Low-Energy p-p Scattering Phase Shifts and Dispersion Relations*

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Nucleon-nucleon dispersion relations have been used to discriminate among various sets of solutions given by MacGregor for low-energy proton-proton scattering. This has greatly reduced the ambiguity existing in the phase-shift analysis at low energy.

I

A PHASE-SHIFT analysis of proton-proton scattering experiments below 40 Mev has recently been given by MacGregor.[1] In this energy range the only accurate measurements are angular distribution measurements. MacGregor's analysis has shown that there are four different types of solution that fit the differential cross-section data equally well. These four types of solutions have been listed by MacGregor as I, II, III, and IV and they differ chiefly in the magnitude and sign of the three P-wave phase shifts used. Within each type of solution a great ambiguity still exists owing to the multiplicity of S-D phase-shift combinations for each set.

Solution types[1] I and III have positive nuclear polarization, while II and IV have negative polarizations. An extrapolation of the high-energy polarization measurements indicates that at low energies the nuclear polarization is small and positive. Thus the ambiguity is reduced from 4 to 2 semi-infinite phase-shift sets, i.e., types II and IV are ruled out by polarization arguments. We may mention here that the recent measurement of polarization at 16 Mev and 25 degrees in the lab system by Blanpied[2] supports the above conclusion as this polarization is positive and equal to $(0.6\pm0.5)\%$.

Polarization measurements being able to remove the ambiguity partly, accurate angular distribution measurements of polarization as well as triple scattering experiments are required even at low energies, and these may be difficult to perform. Since no data from triple scattering experiments exist in this energy region at the present time, it may be interesting to see whether the dispersion relations for nucleon-nucleon scattering of the type written by Goldberger, Nambu, and Oehme[3] and the present author[4] can permit a choice between the types I and III (II and IV being ruled out by polarization arguments). We show in this paper that this, in fact, is possible. If we go through the solution types I and III, we observe that the phase-shift combination $(2\delta_2{}^N+\delta_0{}^N)$, where $\delta_2{}^N$ and $\delta_0{}^N$ are, respectively, the nuclear ${}^3P_2$ and ${}^3P_0$ phase shifts, is positive for type I and negative for type III. Thus if we can write a dispersion relation for that part of the scattering amplitude whose real part at low energies is proportional to the above combination of the phase shifts, it may be possible to discriminate between types I and III. In Sec. II we show that it is possible and dispersion relations, in fact, favor solution type I.

Having selected type I as acceptable solution, the problem now is how one can be able to assign precise values to S and D phase shifts in the solution type I. In this connection also dispersion relations can be helpful. In Sec. III we show that dispersion relations somewhat restrict the values of S and D phase shifts. In fact we show that the average P-wave phase shift $[\bar{\delta}^N=(\delta_0{}^N+3\delta_1{}^N+5\delta_2{}^N)/9]$ should be less than the one-pion exchange contribution (OPEC). This in turn puts an upper limit on the D-wave phase shift and rules out the solutions giving large polarization.

Before closing this section we give a summary of the dispersion relations we shall employ. As is well known, the scattering amplitude for nucleon-nucleon scattering in the c.m. frame can be written as follows:

$$T=(m/E)[H_1+\boldsymbol{\sigma}_1\cdot\mathbf{l}\boldsymbol{\sigma}_2\cdot\mathbf{l}H_2+i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{l}H_3 +\boldsymbol{\sigma}_1\cdot\mathbf{m}\boldsymbol{\sigma}_2\cdot\mathbf{m}H_4+\boldsymbol{\sigma}_1\cdot\mathbf{n}\boldsymbol{\sigma}_2\cdot\mathbf{n}H_5],$$

where m is the nucleon mass and E is the c.m. energy. $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{n}$ are unit vectors in the directions of $\mathbf{k}\times\mathbf{k}'$, $\mathbf{k}'-\mathbf{k}$, and $\mathbf{k}'+\mathbf{k}$, respectively. $\mathbf{k}$ and $\mathbf{k}'$ are the c.m. momenta in initial and final states. In references 3 and 4 it has been shown that simple relativistic dispersion relations can be written for H_1, H_2, H_4, and H_5 in the forward direction. These relations for p-p scattering are:

$$\mathrm{Re}H_i(k^2)=(-1)^{i-1}\frac{f_1{}^2}{2}\frac{m}{4k^2+\mu^2}+\frac{1}{\pi}\int_0^\infty dk'^2\frac{\mathrm{Im}H_i(k'^2)}{k'^2-k^2}\pm\frac{1}{\pi}\int_{\mu^2-m^2}^\infty dk'^2\frac{\mathrm{Im}\bar{H}_i(k'^2)}{k'^2+k^2+m^2}+C_i, \quad (1)$$

