Wave Zone in General Relativity

R. Arnowitt,* S. Deser,† and C. W. Misner‡
Syracuse University, Syracuse, New York, and Brandeis University, Waltham, Massachusetts
(Received August 18, 1960)

It is shown that in general relativity a "wave zone" may be defined for systems which are asymptotically flat. In this region, gravitational radiation propagates freely, independent of its interior sources, and obeys the superposition principle. The independent dynamical variables of the full theory which describe the radiation are shown to be coordinate invariant in the wave zone and to satisfy the linearized theory's equations there. Thus, the basic properties of free waves in linear field theories (e.g., electrodynamics) are reproduced for the gravitational case. True waves are also clearly distinguished from so-called "coordinate waves." Reduction to asymptotic form (taking leading powers of $1/r$), is *not* identical to linearization, since, for example, the Newtonian-like $1/r$ part of the metric begins quadratically in the linear theory's variables. The Poynting vector of the full theory, which measures energy flux in the wave zone, is correspondingly shown to be given by the linearized theory's formula. This Poynting vector is also shown to be coordinate-invariant in the wave zone. All the physical quantities may therefore be evaluated in any frame becoming rectangular sufficiently rapidly. A brief discussion of measurements of the canonical variables in the wave zone is given. The relation between the present work and other treatments of gravitational radiation is examined.

I. INTRODUCTION; DEFINITION OF THE WAVE ZONE

It is well known that in the linearized approximation of the full theory of relativity there exist wave solutions whose physical interpretation can be framed in terms identical to those used in classical electrodynamics. However, it has also been realized that a gravitational wave carries energy, which should therefore give rise to a Newtonian gravitational field at infinity. Such a term is not present in the free theory's linearized approximation (the energy being quadratic in the amplitudes). It is therefore necessary to investigate the validity of the linearized approximation as a description of waves escaping from a strong field interior region. In this paper, we will examine the asymptotic behavior of the full theory and compare it with the linearized form. Our procedure will differ from that commonly used in electrodynamics; it cannot be based on retarded Green's functions relating the radiation amplitudes to the sources, since in general relativity neither these Green's functions nor equivalent existence theorems are available. We shall therefore base our approach on a study of the Einstein equations and their solutions as specified by Cauchy data (a procedure also applicable in electrodynamics). As a result, no relations between the sources and the radiation field will be obtained. However, the behavior of the radiation field after its emission will be specified. We will see that one can define a "wave zone" in which the canonical variables[1] (see III) describing the radiation *do* obey the linear theory's equations. Beyond the wave front, there remain the (coordinate-independent) Newtonian parts of the metric (which include the nonlinear effects resulting from the interior domain), as well as coordinate-dependent parts of the metric.

The "wave zone" is defined in analogy with electrodynamics. We consider a general situation in which the gravitational canonical modes behave asymptotically as $\sim fe^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}/r$ in some region; beyond this region (i.e., past the wave front) they are assumed to vanish more rapidly. This assumption is made to insure that the total energy contained in the wave be finite. The first requirement on the wave zone is the familiar one that $kr\gg1$; this implies that gradients and time derivatives acting on the canonical modes also fall off as $\sim1/r$. However, two further requirements, not made in classical electrodynamics, must be imposed in view of the nonlinear nature of the field. We first demand that all components of the metric $g_{\mu\nu}$ deviate from the Lorentz metric $\eta_{\mu\nu}$ by terms small compared to unity, and decrease at least as $1/r$ in the wave zone. (Note that $g_{\mu\nu}-\eta_{\mu\nu}\sim t/r$ is forbidden, since t/r is not small in every Lorentz frame.) This requirement can always be met when radiation is escaping to infinity by taking r sufficiently large and waiting for the wave to reach this distance. There are three different parts of[2] $g_{\mu\nu}$: (1) the canonical variables for which the above condition implies $f/r\ll1$; (2) the Newtonian-like parts (which behave as $\sim M/r$), so that by $M/r\ll1$, the wave

* Research supported in part by Wright Air Development Division, Air Research and Development Command, U. S. Air Force, and the Air Force Office of Scientific Research.

† Research supported in part by National Science Foundation and by Air Force Office of Scientific Research.

‡ Alfred P. Sloan Research Fellow, on leave from Palmer Physical Laboratory, Princeton University, Princeton, New Jersey.

[1] Related previous papers by the authors are referred to in text by Roman numerals: I—R. Arnowitt and S. Deser, Phys. Rev. **113**, 745 (1959); III—R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. **117**, 1595 (1960); IV—Phys. Rev. **118**, 1100 (1960); V—Phys. Rev. **120**, 313, 321 (1960); IV c—R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. (to be published); IV b—present paper.

[2] Notation and units are as in III, etc.: $\kappa\equiv16\pi\gamma c^{-4}=1=c$, where γ is the Newtonian gravitational constant. Latin indices run from 1 to 3, Greek from 0 to 3, and $x^0=t$. All tensors and covariant operations are *three*-dimensional unless specified, g^{ij} being the matrix inverse to g_{ij} and the subscript vertical bar indicating covariant differentiation with respect to g_{ij} (not ${}^4g_{\mu\nu}$). Partial differentiation is denoted by a comma or by ∂_μ.

zone must lie well beyond the gravitational radius of the *total* system; and (3) the gauge variables whose specification fixes the coordinate frame being used. For this last case, the condition defines how rapidly the coordinate frame must become rectangular at infinity. We shall see that derivatives of the Newtonian part go as $\sim 1/r^2$ and are negligible, while derivatives of the gauge parts may or may not be negligible, depending on the amount of "coordinate waves" present. Further, it will be shown that such "coordinate waves" may be completely isolated in the wave zone and beyond it.

The final requirement can be stated in terms of the derivatives of the metric: it is necessary that $|\partial g/\partial(kx)|^2 \ll |g-\eta|$ in order that a wave of frequency (or wave number) k will behave as free radiation. Alternately, frequencies above some $k_{\min}$ will behave as free radiation provided $k_{\min} \gg k_{\max}(a/r)^{\frac{1}{2}}$, where $|g-\eta| \sim a/r$ and $k_{\max}$ is the maximum frequency appearing in the metric. This condition can always be fulfilled for any desired $k_{\min}$ by taking r large enough, and waiting for the radiation to reach the region. For fixed r, the condition represents a lower bound on frequencies which may be treated in this part of the wave zone.

