackets represents the evaporation-neutron spectrum. The second term is the contribution from the directly ejected neutrons, leaving the residual nucleus in the ground state. This formulation, with constant b, assumes the cross section for direct ejection is proportional to the total (γ,n) cross section. Even if this assumption is invalid in the heavier nuclei, this term represents only a small correction and its magnitude is adjusted to force agreement between the theoretical spectrum and measurements.

To simplify the data reduction, a program has been set up for an IBM 704 digital computer to evaluate the foregoing distribution for various nuclei and various bremsstrahlung energies. This program also evaluates the quantities:

$$\bar{E}_n = \int_0^{E_0-E_t} dE_n\, F(E_n)E_n \Big/ \int_0^{E_0-E_t} dE_n\, F(E_n),$$

$$\bar{k} = \int_{E_t}^{E_0} dk\, \sigma_{\gamma n}(k)\phi(E_0,k) \Big/ \int_{E_t}^{E_0} dk\, \sigma_{\gamma n}(k)\phi(E_0,k)/k,$$

where $\bar{E}_n$ is the average photoneutron energy and $\bar{k}$ is the average energy of the gamma rays which produce the photoneutron reactions.

In the experiments, the recoil nuclei are generated uniformly throughout the thickness of the target foils, but they are not emitted isotropically because of the momentum of the absorbed gamma rays. Furthermore, the range spectrum is very broad as a result of the large variation in the energies of the absorbed gamma rays and emitted neutrons. These effects must be considered in computing the fraction of the atoms which recoil out of the target foils for comparison with experimental data.

In the Appendix, it is shown that for the absorption of a monoenergetic gamma ray of energy k, followed by emission of a monoenergetic neutron of momentum P_n, the fraction f_ω of the atoms recoiling out of a foil of thickness t oriented with its outward perpendicular at an angle ω to the gamma ray is approximately

$$f_\omega = (1/4t)C_\omega R(E_\omega),$$

where, to first order in k/P_n,

$$C_{0,180^\circ} = 1 \pm \tfrac{4}{3}(k/P_n), \qquad C_{90^\circ} = 1,$$

$$E_{0,180^\circ} = \frac{1}{A}[E_n \pm \tfrac{2}{3}k(2E_n/M_n)^{\frac{1}{2}}], \qquad E_{90^\circ} = E_n/A,$$

where E_n is the kinetic energy of the neutron, M_n is its mass, and A is the atomic mass of the recoil nucleus.

The factor C_ω is the correction for the increase or decrease of solid angle in the center-of-mass system corresponding to the laboratory hemisphere. The difference between E_ω and E_n/A is the average contribution of the gamma ray to the recoil energy.

In the general case, when both the gamma and neutron energies are distributed, the recoil fraction should be computed by integrating over these distributions. Within the accuracy of this analysis, it is proper to dispense with the integrals over these distributions and replace k and E_n by their average values, $\bar{k}$ and $\bar{E}_n$. For the case in which the gamma momentum is neglected, this procedure has been justified by an argument presented by Holmes and Leibfried.[7]

EXPERIMENTAL DETAILS

Foil Preparation

Because impurities on the surface of the foils could affect the measurements, the foils were thoroughly cleaned before being assembled into sandwiches. The following procedures were carefully followed in cleaning the various foils:

Aluminum Catcher Foils

The catchers were $1\frac{1}{16}$-in. squares of 0.001-in. thick aluminum foil (Alcoa 1199–0, 99.98% pure). They were cleaned thoroughly with petroleum ether, acetone, and detergent and rinsed with conductivity water. Then they were dabbed dry with lint-free paper tissues (Kay-Pees) and stored in a desiccator (silica gel desiccant) until used in the assembly of sandwiches.

Copper Target Foils

Copper foil, 0.0006 in. thick, was cut into 1-in. squares for targets. The foils were cleaned with petroleum ether and acetone, etched with $3N \cdot HNO_3$, and rinsed with conductivity water. After being dabbed dry with Kay-Pees, they were stored in a desiccator containing P_2O_5.