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[1] M. H. MacGregor, Phys. Rev. **113**, 1559 (1959).
[2] W. A. Blanpied, Phys. Rev. **116**, 738 (1960).
[3] M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. **2**, 226 (1957).
[4] Riazuddin, Ph.D. thesis, University of Cambridge, 1958, (unpublished).

where $i=1, 2, 4, 5$ and in the last term $+$ signs are to be taken for $i=1, 2, 4$ and $-$ signs for $i=5$. $\bar{H}_i$ are the corresponding amplitudes for the proton-antiproton scattering. μ is the pion mass and $f_1^2=f^2/4\pi$ is the pion-nucleon coupling constant. C_i appearing in (1) are just constants which usually appear in unsubtracted relations. We may remark here that in the limit of the forward scattering $H_2=H_4$. For $(H_3/\sin\theta)$ no simple relativistic dispersion relation can be written. But in the nonrelativistic limit a simple relation can be written for $(H_3/\sin\theta)$. This relation for forward scattering is

$$\mathrm{Re}\left(\frac{H_3(k^2,\tau)}{\tau}\right)_{\tau=0}=\frac{k}{\pi}\int_0^\infty dk'^2\frac{\mathrm{Im}(H_3(k'^2,\tau)/\tau)_{\tau=0}}{k'(k'^2-k^2)}+\frac{k}{\pi}\int_{\mu^2-m^2}^\infty dk'^2\frac{\mathrm{Im}(\bar{H}_3(k'^2,\tau)/\tau)_{\tau=0}}{k'(k'^2+k^2+m^2)}+C_3. \quad (2)$$

Here τ is the momentum transfer and C_3 is a constant. In the last integrals of (1) and (2) the interval $\mu^2-m^2<k^2<m^2$ is the unphysical range where more than one meson contribute. The relations (1) and (2) have not yet been proved rigorously.

II

We wish to write a dispersion relation for the amplitude whose real part at low energies (<10 Mev) is proportional to $(2\delta_2{}^N+\delta_0{}^N)$ because such a relation will enable us to distinguish between solutions of type I and III as remarked in the Introduction. From the expressions for H_1, etc., in terms of the phase shifts, we see that such an amplitude is

$$V(k^2)=H_1(k^2)+2H_2(k^2)-H_5(k^2)$$
$$=\frac{E}{m}\frac{1}{2ik}\Big\{\sum_{\mathrm{odd}\ l}[2(l+1)\alpha_{l,l+1}+2l\alpha_{l,l-1}+4((l+1)(l+2))^{\frac{1}{2}}\alpha^{l+1}]\Big\}. \quad (3)$$

The expressions $\alpha_{l,l+1}$, etc., are the same as defined by Stapp *et al.*[5] From Eqs. (1) we see that the dispersion relation for $V(k^2)$ with one subtraction at $k^2=0$ is

$$\mathrm{Re}V(k^2)/k^2=f_1^2\frac{m}{\mu^2(k^2+\mu^2/4)}+\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}V(k'^2)}{k'^2(k'^2-k^2)}dk'^2-\frac{1}{\pi}\int_{\mu^2-m^2}^\infty\frac{\Omega(k'^2)}{(k'^2+m^2)(k'^2+k^2+m^2)}dk'^2, \quad (4)$$

where

$$\Omega=\mathrm{Im}(\bar{H}_1+2\bar{H}_2+\bar{H}_5).$$

[5] H. P. Stapp *et al.*, Phys. Rev. **105**, 302 (1957).