The requirements that $|g_{\mu\nu}-\eta_{\mu\nu}| \ll 1$, $|\partial g/\partial(kx)|^2 \ll |g-\eta|$ are necessary so that usual definitions of radiation, first stated for linear fields, be applicable here. These definitions presume that the radiation is "free", i.e., that it propagates independently of the sources and that superposition holds. In relativity, then, one can only meaningfully speak of radiation if the energy density in the wave zone is small. Otherwise, the self-coupling of the theory comes into play and effectively acts as a source in the region, giving rise, for example, to scattering of waves by the Newtonian-like parts of the field as well as to the scattering of two waves of high frequency to yield waves of other frequencies. A similar situation exists in quantum electrodynamics, due to vacuum polarization. There, one has an effective nonlinear $\delta T^0{}_0 \sim \alpha^2(\mathbf{E}^2-\mathbf{B}^2)+O(\alpha^3)$ for the Maxwell field, so that in the presence of an arbitrary external electromagnetic field, the self-coupling can produce distortion of "waves" (Delbrück scattering). Also, the $\delta T^0{}_0$ term produces scattering of light by light. The usual definitions of electromagnetic radiation would thus fail in a domain where $\delta T^0{}_0$ is significant, and in fact the wave zone is then defined only when the self-coupling has no physical effects. Of course, numerically, the nonlinear effects we are forbidding are automatically small in most conceivable situations for gravitation.

It should be remarked that our restrictions on $|g_{\mu\nu}-\eta_{\mu\nu}|$ and its first derivatives have been imposed on all components, and so, in particular, on the parts depending on the choice of coordinates. Thus, for the same physical situation, it may be necessary to go to larger distances in some frames than in others in order to reach a region where the equations of motion of the dynamical modes are flat-space wave equations. However, we will see that one can distinguish coordinate waves from the true physical waves and that the former may be transformed away if desired. The isolation of coordinate waves will be discussed in Sec. III.

With the above definition of the wave zone, we shall find that the full field equations reduce to those of linear theory for the canonical variables, so that these variables propagate as free radiation. We shall see, further, that the energy flux of this radiation is measured by the Poynting vector of the linearized theory. Both this vector and the canonical variables themselves are *coordinate-independent* in the wave zone, and thus form a suitable *invariant* basis for the analysis of radiation. One can, in fact, obtain these quantities invariantly from measurements of the spatial part of the metric and its first time derivatives.

Beyond the wave front, the dynamical variables vanish, leaving only the Newtonian-like and coordinate-dependent parts. In a subsequent paper (IV c), we shall show that from the Newtonian-like parts of the field one may obtain a *coordinate-invariant* definition of the energy-momentum vector of the total system.

II. FIELD EQUATIONS IN THE WAVE ZONE

A complete set of Cauchy data specifying the state of the gravitational field[3] on a space-like surface (which we take to be $t=\text{const}$) is provided by the variables[2] g_{ij} and π^{ij}. Here g_{ij} is the spatial part of the metric and defines the intrinsic geometry of the surface, while π^{ij} is effectively the first time derivative of g_{ij}: $\pi^{ij} \equiv (-{}^4g)^{\frac{1}{2}} \times ({}^4\Gamma^0{}_{lm}g^{li}g^{mj} - g^{ij}\,{}^4\Gamma^0{}_{lm}g^{lm})$. Equivalently, π^{ij} is related to the second fundamental form,[4] K_{ij}, by $\pi^{ij} \equiv -({}^3g)^{\frac{1}{2}} \times (K^{ij}-g^{ij}K^l{}_l)$, and thus describes the external curvature of the surface as imbedded in the four-space. This set of Cauchy data determines not only the state of the system, but the four coordinates of the surface as well. Thus, four of these twelve quantities fix the coordinates, leaving four dynamical variables, and four Newtonian-like components determined as functions of the other eight by the constraint equations $G^0{}_\mu \equiv {}^4R^0{}_\mu - \delta^0{}_\mu\,{}^4R = 0$. The division is best seen in terms of the orthogonal breakup of the symmetric g_{ij} and π^{ij} into transverse and longitudinal parts (as in III):

$$g_{ij}-\delta_{ij} = g_{ij}{}^{TT} + g_{ij}{}^T + (g_{i,j}+g_{j,i}), \tag{2.1a}$$

$$\pi^{ij} = \pi^{ijTT} + \pi^{ijT} + (\pi^i{}_{,j}+\pi^j{}_{,i}). \tag{2.1b}$$

The quantity $g_{ij}{}^T \equiv \frac{1}{2}[\delta_{ij}g^T - (1/\nabla^2)g^T{}_{,ij}]$ is determined by the single component g^T, where $1/\nabla^2$ is the inverse of the flat space Laplacian with solutions vanishing at spatial infinity. In a coordinate system specified by

[3] The first paragraph of this section is a brief review of the results of III.

[4] See, for example, L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1949).

$g_i=0=\pi^T$, or equivalently by

$$g_{ij,j}=0, \tag{2.2a}$$

$$\pi^{ij}{}_{,ij}-\pi^{ii}{}_{,jj}=0, \tag{2.2b}$$

it was shown in III that $g_{ij}{}^{TT}$, π^{ijTT} form two pairs of canonically conjugate variables, while g^T and π^i are the Newtonian-like parts. The g^T and π^i are obtained as functions of the canonical variables by solving the four constraint equations

$$^3R+\tfrac{1}{2}\pi^2-\pi^{ij}\pi_{ij}=0, \quad \pi\equiv\pi^l{}_l, \tag{2.3a}$$

$$\pi^{ij}{}_{|j}\equiv\pi^{ij}{}_{,j}+\pi^{lm}\Gamma^i{}_{lm}=0. \tag{2.3b}$$

In the wave zone, the conditions of Sec. 1 imply that, when one wishes to use a frame different from (2.2), the new metric $g_{\mu\nu}'$ differs from $\eta_{\mu\nu}$ by terms of order $a/r\ll 1$. Therefore in the coordinate transformation $x'^\mu=x^\mu-\xi^\mu$, the quantities $\xi^\mu{}_{,\nu}$ must go as a/r. The coordinate changes themselves, ξ^μ, can then behave as $\sim a\ln r$, ax^i/r, $a\exp(q_\mu x^\mu)/r$ (as well as, of course, higher structures like $1/r$). When differentiated, these forms all give $\xi^\mu{}_{,\nu}\sim a/r$ (this behavior is also permitted past the wave front). In this connection, we may distinguish two classes of functions: those whose derivatives are $\sim 1/r$ smaller than the functions (which we will call "static") and those whose derivatives are of the same order (which we will call "oscillatory"). Thus, the structures $\ln r$, x^i/r, b/r fall into the former class; such ξ^μ appear, for example, in transforming from Schwarzschild to isotropic coordinates (which involves $\xi^i\sim mx^i/r$). The $\exp(iq_\mu x^\mu)/r$ form falls into the second class, representing a "coordinate wave" (which can also exist past the wave front). To order $\sim a/r$, the g_{ij} and π^{ij} transform according to the linearized coordinate transformation law:

$$g_{ij}'=g_{ij}+\xi^i{}_{,j}+\xi^j{}_{,i}, \tag{2.4a}$$

$$\pi^{ij\prime}=\pi^{ij}+(\xi^0{}_{,ij}-\delta_{ij}\xi^0{}_{,ll}), \tag{2.4b}$$

the nonlinear terms being $O(1/r^2)$ or higher. (Correspondingly, $g_{00}'=g_{00}-2\xi^0{}_{,0}$ and $g_{0i}'=g_{0i}+\xi^i{}_{,0}-\xi^0{}_{,i}$.) Comparing Eqs. (2.4) with (2.1), we see that only $g_{i,j}$ and π^{ijT} are affected by the linearized transformation. To make sure that the remaining quantities $g_{ij}{}^{TT}$, π^{ijTT}, g^T, and $\pi^i{}_{,j}$ are invariants in the wave zone, one must show that the neglected $O(1/r^2)$ terms do not have $O(1/r)$ effects on these components. Such terms arise in the nonlinear parts of the coordinate transformations, and in Appendix B it is shown that they indeed do not affect, to $O(1/r)$, the $g_{ij}{}^{TT}$, π^{ijTT}, g^T, and $\pi^i{}_{,j}$. This is in spite of the appearance of $1/\nabla^2$ operators in the definitions of the relevant orthogonal components.

To illustrate the behavior of the various parts of the metric in the wave zone, let us first examine them in the frame of Eqs. (2.2). The canonical variable $g_{ij}{}^{TT}$ may, in the interior, have arbitrary amplitude, but will, for a radiation case, fall off as a/r in the wave zone (e.g., $\sim f\exp(ik_\mu x^\mu)/r$). Note that in this region both space and time derivatives acting on $g_{ij}{}^{TT}$ maintain its $\sim 1/r$ character. Beyond the wave front, however, $g_{ij}{}^{TT}$ vanishes more rapidly, since the system is bounded. The g^T component is determined from the constraint equations (2.3) to be $g^T=(1/\nabla^2)\mathcal{T}^0{}_0[g^{TT},\pi^{TT}]$, where $-\int d^3r\,\mathcal{T}^0{}_0[g^{TT},\pi^{TT}]$ is the Hamiltonian of the gravitational field. Beyond the wave front, g^T then has a static behavior as $\sim E/r$, where E is the *total* energy of the system (including the waves). In the wave zone, additional $1/r^2$ terms (both static and wave-like) enter in g^T, as is shown in Appendix C. While g^T is the dominant part of the metric beyond the wave front in the full theory, the linearized theory neglects g^T everywhere, since g^T begins quadratically in the canonical variables. [That this is the case is seen by solving for g^T in the constraint equations (2.3) in terms of the g^{TT}, π^{TT}.] The situation for the conjugate variables

$$\pi^{ij}=\pi^{ijTT}+(\pi^i{}_{,j}+\pi^j{}_{,i}) \tag{2.5}$$

is in close parallel. Here the canonical variables π^{ijTT} go as kfe^{ikx}/r in the wave zone, and vanish rapidly beyond it. On the other hand, the constraint variables π^i (which are quadratic in g^{TT}, π^{TT} and hence are neglected in linearized theory) are static and go as $\sim P^i/r$ (where P^i is the total momentum) outside the wave zone [from the constraint equations, $\pi^i\approx(1/\nabla^2)\mathcal{T}^0{}_i$] and so $\pi^i{}_{,j}\sim P^i/r^2$ there (for details, see end of Appendix C). In the wave zone, other terms of comparable magnitude appear in $\pi^i{}_{,j}$ (e.g., $\sim kf^2e^{ikx}/r^2$). Finally, we will see, in Appendix C, that the remaining components, $g_{0\mu}-\eta_{0\mu}$, of the metric correctly behave as $O(a/r)$.

We shall now show that, *in the wave zone*, the dynamics of the full theory rigorously reduces to that of the linearized approximation. This is not a trivial fact, since, as we have seen, g^T and π^i go as $\sim a/r$ in the wave zone, and are not zero, as they would be in linearized theory. The distinction here is clearly due to the difference between linearizing (expanding to first order in $g_{\mu\nu}-\eta_{\mu\nu}$) and going to the asymptotic form (expanding in powers of $1/r$). The quantities that will obey linear equations of motion are $g_{ij}{}^{TT}$ and π^{ijTT}. We have seen previously that in the frame of Eqs. (2.2), these quantities represented the canonical variables of the theory; further, as was discussed above, they are coordinate-invariant to $O(1/r)$ in the wave zone. They are therefore a suitable complete set of variables to describe the dynamics of the radiation. The proof itself will be given in an arbitrary frame and is subject only to the three inequalities characterizing the wave zone, as stated in Sec. 1. The field equations

of the full theory are Eqs. (2.3) and:

$$\partial_0 g_{ij}=2N({}^3g)^{-\frac{1}{2}}(\pi_{ij}-\tfrac{1}{2}g_{ij}\pi)+\eta_{i|j}+\eta_{j|i}, \tag{2.6a}$$

$$\begin{aligned}\partial_0\pi^{ij}=&-N({}^3g)^{\frac{1}{2}}({}^3R^{ij}-\tfrac{1}{2}g^{ij}\,{}^3R)\\&+\tfrac{1}{2}N({}^3g)^{-\frac{1}{2}}g^{ij}(\pi^{mn}\pi_{mn}-\tfrac{1}{2}\pi^2)\\&-2N({}^3g)^{-\frac{1}{2}}(\pi^{im}\pi_m{}^j-\tfrac{1}{2}\pi\pi^{ij})\\&+({}^3g)^{\frac{1}{2}}(N^{|ij}-g^{ij}N^{|m}{}_{|m})\\&+[(\pi^{ij}\eta^m)_{|m}-\eta^i{}_{|m}\pi^{mj}-\eta^j{}_{|m}\pi^{mi}].\end{aligned} \tag{2.6b}$$