[6] G. Cortini *et al.*, Nuovo cimento **14**, 54 (1959).

[7] D. K. Holmes and G. Leibfried, J. Appl. Phys. **31**, 1046 (1960).

Silver Target Foils

The silver foils were 0.001 in. thick and were cleaned, etched, and stored identically as in the copper foil treatment.

Gold Target Foils

The gold target foils were 1×1×0.0005 in. They were cleaned with petroleum ether, acetone, and detergent and were thoroughly rinsed with conductivity water. They were also stored in a desiccator until ready for sandwich assembly.

Sandwich Assembly

A typical irradiation sandwich assembly consisted of the following components: twenty target foils, each one between two aluminum catchers; twenty aluminum foils, prepared exactly as the catcher foils, but not placed next to any target foils, to serve as background monitors; two copper monitor foils, placed on the front and back of the foil sandwich (the 12.8-hr Cu^{64} activity in these foils was used to measure the integrated gamma exposure of the sandwich); one aluminum clamping plate with $\frac{1}{16}$-in. thick center section, provided with water cooling around its periphery; and one aluminum clamping plate, $\frac{1}{16}$ in. thick.

Accelerator Facility and Irradiation

Figure 1 illustrates the arrangement of the General Atomic electron linear accelerator which was used to perform the irradiations. The following procedure was used to adjust various magnets and calibrate the output energy.

The accelerator and magnets were tuned at one electron energy to deliver a beam of minimum cross-sectional area ($\sim\frac{1}{4}$ in. diameter) in the center of the output window. For any other electron energy the steering, deflecting, and focussing magnets were adjusted to a new current which was chosen to be proportional to the electron's momentum. The accelerator alone was then retuned to deliver the optimum beam in the center of the output window. The energy calibration was performed by irradiating copper foils with bremsstrahlung spectra of various energies from 10 to 20 Mev. A plot of square root of the 9.9-min Cu^{62} activity of the foils versus the magnet settings was constructed.[8] The energy intercept of the straight line best fitting the data was assumed to be 10.8 Mev.[8] The same experiment was repeated with polystyrene foils near the 18.7-Mev threshold of the $C^{12}(\gamma,n)C^{11}$ (20-min) reaction. For the sandwich irradiations the assumed bremsstrahlung energy was adjusted to take account of electron-energy loss in the Pt converter, $(e,e'n)$ reactions in the target foils, and bremsstrahlung production in the sandwich assembly.

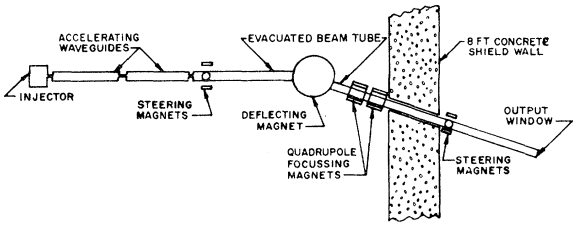

FIG. 1. Accelerator facility.

The irradiation geometry used is illustrated in Fig. 2.

The irradiations usually lasted 20 min. The average bremsstrahlung intensities incident on the sample during this period were between 5×10^4 and 2×10^5 R/min. After irradiation the sandwiches were disassembled and individual components were placed on $3\times3\times\frac{1}{16}$ in. aluminum cards and covered with 0.00025-in. Mylar foil, which was taped in place. Individual samples for counting included all forward catchers, all backward catchers, all background monitors, the central target foil and the two intensity monitors.

Counting was performed with a $1\frac{3}{4}\times2$ in. NaI(Tl) scintillation crystal, observed by a Dumont 6292 photomultiplier. To decrease counting background, this unit was enclosed in a 4-in. thick steel shield. The associated electronics consisted of a conventional single-channel differential pulse-height analyzer, with a high-speed scaler and electronic timer.