For low energies, where P-wave phase shifts are small,

$$\mathrm{Re}V(k^2)=\frac{2}{k}[2\delta_2{}^N+\delta_0{}^N],$$
$$\mathrm{Im}V(k^2)=\frac{2}{k}[2(\delta_2{}^N)^2+(\delta_0{}^N)^2+2\phi_1(2\delta_2{}^N+\delta_0{}^N)]\equiv U(k^2), \quad (5)$$

where $\delta_2{}^N$ and $\delta_0{}^N$ are nuclear $^3P_2$ and $^3P_0$ phase shifts and ϕ_1 is the Coulomb phase shift for $l=1$ and is given by

$$\phi_1=\tan^{-1}\eta,\quad \eta=e^2/\hbar v.$$

The last integral in (4) involves unphysical contribution for the region $\mu^2-m^2<k'^2<m^2$ and the physical contribution from the proton-antiproton scattering for $k'^2\geqslant m^2$. The latter should not be so important because of the large denominator $(k'^2+m^2)(k'^2+k^2+m^2)$ appearing in it and we shall neglect it. The unphysical contribution comes from more than one meson and it has been conjectured by Goldberger *et al.*[3] that this contribution is the Fourier transform of nuclear potential of order more than two. Now our relation involves P or higher waves. The impact parameter for P waves at 4 Mev [where we shall apply relation (4)] is $3.3\mu^{-1}$ and so these are not much affected by the potentials in the region $<\mu^{-1}$ where the fourth and higher order potentials are important. So it may be justified to neglect the unphysical contribution from the region $\mu^2-m^2<k^2<m^2$ and we shall do so. We shall, however, show later that the conclusions which we draw are not affected by the inclusion of the two-meson contribution on the right-hand side of Eq. (4). We may remark here that Lapidus[6] has shown by direct calculation that for low energies the entire contribution from nucleon-antinucleon scattering and the unphysical region is negligible, at least for the amplitude H_1. Neglecting the above-mentioned contributions, we can write relation (4) on using (5), as

$$F(k^2)=f_1^2\frac{mk^3}{\mu^2(k^2+\mu^2/4)}+\frac{2k^3}{\pi}\int_{k\mathrm{max}}^\infty dk'\frac{\mathrm{Im}V(k'^2)}{k'(k'^2-k^2)}, \quad (6)$$

where $\mathrm{Im}V(k^2)$ is given by (3) and

$$F(k^2)=2(2\delta_2{}^N+\delta_0{}^N)-\frac{2k^3}{\pi}\int_0^{k_{\mathrm{max}}}dk'\frac{U(k'^2)}{k'(k'^2-k^2)}. \quad (7)$$

We apply (6) to the phase-shift set solutions given by MacGregor at 4.203 Mev. For this energy $k=(0.0236)^{\frac{1}{2}}\times(4.203)^{\frac{1}{2}}\mu$ and we take $k_{\mathrm{max}}=(0.0236)^{\frac{1}{2}}5^{\frac{1}{2}}\mu$ (lab energy 5 Mev). $U(k^2)$ is given by (5). The integral in (7) is evaluated as follows: We take the energy dependence of $\delta_2{}^N$ and $\delta_0{}^N$ below 5 Mev as

$$\delta_2{}^N=a_2k^3,\quad \delta_0{}^N=a_0k^3,$$

[6] L. I. Lapidus, Zhur. Eksp. i Teoret. Fiz. **36**, 283 (1959) [translation: Soviet Phys. JETP **36**, 193 (1959)].

and after integration we again use these equations to get a_2 and a_0 in terms of δ_2^N and δ_0^N. The energy dependence of the Coulomb phase shift is taken as

$$\phi_1=\tan^{-1}\eta=\tan^{-1}(1/2Rk), \quad \text{with} \quad R=\hbar^2/me^2\approx 20.6\mu^{-1},$$

for

$$0<k<k_0=(0.0236)^{\frac{1}{2}}(0.2)^{\frac{1}{2}}\mu$$

and

$$\phi_1=1/2Rk, \quad \text{for} \quad k_0<k<k_{\max}.$$

Then

$$F(k^2)=2(2\delta_2^N+\delta_0^N)b_1-2[2(\delta_2^N)^2+(\delta_0^N)^2]b_2, \quad (8)$$