In the wave zone, all components of $g_{\mu\nu}-\eta_{\mu\nu}$ and π^{ij} go at most as $\sim a/r$; as a result, all nonlinear terms are $O(1/r^2)$ and Eqs. (2.6) reduce to

$$\partial_0 g_{ij}=2(\pi^{ij}-\tfrac{1}{2}\delta_{ij}\pi)+\eta_{i,j}+\eta_{j,i}+O(1/r^2), \tag{2.7a}$$

$$\begin{aligned}\partial_0\pi^{ij}=\tfrac{1}{2}(g_{ij,kk}+g_{kk,ij}-g_{ik,kj}-g_{jk,ki})+(N_{,ij}-\delta^{ij}N_{,ll})\\+\tfrac{1}{2}\delta^{ij}(g_{lk,lk}-g_{ll,kk})+O(1/r^2).\end{aligned} \tag{2.7b}$$

To examine the motion of the dynamical modes, we perform the orthogonal breakup of Eq. (2.1) on Eqs. (2.7), noting that it commutes with both ∂_0 and ∂_i. Formally, the "TT" parts of the equations are obtained by applying the linear operation,

$$\begin{aligned}f_{ij}{}^{TT}\equiv f_{ij}-\tfrac{1}{2}\delta_{ij}f_{ll}-(1/\nabla^2)[f_{il,lj}+f_{jl,li}\\-\tfrac{1}{2}\delta_{ij}f_{lm,lm}-\tfrac{1}{2}f_{ll,ij}-(1/2\nabla^2)f_{lm,lmij}].\end{aligned} \tag{2.8}$$

As is shown in Appendix B, this operation is well-defined even for $f_{ij}\sim 1/r$, $1/r^2$ and yields then $f_{ij}{}^{TT}\sim 1/r$, $1/r^2$, respectively. The equations obeyed by $g_{ij}{}^{TT}$ and π^{ijTT} to order $1/r$ are just

$$\partial_0 g_{ij}{}^{TT}=2\pi^{ijTT}, \tag{2.9a}$$

$$\partial_0\pi^{ijTT}=\tfrac{1}{2}\nabla^2 g_{ij}{}^{TT}. \tag{2.9b}$$

To complete the proof, it remains to be shown that the terms of order $1/r^2$ dropped in Eqs. (2.9) are indeed negligible. First, if one is to neglect $\sim 1/r^2$ terms, then one must also neglect $1/r^2$ terms in derivatives of g^{TT} or π^{TT}. With g^{TT}, $\pi^{TT}\sim e^{ikx}/r$ this means neglecting e^{ikx}/r^2 with respect to $(k/r)e^{ikx}$. Our first condition,

$$kr\gg 1, \tag{2.10a}$$

guarantees this. Correspondingly, Eqs. (2.9) yield information only about Fourier components satisfying the condition $kr\gg 1$, as the lower frequency parts give a contribution comparable to the $1/r^2$ terms. Thus, Eq. (2.10a) has arisen just as in electrodynamics, where this condition is needed if one is to neglect the source terms. Turning to the $O(1/r^2)$ terms in Eqs. (2.7), we first consider nonlinearities arising from undifferentiated $g_{\mu\nu}-\eta_{\mu\nu}$ (as in expansions of $g^{\mu\nu}$ in terms of $g_{\mu\nu}$). These give rise to quadratic terms at least $|g_{\mu\nu}-\eta_{\mu\nu}|\sim a/r$ times the leading linear ones. By our second condition,

$$|g_{\mu\nu}-\eta_{\mu\nu}|\sim a/r\ll 1, \tag{2.10b}$$

such terms (as well as all higher powers of $g_{\mu\nu}-\eta_{\mu\nu}$) are negligible. The last class of $\sim 1/r^2$ terms involves derivatives of $g_{\mu\nu}$, which are $\lesssim k_{\max}|g_{\mu\nu}-\eta_{\mu\nu}|\sim k_{\max}a/r$. We wish to show that a component of $g_{ij}\sim ae^{ikr}/r$ can be treated by linearized theory in the presence of other Fourier components extending up to frequencies $\sim k_{\max}$. Equation (2.7a) then shows that the k frequency part of π^{ij} is $\sim kae^{ikx}/r$, which we will use in estimating the $O(1/r^2)$ terms of Eq. (2.7b). Every term in Eq. (2.7b) contains two derivatives of $g_{\mu\nu}$ (counting π^{ij} as ∂g). A k frequency component in the leading linear terms is then $\sim k^2ae^{ikx}/r$. In the nonlinear terms, factors containing frequencies up to $k_{\max}$ may interfere to produce a k frequency component. The maximum value of such a term is $\sim k_{\max}{}^2(a/r)^2e^{ikr}$. [Higher order nonlinear terms, which contain no additional derivatives, are smaller by higher powers of (a/r).] By the third condition,

$$k\gg k_{\max}(a/r)^{\frac{1}{2}}, \tag{2.10c}$$

these too are negligible. Similar arguments then show that the $O(1/r^2)$ term in Eq. (2.7a) is smaller than the leading term there by a factor $(k_{\max}/k)(a/r)\ll 1$. This shows that the $O(1/r^2)$ terms in Eqs. (2.7) are negligible compared to the leading linear $1/r$ structures.

Equations (2.9) were obtained by applying the "TT" operation of Eqs. (2.8) to Eqs. (2.7). As we have mentioned, Eqs. (2.9) are only meaningful for the high frequency components. For these components, the significant point (shown in Appendix B) is that the "TT" operation preserves magnitudes so that the $O(1/r^2)$ terms yield a negligible contribution to Eqs. (2.9). This completes the derivation that the dynamical modes in the wave zone obey the linearized equations (2.9) at a fixed time.