The isotopes 9.9-min Cu^{62} and 24-min Ag^{106m} are both positron emitters. In these cases, the sample was placed

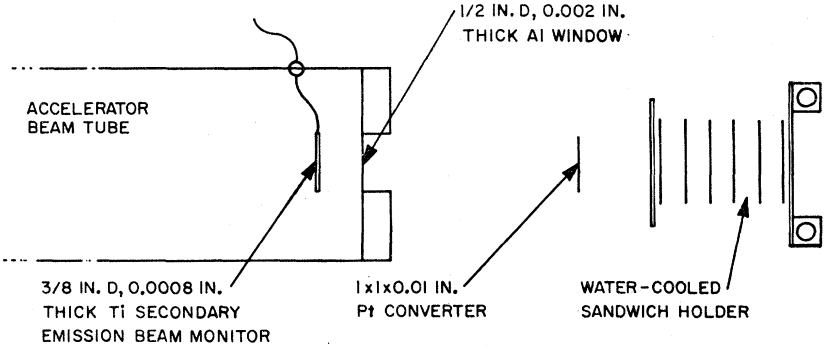

FIG. 2. High intensity irradiation geometry.

[8] A. S. Penfold and E. L. Garvin, Phys. Rev. **115**, 420 (1959).

TABLE I. Copper data.

Bremsstrahlung energy E_0 (Mev)	Catcher angle ω (deg)	Measured range $4tf$ (A)	Corrected range $4tf/C$ (A)	Calculated recoil energy $\bar{E}$ (kev)
11.9	0	95±3	60±2	9.2
	75	70±4	61±3	6.7
	105	54±5	64±6	4.9
	180	31±5	74±12	2.4
14.0	0	180±7	125±7	21.9
	75	94±7	85±6	16.9
	105	81±6	91±7	13.5
	180	69±5	123±9	8.5
16.9	0	254±12	190±9	36.0
	75	142±7	130±6	29.2
	105	121±6	133±7	24.4
	180	101±5	153±8	17.7
17.5	0	258±5	194±4	38.6
	75	184±5	169±5	31.6
	105	162±6	178±7	26.4
	180	100±2	150±3	19.5
20.8	0	274±10	209±8	54.0
	75	194±12	180±11	44.5
	105	154±12	167±13	38.0
	180	108±6	156±9	28.4
22.5	0	270±24	206±18	55.0
	75	278±28	258±26	45.4
	105	142±16	154±17	38.6
	180	110±8	159±12	29.0
26.0	0	305±20	231±15	57.5
	75	250±13	232±12	47.0
	105	211±11	230±12	40.0
	180	121±8	178±12	29.6

between aluminum absorbers on top of the scintillation crystal. The detector system was biased to accept pulses corresponding to gamma energies between 0.46 and 0.56 Mev. Therefore, the radiation detected was predominately the 0.511-Mev gamma rays from the annihilation of the positrons. The positrons emitted by Cu^{62} are very energetic—up to 2.9 Mev. Since the individual samples were of slightly different thicknesses, counting errors were reduced by placing the samples between $\frac{1}{2}$ in. of aluminum (which was on top of the crystal) and $\frac{1}{8}$ in. of aluminum. In all cases, more than 99% of the positrons were annihilated in this counting geometry and none penetrated into the NaI crystal.

TABLE II. Silver data.

Bremsstrahlung energy E_0 (Mev)	Catcher angle ω (deg)	Measured range $4tf$ (A)	Corrected range $4tf/C$ (A)	Calculated recoil energy $\bar{E}$ (kev)
11.0	0	47±4	33±3	7.3
	180	12±1	21±2	2.9
13.1	0	85±5	63±4	13.9
	180	33±2	50±3	6.9
16.8	0	117±8	89±6	22.2
	75	87±4	81±4	18.3
	105	63±3	68±3	15.6
	180	46±3	67±4	11.7
17.4	0	103±4	79±3	23.2
	180	43±3	62±4	12.2
20.6	0	132±3	101±2	26.4
	180	49±3	71±4	14.0
23.6	0	124±4	95±3	27.0
	180	51±2	74±3	14.3
25.8	0	129±4	99±3	27.2
	75	83±3	77±3	22.4
	105	70±2	76±2	19.2
	180	55±2	80±3	14.4

The product of the (γ,n) reaction in gold is 5.6-day Au^{196}. The gamma rays following weak negatron or electron capture decay have energies of 0.354 and 0.426 Mev. Therefore, gamma-ray pulses between 0.30 and 0.50 Mev were counted in the scintillation spectrometer.