where

$$b_1=\left[1-\frac{2}{\pi}\eta\ln\frac{k_{\max}-k}{k_{\max}+k}\frac{k+k_0}{k-k_0}\right.$$
$$\left.-\left(\eta^2-\frac{2}{\pi}[(k_0/k)^2\tan^{-1}\eta_0+\eta^2\tan^{-1}\eta_0+(k_0/k)\eta]\right)\right],$$

$$b_2=\frac{1}{\pi}\left[\ln\frac{k_{\max}-k}{k_{\max}+k}+\tfrac{2}{3}(k_{\max}/k)^3+2(k_{\max}/k)\right],$$

where $\eta_0=1/2Rk_0\approx 0.35$. At 4.203 Mev, $b_1\approx 1.15$ and $b_2\approx -0.03$.

$F(k^2)$ is positive for the solution type I at 4.203 Mev and negative for the solution type III. Now the first term on the right-hand side of (6) is certainly positive. If we can show that the second term is also positive, then only the solution type I will be compatible with the dispersion relations and the solution type III will be thrown out. The second term on the right-hand side of (6) will certainly be positive if $\operatorname{Im}V(k'^2)$ is positive definite, because in the range of integration k' is always greater than k. If we follow MacGregor's analysis of the p-p data below 40 Mev, we see that all his solutions (except possibly type IV which, however, we have ruled out on polarization arguments) give positive $\operatorname{Im}V(k'^2)$. Above 40 Mev, Coulomb effects are negligible and in the notation of Stapp *et al.*, $\operatorname{Im}V(k'^2)$ can be written as

$$\begin{aligned}\operatorname{Im}V(k'^2)&\\=\frac{E'}{m}\left(\frac{1}{k'}\right)&[2\sin^2{}^3P_0+\sum_{\text{odd } l}\{2(2l+3)\sin^2\bar{\epsilon}_{l+1}\\&+2(l+1)^{\frac{1}{2}}[(l+1)^{\frac{1}{2}}\cos 2\bar{\epsilon}_{l+1}\sin^2\bar{\delta}_{l,l+1}\\&+(l+2)^{\frac{1}{2}}\sin 2\bar{\epsilon}_{l+1}\sin(\bar{\delta}_{l,l+1}+\bar{\delta}_{l+2,l+1})]\\&+2(l+2)\cos 2\bar{\epsilon}_{l+1}\sin^2\delta_{l+2,l+1}\}]. \quad (9)\end{aligned}$$

Each term with the exception of one in the above summation is positive definite. The exception is the amplitude-mixing term

$$\sin 2\bar{\epsilon}_{l+1}\sin(\bar{\delta}_{l,l+1}+\bar{\delta}_{l+2,l+1}),$$

which may be positive or negative. But this term in general is much smaller than $\cos 2\bar{\epsilon}_{l+1}\sin^2\bar{\delta}_{l,l+1}$. This is the case for all the five solutions of Cziffra *et al.*[7] for the p-p scattering at 310 Mev. This was the case below 40 Mev, as already seen. Therefore we can conclude that $\operatorname{Im}V(k'^2)$ is almost certainly positive. Hence the right-hand side of Eq. (6) is positive. Hence the dispersion relations throw out solution type III. We may remark here that solution type IV gives negative $F(k^2)$ and is therefore thrown out by the above argument also. Solution type II is compatible with (6) but we have ruled it out on polarization arguments.

One further remark: Data at 4.203 Mev were also analyzed by Hull and Shapiro.[8] They gave two types of solution sets, for both of which $F(k^2)$ is negative, and they are, therefore, also thrown out by the above argument.

We now come to the consideration of the two-meson contribution arising from the second integral in Eq. (4) which we have neglected. This contribution should be added to the right-hand side of Eq. (6). It has been shown by Grashin and Kohsarev[9] that at 10 Mev the two-meson contribution to δ_0^N is about 6% of the one-meson contribution while to δ_2^N it can be as large as that of one meson. But the two-meson contribution to both δ_0^N and δ_2^N is of the same sign as that of one meson, which is positive for δ_0^N and is negligible for δ_2^N at 5 Mev (see reference 7). Thus the two-meson contribution to our amplitude, which is proportional to $(2\delta_2^N+\delta_0^N)$, is of the same sign as the one-meson contribution which is positive. Hence the conclusions which we have drawn are not altered by the addition of the two-meson contribution to the right-hand side of (6).