Finally, we will show that these modes *continue* to propagate according to the linearized equations for a sufficiently long time. To do this, we show that the neglected $O(1/r^2)$ terms (which were negligible initially) do not produce large cumulative effects upon time integration. We proceed by iteration, that is, we integrate the linear equations, and use these solutions to estimate the integrated effect of the higher terms. For example, we take Eq. (2.9b), and consider the $\mathbf{k}$ wave number component:

$$\partial_0\pi^{ijTT}=-\tfrac{1}{2}\mathbf{k}^2 g_{ij}{}^{TT}+O_k(1/r^2). \tag{2.9b$_k$}$$

The linear solution has frequency $\omega=|\mathbf{k}|$, but the $O_k(1/r^2)$ terms may be time independent even with *wave number* $\mathbf{k}$. Such terms, in fact, being secular, will grow fastest in time integration. The ratio, λ, before integration, of O_k to k^2g^{TT} is $\lambda\sim f/r\ll 1$ by Eq. (2.10b). Integrating for a time τ, the leading term is multiplied by $\omega^{-1}=|\mathbf{k}|^{-1}$, the secular O_k term by τ. Thus the ratio becomes $\lambda\sim k\tau(f/r)$ and thus remains much less than one for many periods. We now see that this available time is sufficient for our purposes. Initial data are available for Eqs. (2.9) only in the wave zone, $r>R$. Thus at a point $r=R+L$, one can only integrate the linearized equations up to a time L later (since after a longer interval one would be within the light cone of

the interior region). Thus the error becomes at most $\lambda \sim (kf)L/r$. To estimate kf, we note the mass contained within $R<r<R+L$ is, by the linearized Hamiltonian, $\int_R^{R+L} d^3r[\frac{1}{4}(\nabla g^{TT})^2+(\pi^{TT})^2] \sim k^2f^2L \leqslant m_{\text{tot}}$. Thus $\lambda \leqslant (L/r)^{\frac{1}{2}}(m_{\text{tot}}/r)^{\frac{1}{2}}$, and since $L/r<1$, and $m_{\text{tot}}/r \ll 1$, λ is quite small.

The remaining field equations in the wave zone, i.e., the equations for $\partial_0 g^T$, $\partial_0 g_{i,j}$, $\partial_0 \pi^i{}_{,j}$, and $\partial_0 \pi^T$ can be obtained by similar arguments. They are

$$\partial_0 g^T = [-\pi^i{}_{,i}], \tag{2.11a}$$

$$\partial_0 \pi^i{}_{,j} = 0, \tag{2.11b}$$

$$\partial_0 g_{i,j} = \eta_{i,j} - \tfrac{1}{2}(1/\nabla^2)\pi^T{}_{,ij} + [2\pi^i{}_{,j} - (1/\nabla^2)\pi^l{}_{,lij}], \tag{2.12a}$$

$$\partial_0 \pi^T = -2\nabla^2 N - [\tfrac{1}{2}\nabla^2 g^T], \tag{2.12b}$$

and again are meaningful only for high-frequency components. From the constraint equations (2.3), one can see that $\pi^i{}_{,j}$, $\nabla^2 g^T \sim 1/r^2$. Thus the bracketed terms in Eqs. (2.11, 2.12) are to be discarded. Note that Eqs. (2.11) are Bianchi identities, while Eqs. (2.12) determine N and η_i when coordinate conditions are imposed on π^T and g_i. In Appendix C, an alternate derivation of Eqs. (2.9) is given under slightly stronger assumptions than those used here. The extra assumptions are of the type needed in order that higher multipole terms in $(1/\nabla^2)\mathcal{T}^0{}_\mu$ be negligible compared to the monopole term in the wave zone (which can always be satisfied by suitably increasing the radius of the wave zone). The derivation then shows that Eqs. (2.9, 2.11, 2.12) are valid also for low frequencies.

III. OBSERVABLES IN THE WAVE ZONE

The main result of the previous section was embodied in Eqs. (2.9) for the $1/r$ parts of the dynamical modes of the full theory, g^{TT} and π^{TT}, in the wave zone. Equations (2.9) are identical in form to the equations for the canonical variables in linearized theory and therefore correspond to the linearized Hamiltonian density $\mathcal{H}_{\text{lin}} = \frac{1}{4}(\nabla g^{TT})^2 + (\pi^{TT})^2$. These equations are, of course, linear, ensuing that superposition holds; further, they are source-free, indicating that the radiation propagates independently of its origin, and without any self-interaction or dependence on the Newtonian-like part of the field. Thus, the flat-space wave equation $\square^2{}_{\text{flat}} g_{ij}{}^{TT}=0$ represented by (2.9) shows the absence of curvature effects on the radiation. Note also that coordinate effects have disappeared from the wave equation, since the coordinate-dependent parts of the field, g_i, π^T, and $g_{0\mu}$ are no longer present. Since $g_{ij}{}^{TT}$ and π^{ijTT} are invariant to order $1/r$ under coordinate transformations which leave the metric asymptotically flat (and which do not involve a Lorentz transformation at infinity), they represent a coordinate-independent description of the radiation in a fixed Lorentz frame. When a Lorentz transformation is performed, one must use the "TT" variables in the new Lorentz frame to describe the radiation, just as in electrodynamics one must use the transverse parts of the transformed $\mathbf{A}^T$ and $\mathbf{E}^T$.

In the presence of matter (e.g., the Maxwell field), the above derivation may be carried through equally well, provided, of course, the matter system is contained in the interior region (and has finite energy content). Under these circumstances, any gravitational waves in the wave zone are still independent of the matter sources. Even if electromagnetic waves are propagating in the wave zone with small amplitude (and $\sim 1/r$), the gravitational radiation is still unaffected by the electromagnetic waves, since the latter couple quadratically ($\sim 1/r^2$) into the field equations (2.6). Thus in the common wave zone, the two systems are correctly independent.

Our discussion has so far been concentrated on the dynamical ("TT") modes in the wave zone, where they are coordinate-invariant and represent the physical waves. We can also examine coordinate waves in this region; the only components of g_{ij} and π^{ij} affected by coordinate transformations to $\sim 1/r$ are $g_{i,j}$ and π^T (as are also $g_{0\mu}$, by the equations of motion). Therefore, $g_{i,j}$ and π^T carry the effects of the choice of coordinate system. One may then adopt the convention that a frame for which these quantities are nonoscillatory (in order $1/r$) has no coordinate waves. The frame of Eqs. (2.2), for example (where $g_{i,j}=0=\pi^T$), falls within this class, as does the "isotropic" frame (see V) specified by $g_{i,j} - (1/4\nabla^2)g^T{}_{,ij} = 0 = \pi^T + 2\pi^i{}_{,i}$. For such "static" frames, the frequency $k_{\max}$ appearing in condition (2.10c) is then a physical wave frequency rather than one connected with a coordinate wave. However, in an "oscillatory" frame, where $g_{i,j}$ or π^T are not static, the highest frequency in the full metric may reside in the coordinate waves. In this case, condition (2.10c) may be unnecessarily restrictive, due to the choice of coordinates. Such a situation may be recognized independent of our convention simply by noting that the full metric contains frequencies higher than those in the canonical modes. One has then defined the wave zone to be further out than is necessary. By making the coordinate transformation $x^{i\prime} = x^i + g_i$, $\bar{t} = t - (1/2\nabla^2)\pi^T$, one may remove these excessively high-frequency coordinate waves (of course, only the high-frequency parts of g_i and π^T need be included in the transformation). Similarly, condition (2.10b) may also be too restrictive if the size of the metric is governed by its coordinate parts g_i and π^T.