TABLE III. Gold data.

Bremsstrahlung energy E_0 (Mev)	Catcher angle ω (deg)	Measured range $4tf$ (A)	Corrected range $4tf/C$ (A)	Calculated recoil energy $\bar{E}$ (kev)
13.1	0	31±2	24±2	10.7
	180	12±1	16±1	6.1
16.5	0	30±2	23±2	12.4
	75	31±3	29±3	10.5
	105	25±2	27±2	9.1
	180	13±1	18±1	7.0
17.0	0	38±2	29±2	12.8
	180	16±1	23±1	7.0
19.5	0	42±2	33±2	13.8
	180	17±1	24±1	7.6
22.6	0	44±3	34±2	14.8
	180	19±1	27±1	8.2
24.2	0	33±2	26±2	15.1
	75	23±2	22±2	12.6
	105	19±1	20±1	11.0
	180	13±1	18±1	8.4

Various check experiments have been performed. It has been established that less than 0.5% of the Cu^{62}, less than 1.6% of the Ag^{106m}, and less than 2.4% of the Au^{196} atoms incident on the aluminum catchers are reflected. Furthermore, by neutron activation analysis of the catchers, it has been shown that less than 0.3% of the activity on the catchers could be ascribed to material which may have rubbed off the target onto the catchers. It was further established by irradiating sandwiches in vacuum that the presence of small amounts of air between the target and catcher foils has no effect on the range measurement, which were normally performed on tightly clamped sandwiches in normal atmosphere.

EXPERIMENTAL RESULTS

Tables I, II, and III summarize the results of the range measurements on copper, silver, and gold,

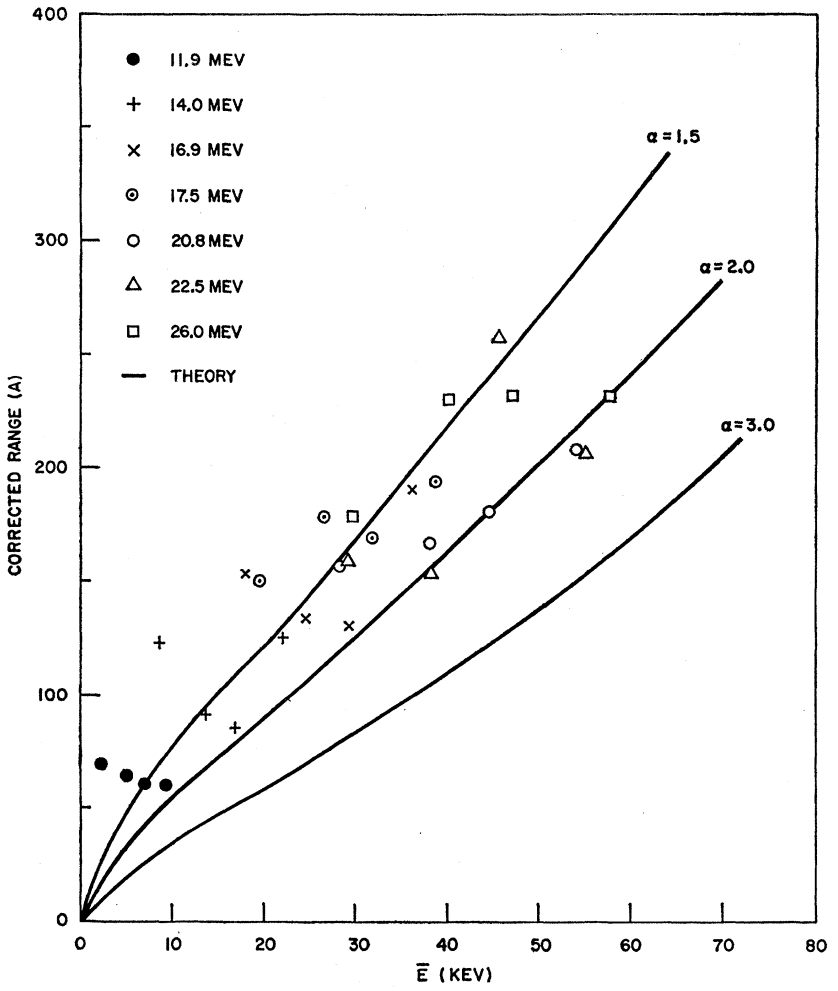

FIG. 3. Copper data.