We conclude this section with the following observation: As we have seen, the amplitude $\operatorname{Re}V(k^2)$ is positive for type I at low energies and negative for type III, and it is the type I which is compatible with the dispersion relations. If we calculate $\operatorname{Re}V(k^2)$ for the phase-shift sets I and II of the modified analysis by Cziffra *et al.*[7] of p-p scattering at 310 Mev, we see that it is the solution set I which gives positive $\operatorname{Re}V(k^2)$ while the set II gives negative $\operatorname{Re}V(k^2)$. Thus in this respect the phase-shift sets I and II of Cziffra *et al.* at 310 Mev resembles, respectively, the solution types I and III of MacGregor at low energies. But the solution type III of MacGregor which is similar to solution set II of Cziffra *et al.*, is rejected by dispersion relations. This may be an evidence in favor of solution set I of Cziffra *et al.*

III

We now show that the dispersion relations can also put an upper limit on $\bar{\delta}^N$ which in turn puts an upper limit on the D-wave phase shift. For this purpose we write a dispersion relation for the triplet part of H_1

[7] P. Cziffra *et al.*, Phys. Rev. **114**, 880 (1959); M. H. MacGregor *et al.*, Phys. Rev. **116**, 1248 (1959).
[8] M. H. Hull and J. Shapiro, Phys. Rev. **109**, 846 (1958).
[9] A. F. Grashin and I. Kohsarev, Nuclear Phys. **17**, 181 (1960).

which is given by

$$T(k^2)=\tfrac{3}{4}H_1(k^2)+\tfrac{1}{4}[2H_2(k^2)+H_5(k^2)]$$
$$=-\frac{E}{m}\frac{1}{4ik}\sum_{\text{odd } l}[(2l+3)\alpha_{l,l+1}+(2l-1)\alpha_{l,l-1}+(2l+1)\alpha_{l,l}]. \quad (10)$$

At low energies,

$$\mathrm{Re}T=9\bar{\delta}^N/2k,$$
$$\mathrm{Im}T=\frac{1}{2k}[9\langle\delta^{N2}\rangle_{\mathrm{av}}+18\phi_1\bar{\delta}^N]\equiv U_1(k^2), \quad (11)$$

where

$$\bar{\delta}^N=(1/9)(5\delta_2{}^N+3\delta_1{}^N+\delta_0{}^N),$$
$$\langle\delta^{N2}\rangle_{\mathrm{av}}=(1/9)[5(\delta_2{}^N)^2+3(\delta_1{}^N)^2+(\delta_0{}^N)^2].$$

From Eqs. (1) we see that the dispersion relation for $T(k^2)$ with one subtraction at $k^2=0$ is

$$\mathrm{Re}T(k^2)/k^2=-f_1{}^2\frac{m}{4}\frac{1}{\mu^2(k^2+\mu^2/4)}+\frac{1}{\pi}\int_0^\infty dk'^2\frac{\mathrm{Im}T(k'^2)}{k'^2(k'^2-k^2)}-\frac{1}{\pi}\int_{\mu^2-m^2}^\infty dk'^2\frac{\Omega_1(k'^2)}{(k'^2+m^2)(k'^2+k^2+m^2)}, \quad (12)$$

where

$$\Omega_1=\mathrm{Im}\tfrac{1}{4}(3\bar{H}_1+2\bar{H}_2-\bar{H}_5).$$

As remarked in Sec. II, we shall again neglect the second integral on the right-hand side of Eq. (12). We now apply the truncated relation thus obtained at low energies (<5 Mev) and can write it as

$$F_1(k^2)=f_1{}^2\frac{mk^3}{18\mu^2(k^2+\mu^2/4)}-\frac{k^3}{9\pi^2}\int_{k_{\max}}^\infty dk'\frac{[\sigma(k'^2)-\frac{1}{4}\sigma_s(k'^2)]}{k'^2-k^2}, \quad (13)$$