We turn next to the question of energy flux in the wave zone, i.e., to the Poynting vector $\mathcal{T}^{i0}$. In the canonical formalism, as developed in III and IV, an expression for $\mathcal{T}^{i0}$ has not been derived. It is not *a priori* the same as the momentum density $\mathcal{T}^{0i}$, which was shown to be $-2\pi^{ij}{}_{,j}$ in the frame (2.2). However, a simple physical argument proves that, for the wave zone, at least, $\mathcal{T}^{i0} = \mathcal{T}^{0i} = -2\pi^{ij}{}_{,j} = 2\pi^{lm}\Gamma^i{}_{lm}$. We have

seen that, in the wave zone, the full metric $g_{\mu\nu}$ satisfies the linearized equations (2.7), along with linearized form of the constraint equations (2.3):

$$g_{ij,ij}-g_{ii,jj}=O(1/r^2)\sim 0, \quad -2\pi^{ij}{}_{,j}=O(1/r^2)\sim 0. \quad (3.1)$$

Thus, if one imagines a dilute absorber of energy in this region (i.e., one whose energy content is not so highly concentrated as to produce strong fields), then the behavior of such an absorber is governed by linearized theory.[5] Since linearized theory is a standard Lorentz-covariant theory, its symmetric stress-tensor can be obtained by conventional techniques; the result has been recorded in I. In particular, the energy absorbed is given by this T^{i0}. One may also carry out the proof of this result from the point of view of emission of the energy rather than its absorption. Although the physical metric we are considering satisfies Eqs. (2.7) and (3.1) without the $O(1/r^2)$ terms, only in the wave zone, it is possible to construct a parallel situation in which the field is everywhere weak, but agrees with the first in the wave zone. Since it is weak, the parallel field satisfies linearized equations everywhere. However, in order to obtain agreement between the corresponding *solutions* in the wave zone, one must introduce fictitious matter sources in the parallel situation (whether or not there were any in the original case). This is needed in order to simulate the effects in the wave zone solutions produced by the $O(1/r^2)$ terms. Thus, while the interior $O(1/r^2)$ terms do not affect the high-frequency components of the solutions (Sec. 2), they can conceivably modify the low-frequency parts, which include the dominant parts of g^T and π^i. For example, Eqs. (3.1) for the parallel situation read:

$$g_{ii,jj}-g_{ij,ij}\equiv\nabla^2 g^T=\mathrm{T}_S{}^0{}_0, \quad (3.2)$$

$$-2\pi^{ij}{}_{,j}\equiv-2(\pi^i{}_{,jj}+\pi^j{}_{,ij})=\mathrm{T}_S{}^0{}_i, \quad (3.3)$$

where the simulating sources $T_S{}^0{}_\mu$ vanish in the wave zone and are to be distributed sufficiently thinly within the interior region so that linearized theory remains valid. To show that the simulation can be achieved by dilute sources, consider for example, Eq. (3.2). We wish to reproduce the physical g^T by $(1/\nabla^2)\mathrm{T}_S{}^0{}_0$. Since g^T satisfies the Laplace equation in the wave zone (and beyond), is of the form $\sum r^{-l-1}Y_{lm}(\theta,\varphi)M_{lm}$. Therefore, one need only choose $\mathrm{T}_S{}^0{}_0$ such that $\int_0^\infty d^3r\, r^l Y_{lm}\mathrm{T}_S{}^0{}_0 = M_{lm}$. By the condition (2.10b), $M_{lm}/r^{l+1}\sim a/r\ll 1$, i.e., $M_{lm}\sim ar^l$. Thus, the conditions on $\mathrm{T}_S{}^0{}_0$ can be satisfied with a $|\mathrm{T}_S{}^0{}_0|$ of the order of magnitude a/R^3 (where R is the inside radius of the wave zone) up to $r\sim R$ and zero outside. For such a source, $(1/\nabla^2)\mathrm{T}_S{}^0{}_0\sim a/R\ll 1$ everywhere, so that the analog g^T is kept small throughout. Equation (3.3) can be similarly analyzed for π^i The fictitious $\mathrm{T}_S{}^0{}_\mu$ we have introduced will require a $\mathrm{T}_S{}^i{}_j$ such that $\mathrm{T}_S{}^\mu{}_{\nu,\mu}=0$ to maintain the Bianchi identities. This $\mathrm{T}_S{}^i{}_j$ will also be small; it will appear in the analog of Eq. (2.7b) and, like $\mathrm{T}_S{}^0{}_\mu$, vanishes in the wave zone. With the analog g^T and $\pi^i{}_{,j}$ now guaranteed weak everywhere, we may specify the remaining analog components of g_{ij} and π^{ij} initially to agree with the physical field in the wave zone and to be weak in the interior. The components $g_{0\mu}$ may be similarly specified for all time, since no time derivatives of them appear. The linearized version of Eqs. (2.6) [i.e., Eqs. (2.7)] now holds rigorously for the analog metric for all space. The latter then propagates maintaining smallness, and hence, according to linearized theory, for sufficiently long times,[6] i.e., $t\sim R$. We have now guaranteed the identity of the two situations in the wave zone. Note that the analog metric is a solution of the full theory's equations; due to its smallness it was seen to be a solution of the linearized equations as well. An absorber of gravitational radiation in the wave zone, then, clearly cannot distinguish between the two situations, and we can therefore use the Poynting vector of linearized theory to measure the energy flux of the radiation.