respectively. The calculated average recoil energy, $\bar{E}$, shown in the last columns of these tables is derived from the average neutron energy, $\bar{E}_n$, and the average absorbed-gamma energy, $\bar{k}$, and corresponds to the average recoil energy in the particular direction. The measured range values have been divided by the factor C_ω to calculate the corrected range values. The second-order terms proportional to $(k/P_n)^2$ have been neglected. At angles other than $\omega=0°$, 90°, and 180° the calculation of the C_ω and E_ω has not been performed, but an interpolation was used assuming the correction terms to be proportional to $\cos\omega$. This interpolation has the proper value for the calculated angles and is sufficiently accurate compared to the other assumptions which have been made.

The range-energy data which result from these experiments have been plotted in Figs. 3–5 for copper, silver, and gold, respectively.

The errors which have been indicated in Tables I–III represent one standard deviation of the counting statistics only. In most cases, the counting statistics are the most important source of error. Other possible sources of error in the recoil fraction and their estimated magnitude are as follows: 1. Thickness of target foils derived from their weight and area: $\pm 1\%$. 2. Possible reflection of recoil atoms from catcher foils: $<0.5\%$

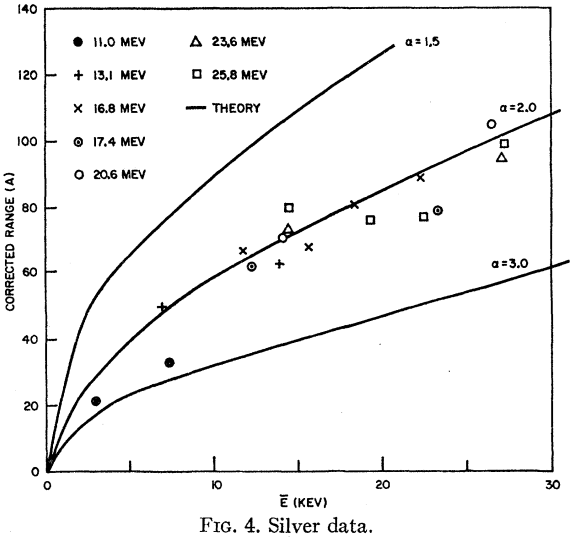

FIG. 4. Silver data.

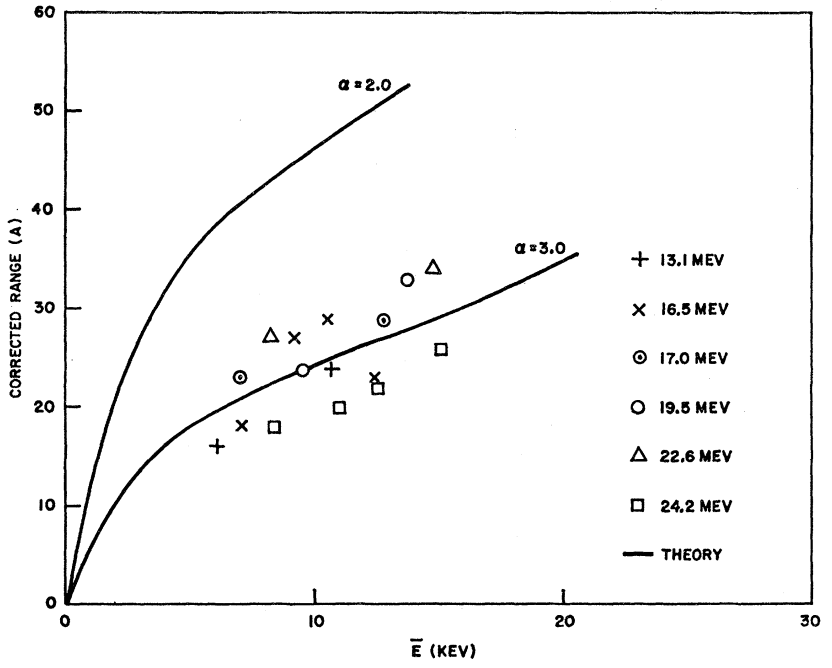

FIG. 5. Gold data.