where

$$F_1(k^2)=-\bar{\delta}^N+\frac{4k^3}{9\pi}\int_0^{k_{\max}}\frac{U_1(k'^2)}{k'(k'^2-k^2)}dk', \quad (14)$$

with $k_{\max}$ and k as defined in Sec. II. In writing (13) we have also made use of the optical theorem

$$\mathrm{Im}T(k^2)=(k/4\pi)[\sigma(k^2)-\tfrac{1}{4}\sigma_s(k^2)],$$

where σ is the total cross section and σ_s is the singlet part of the cross section. The integral in (14) is evaluated as in Sec. II and we get

$$F_1(k^2)=(-\bar{\delta}^N)b_1+\langle\delta^{N2}\rangle_{\mathrm{av}}b_2. \quad (15)$$

The last integral in (13) is positive as $\sigma(k'^2)-\frac{1}{4}\sigma_s(k'^2)>0$ and $k<k'$. Thus

$$F_1(k^2)<f_1{}^2\frac{mk^3}{18\mu^2(k^2+\mu^2/4)}. \quad (16)$$

The quantity on the right-hand side of the inequality (14) is the one-pion exchange contribution to $F_1(k^2)$. From (13), (14), and (15) we get

$$b_1(-\bar{\delta}^N)=f_1{}^2\frac{mk^3}{18\mu^2(k^2+\mu^2/4)}-\frac{k^3}{9\pi^2}\int_{k_{\max}}^\infty dk'\frac{\sigma(k'^2)-\frac{1}{4}\sigma_s(k'^2)}{k'^2-k^2}-b_2\langle\delta^{N2}\rangle_{\mathrm{av}}. \quad (17)$$

The integral in (17) is positive, as remarked earlier. Also $\langle\delta^{N2}\rangle_{\mathrm{av}}$ is positive and b_2 is negative but very small. For k and $k_{\max}$ given in Sec. II, $b_2\approx-0.03$ and $b_1\approx1.15$. Even for those solutions of MacGregor at 4.203 Mev which give the largest $\langle\delta^{N2}\rangle_{\mathrm{av}}$, $-b_2\langle\delta^{N2}\rangle_{\mathrm{av}}$ is not more than 0.011 if $\bar{\delta}^N$ is measured in degrees. Thus Eq. (17) gives an upper limit on $(-\bar{\delta}^N)$ which is

$$(-\bar{\delta}^N)_{\text{in degrees}}<f_1{}^2\frac{mk^3}{18\mu^2(k^2+\mu^2/4)}\frac{180}{\pi}+0.011;$$

i.e., essentially $(-\bar{\delta}^N)$ is less than the one-pion exchange contribution which is 0.1526 at 4.203 Mev, $\bar{\delta}^N$ being measured in degrees. This limit is also given by meson-theoretic potentials.[9] For 4.203 Mev and $f_1{}^2\approx0.08$, the above inequality gives an upper limit on $(-\bar{\delta}^N)$ measured in degrees as 0.142. Now if we go through the solutions of MacGregor at 4.203 Mev, we see that $(-\bar{\delta}^N)$ increases with increasing *D*-wave phase shift. Thus upper limit on $(-\bar{\delta}^N)$ puts an upper limit on the *D*-wave phase shift. For solution type I which is the only solution compatible with the dispersion relations and polarization arguments, the upper limit on the *D*-wave phase shift is 0.2°. The upper limit on $(-\bar{\delta}^N)$ or *D*-wave phase shift given by OPEC at once rules out the phase-shift solutions giving large polarization (given, for example, by Hull and Shapiro[8]).

We now sum up the conclusions of the paper. The dispersion relations can discriminate between the solution types I, II and III, IV given by MacGregor and it is the former types which are compatible with them. This combined with the polarization arguments, gives solution type I as the only acceptable one. Further, the dispersion relations put an upper limit on the *D*-wave phase shift. This to some extent restricts the values of the *D*-wave phase shift which are otherwise possible within the solution type I and rules out solutions giving large polarization.

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[9] J. Iwadare (private communication).