The Poynting vector as given by linearized theory, must of course be time-averaged, as in electrodynamics. However, since coordinate waves do not necessarily have frequency related to wave number, one must also average over wavelengths to obtain the physical $\mathcal{T}^{i0}$ in the presence of such waves.[7] From I, one finds that

$$\mathcal{T}_{\mathrm{lin}}{}^{i0}=\pi^{lmTT}(2g_{il}{}^{TT}{}_{,m}-g_{lm}{}^{TT}{}_{,i}) \quad (3.4)$$

to $O(1/r^2)$, where the averaging is understood. In establishing Eq. (3.4), it is easiest first to verify it in the coordinate system (2.2), where $g_i=0=\pi^T$. One may then show that $\mathcal{T}_{\mathrm{lin}}{}^{i0}$ in any other frame differs from its value in this frame by divergences of quadratic structures such as $(\xi^0{}_{,ll}\xi^i{}_{,j})_{,j}$, as well by terms manifestly $O(1/r^3)$. If $\xi^\mu{}_{,\nu}$ represents a coordinate wave (fe^{ipx}/r), then the divergence is either oscillatory and $\sim 1/r^2$ or the phases cancel, in which case it is $\sim(1/r^2)_{,j}\sim 1/r^3$. Thus, with the averaging over oscillations, $\mathcal{T}^{i0}$ is coordinate-independent through $O(1/r^2)$. The right-hand side of Eq. (3.4) is obviously coordinate-invariant through $O(1/r^2)$ since π^{TT} and g^{TT} are invariant through $O(1/r)$ and the change due to transforming the gradients is also $\sim 1/r^3$. Hence Eq. (3.4) provides an invariant formula for the wave-zone Poynting vector. Note that g^T and π^i do not enter to $O(1/r^2)$, showing that the Newtonian-like parts do not affect the energy flux of radiation. Similarly, in electrodynamics, $\mathbf{E}^L$ does not contribute to $\mathbf{E}\times\mathbf{B}$ since $\mathbf{E}^L\sim 1/r^2$.

[5] To analyze the behavior of the dilute absorber, consider initial Cauchy data for both the field and the absorber specified in a region many wavelengths in size around the absorber. These data are sufficient to determine the state of the absorber for a time many periods long, during which the absorber's backward light cone intersects only this region. Since the system is in the wave zone, the final state of the absorber is given by linearized theory [by the discussion of Eq. (2.9b$_k$)].

[6] The derivation is similar to the one used in the discussion Eq. (2.9b$_k$).

[7] One can always recognize the existence of such coordinate waves in the Poynting vector since, starting from the frame (2.2), they are found entirely in g_i and π^T. In frame (2.2), $\mathcal{T}_{\mathrm{lin}}{}^{i0}$ is just the right-hand side of Eq. (3.4) *without* spatial averaging.

We now show that the Poynting vector is indeed $-2\pi^{ij}{}_{,j}=2\pi^{lm}\Gamma^i{}_{lm}$. The right-hand side of (3.4) is just $2\pi^{lm}\Gamma^i{}_{lm}+O(1/r^3)$ in the frame of Eqs. (2.2), and $2\pi^{lm}\Gamma^i{}_{lm}$ can be verified to be coordinate invariant through $O(1/r^2)$ as above (the transformation terms forming divergences of quadratic structures). Finally, $-2\pi^{ij}{}_{,j}=2\pi^{lm}\Gamma^i{}_{lm}$ in all frames. (The invariance of $\pi^{ij}{}_{,j}$ is also directly verifiable from its definition.) The invariance of the Poynting vector holds only if the Lorentz frame at infinity is left unchanged by the transformation, since the invariant components of g_{ij} and π^{ij} discussed earlier in this section would otherwise be altered. Under Lorentz transformations, $\mathcal{T}^{i0}$ transforms, of course, like the $(i0)$ component of a Lorentz tensor, as can be seen from its linearized theory definition.

In the presence of matter, the constraint equation (2.3b) used above is modified to become

$$-2\pi^{ij}{}_{,j}=2\pi^{lm}\Gamma^i{}_{lm}+\mathcal{T}_M{}^{i0}, \tag{3.5}$$

where to $O(1/r^2)$, any metric dependence in $\mathcal{T}_M{}^{i0}$ may be replaced by its flat space value. The total Poynting vector, i.e., the energy flux carried by all fields, including the gravitational field, is

$$\mathcal{T}_{\text{tot}}{}^{i0}=-2\pi^{ij}{}_{,j}, \tag{3.6a}$$

while the measure of purely gravitational energy flux remains

$$\mathcal{T}_{\text{grav}}{}^{i0}=2\pi^{lm}\Gamma^i{}_{lm}. \tag{3.6b}$$

The separate fluxes are clearly additive.

The canonical variables can be measured in any frame by means of Fourier analysis of the field, since the $kr\gg 1$ condition implies that the wave zone is large enough to single out a particular Fourier component. If one takes such a Fourier component of g_{ij} and $\partial_0 g_{ij}$, $g_{ij}(k)^{TT}$ and $\pi^{ij}(k)^{TT}$ can be extracted algebraically. Alternately, one may measure the curvature tensor ${}^4R_\alpha{}^\mu{}_{\nu\beta}$ to obtain these physical quantities[8]:

$$g_{ij}(k)^{TT}=(2/\mathbf{k}^2)\,{}^4R_i{}^m{}_{mj}(k), \tag{3.7a}$$

$$\pi^{ij}(k)^{TT}=-(ik_m/\mathbf{k}^2)\,{}^4R_i{}^0{}_{mj}(k), \tag{3.7b}$$

where $\mathbf{k}$ is the propagation vector. Such measurements find a parallel in Maxwell theory, where $\mathbf{E}^T$, $\mathbf{A}^T$ are found from $\mathbf{E}$ and $\mathbf{B}$ by

$$\mathbf{E}^T(k)=\mathbf{E}(k)-\mathbf{k}(\mathbf{k}\cdot\mathbf{E})/\mathbf{k}^2, \tag{3.8a}$$

$$\mathbf{A}^T(k)=i\mathbf{k}\times\mathbf{B}(k)/\mathbf{k}^2. \tag{3.8b}$$

While Fourier measurements are, in principle, nonlocal in space, for a fixed desired accuracy, the region required is finite. In fact, electromagnetic wave measurements are commonly of this type, as are recent proposals for measuring gravitational waves.[9]

Finally, it may be noted that "S-matrix" types of measurements made in the asymptotic domain are sufficient to determine a great deal of other information about the interior region. For example, Plebanski[10] has recently shown that the properties of orbits (even if they remain in the strong field interior) can be determined by geodesic projection to the asymptotic domain. Thus, with knowledge of the invariant physical quantities at infinity, one may, in principle, discuss a wide class of characteristics of the system with apparatus located entirely outside strong gravitational fields.