for Cu, $<1.6\%$ for Ag, and $<2.4\%$ for Au. 3. Transport of part of target foil onto catcher foil: $<0.3\%$. 4. Correction for relative counting geometry, including absorption and backscattering, between target and catcher foils: $\pm 4\%$.

Therefore, the errors in the results are limited by counting statistics down to a lower limit of about 4%.

The errors in the average recoil energy are much greater than this value. Errors in the average recoil energy due to various causes are estimated as follows: 1. Calibration of accelerator energy: $\pm 3\%$. 2. Calculation of neutron energy spectrum: $\pm 15\%$. 3. Energy loss of electron beam in converter foil: $\pm 4\%$. 4. Approximations in formula for deriving average recoil energy: $\pm 10\%$.

DISCUSSION

Holmes and Leibfried[7] have deduced a theoretical range-energy relation for recoil atoms moving in a lattice of the same atoms based on an assumed screened Coulomb potential

$$V(r)=(Z^2e^2/r)\exp(-r/a),$$

where r is the internuclear distance, Z is the atomic number of the atom, e is the electronic charge, and a is a screening radius. Bohr[9] suggested that a should have the value,

$$a_{\text{Bohr}}=a_0/(\sqrt{2}Z^{\frac{1}{3}}),$$

where a_0 is the Bohr radius of the hydrogen atom, but in the analysis of Holmes and Leibfried it was left as an adjustable parameter in the form

$$a=\alpha a_{\text{Bohr}}.$$

In analyzing the data of Schmitt and Sharp[1] for copper at 22 Mev, Holmes and Leibfried concluded that the best fit was achieved by assuming $\alpha=2$. In Figs. 4–6 their theory has been used to draw theoretical curves corresponding to $\alpha=1.5$, 2, and 3.

These data indicate that the best fit value of α increases from approximately 1.7 in Cu, to 2.0 in Ag, to 3.0 in Au.

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APPENDIX. CALCULATION OF RECOIL FRACTION

Assume a gamma ray of energy k to be absorbed by a nucleus of mass $A+1$, and a neutron to be emitted with momentum P_n and angle $(\pi-\theta)$ with the incident gamma in the center-of-mass system.[10] At the energies involved, nonrelativistic mechanics can be used for the nucleus and the neutron. The gamma ray is incident in the Z direction. The momentum of the recoiling nucleus in the laboratory system, P, is

$$P_Z=P_n\cos\theta+k,$$

$$P_\perp=P_n\sin\theta,$$

$$P=(P_n^2+2P_nk\cos\theta+k^2)^{\frac{1}{2}}\approx P_n+k\cos\theta+\frac{k^2}{2P_n}\sin^2\theta,$$

[9] N. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **18**, No. 8 (1948).

[10] Units are chosen so that the velocity of light is unity.

where the last equation assumes that terms higher than second order in k/P_n can be neglected. Within the same approximation, the direction of motion of the recoil atom in the laboratory system is determined by

$$\cos\theta_L=(P_n\cos\theta+k)/P \approx\cos\theta+\frac{k}{P_n}\sin^2\theta-\frac{3}{2}\frac{k^2}{P_n^2}\sin^2\theta\cos\theta.$$

This equation can be inverted:

$$\cos\theta\approx\cos\theta_L-\frac{k}{P_n}\sin^2\theta_L-\frac{1}{2}\frac{k^2}{P_n^2}\sin^2\theta_L\cos\theta_L.$$

Assume that the range at any momentum can be described adequately by a first-order Taylor expansion about P_n:

$$R(P)=R_0(P_n)+\left[\frac{dR}{dP}\right]_{p=p_n}(P-P_n).$$

Also assume that on the average a recoil atom will escape from a foil if its range is greater than the distance to the surface along its initial direction of motion. At the lower starting energies this assumption may not be valid because the atom may lose knowledge of its original direction of motion and approach the surface by diffusion.

Using the foregoing assumptions, the fraction of the recoil atoms which escape from a foil of thickness, t, in the forward direction (0°) through a surface perpendicular to the gamma-ray beam is

$$f_{0^\circ}=\frac{1}{4\pi t}\oint_{\text{lab hemisphere}}d\Omega\,R[P(\theta_L)]\cos\theta_L,$$

where $d\Omega$ is the element of solid angle in the center-of-mass system.

$$f_{0^\circ}=\frac{1}{2t}\int_{\mu=0}^{1}d\mu\,\mu\left\{R_0[1+2\mu\eta-\tfrac{1}{2}\eta^2(1-3\mu^2)]+P_n\frac{dR}{dP}[\eta\mu-\tfrac{1}{2}\eta^2(1-5\mu^2)]\right\},$$

where $\mu=\cos\theta_L$ and $\eta=k/P_n$. Integrating, we have

$$f_{0^\circ}=\frac{R_0}{4t}(1+\tfrac{4}{3}\eta+\tfrac{1}{4}\eta^2)+\frac{P_n}{4t}\frac{dR}{dP}(\tfrac{2}{3}\eta+\tfrac{3}{4}\eta^2).$$

For the backward direction, f_{180°, it is necessary only to reverse the sign of η.

If the foil is parallel to the gamma beam, the polar angles describing orientation relative to the normal to the foil (θ',ϕ') are related to θ_L by the equation

$$\cos\theta_L=\sin\theta'\cos\phi',$$

and integration over a hemisphere $0\leq\theta'\leq\pi/2$ yields

$$f_{90^\circ}=\frac{R_0}{4t}(1-\eta^2/8)+\frac{P_n}{32t}\frac{dR}{dP}\eta^2.$$

The form of these answers can be changed for convenience in analyzing experimental data:

$$f_\omega=(1/4t)C_\omega R(P_\omega),$$

where C_ω is a multiplicative constant and the range R is evaluated at the corrected momentum P_ω. Comparing with the foregoing equations:

$$C_{0,180^\circ}=1\pm\tfrac{4}{3}\eta+\tfrac{1}{4}\eta^2,\qquad C_{90^\circ}=1-\tfrac{1}{8}\eta^2,$$

$$P_{0,180^\circ}=P_n[1\pm\tfrac{2}{3}\eta-(5/36)\eta^2],\quad P_{90^\circ}=P_n(1+\tfrac{1}{8}\eta^2),$$

where the upper sign refers to the 0° angle. Within the same approximation the kinetic energy E_ω, corresponding to P_ω, can be calculated.

$$E_{0,180^\circ}=\frac{1}{A}\left[E_n\pm\tfrac{2}{3}k\left(\frac{2E_n}{M_n}\right)^{\frac{1}{2}}+\frac{1}{12}\frac{k^2}{M_n}\right],$$

$$E_{90^\circ}=\frac{1}{A}\left[E_n+\frac{1}{8}\frac{k^2}{M_n}\right],$$

where E_n and M_n are neutron kinetic energy and mass, respectively.

The assumption that the atom's range is measured along its initial direction of motion can be changed to the opposite extreme; namely, the initial direction of motion is immaterial. In other words, no matter which direction the recoil atom moves initially, it is equally likely to come to rest in any direction at a distance determined only by its initial energy. In this case, the effect of the extra momentum imparted by the gamma ray is to broaden the range distribution in all directions, but it does not introduce any angular asymmetry. The average range should still be equal to $R_0(P_n)$. However, it has been shown that the experimental data are inconsistent with this assumption. It is possible that a tendency toward isotropy at low recoil energies can be explained by severe scattering of the recoil atom.