IV. DISCUSSION

In this investigation, we have considered under what circumstances the full self-interacting gravitational field possesses a wave zone in which its wave modes behave as a radiation field propagating independently of the sources and according to the equations of the linearized approximation.[11] To do this, it was necessary to examine the rigorous field equations; it was then found that, if the full metric becomes flat as $1/r$ at spatial infinity, a wave zone always exists. In this region, the basic physical quantities describing the radiation's amplitudes and the energy it carries are invariantly defined. Further, the above boundary condition on the metric necessarily implies that a wave front exists, beyond which the physical waves fall off rapidly. Otherwise, one would find an infinite contribution to the energy from the radiation in the wave zone (using the linearized Hamiltonian (which is valid there), and then from $g^T=(1/\nabla^2)\,\mathcal{T}^0{}_0$, g^T would violate the asymptotic requirement. The necessity for the existence of a wave front has previously been noted by Papapetrou.[12,13]

It is instructive to compare our results with some previous investigations on radiation. In a recent work, Trautman[14] has discussed the generalization of the Sommerfeld outgoing wave conditions to gravitation. The main aim of Trautman's work was to formulate appropriate radiation boundary conditions in the hope that these conditions would be sufficient to replace the initial Cauchy data for the gravitational field. In that case, one would have an analog of the retarded Green's function specification of solutions in linear theories, in which knowledge of the external sources plus outgoing wave boundary conditions are sufficient to determine

[8] Equations (3.7) again demonstrate the invariance of $g_{ij}{}^{TT}$ and π^{ijTT} in the wave zone, since, as is well known, ${}^4R_\alpha{}^\mu{}_{\nu\beta}$ is coordinate invariant in linearized theory.

[9] J. Weber, Phys. Rev. **117**, 306 (1960).

[10] J. Plebanski (private communication).

[11] A. Peres and N. Rosen, Nuovo cimento **13**, 430 (1959), have suggested that linearized theory may not be a valid approximation even when g_{ij} and $\partial_0 g_{ij}$ are weak. In arriving at this, they fail to recognize that the apparently linear quantity $\nabla^2 g^T$ is really of second order, this being the content of the constraint equation (2.3a). Their difficulty is due to their "inconvenient" choice of Cauchy data, g_{ij} and $\partial_0 g_{ij}$ (rather than g_{ij} and π^{ij}), which, as they show, leads to an unstable determination of the solution.

[12] A. Papapetrou, Ann. Physik **2**, 87 (1958).

[13] The possibility of using a wave front to make the energy finite seems to have been overlooked in a calculation similar to Papapetrou's by A. Peres and N. Rosen, Phys. Rev. **115**, 1085 (1959).

[14] A. Trautman, Bull. Acad. Sci. (Poland) **6**, 403, 407 (1958).

the solutions uniquely. Since Trautman was concerned with characterizing outgoing waves, he did not investigate the structure of the full field equations; in this paper, we have stated under what circumstances (i.e., the wave zone criteria) such radiation may exist. Also, the coordinate frames he considered were restricted by asymptotic DeDonder conditions, so that all his coordinate waves satisfied the wave equation asymptotically. Hence, no general separation was made between the invariant wave amplitudes and coordinate effects.

A minimal invariant statement of the outgoing wave boundary conditions is provided by $g_{ij}{}^{TT}$ itself: $(\partial_r+\partial_t)g_{ij}{}^{TT}\sim 1/r^2$; this condition is significant *in the wave zone*, where $g_{ij}{}^{TT}$ is $\sim 1/r$. Since $g_{ij}{}^{TT}$ is expressible in terms of the curvature tensor [Eqs. (3.7)], these two conditions may be rephrased as $k^\mu R_{\mu ij0}\sim 1/r^2$, where k^μ is the outgoing propagation vector, $k_\mu=\partial_\mu(r-t)$. The "pure radiation" conditions of Lichnerowicz[15] are: $k^\mu R_{\mu\alpha\beta\gamma}=0=k_{[\mu}R_{\alpha\beta]\sigma\tau}$, and it is easy to see that these are satisfied in order $1/r$ in the wave zone, using the field equations. It is interesting to note that, on physical grounds, the conjecture that outgoing wave boundary conditions determine the solution uniquely can be related to one made by Papapetrou.[12] In turn, these ideas are connected with the question of the positive-definiteness of the gravitational field's energy.[16] Papapetrou has given arguments for believing that every solution of the Einstein equations satisfying $g_{\mu\nu}-\eta_{\mu\nu}\rightarrow 0$ as $r\rightarrow\infty$ is such that for $t\rightarrow\infty$, the field becomes time-independent in a suitable frame. That is, $g_{\mu\nu}\rightarrow g^S{}_{\mu\nu}+h_{\mu\nu}$ where $g^S{}_{\mu\nu}$ is time-independent, while $h_{\mu\nu}\rightarrow 0$ at $t=\infty$. According to a theorem of Lichnerowicz,[17] the only time-independent solution of the source-free field equations is given by flat space, i.e., $g^S{}_{\mu\nu}=\eta_{\mu\nu}$, the Lorentz metric. Thus, one would expect[18] by this theorem, that the energy (which is, of course, conserved) is positive-definite since the linearized theory's energy expression (see I) $E=\int d^3r[\frac{1}{4}(g_{ij}{}^{TT}{}_{,k})^2+(\pi^{ijTT})^2]\geqslant 0$ is valid as $t\rightarrow\infty$. The physical situation at $t=\infty$, then, is of a spreading radiation field, such as $f(t-r)/r$, which gets more diffuse with time. The conjecture of Papapetrou therefore states that all initial field configurations (however intense) eventually dissipate into free radiation. The contrary possibility, namely a bound state of the field (e.g., a completely stable geon), which Papapetrou excludes from his discussion (since it cannot be treated by perturbation expansions), should physically have negative energy. No such bound-state solutions are known at present. Returning to Trautman's radiation boundary conditions, then, it would seem physically reasonable that such conditions could, in fact, replace initial conditions on the independent gravitational wave modes. For, with the exception of bound states, these modes would eventually spread out to spatial infinity and be determined by the radiation conditions there.

[15] A. Lichnerowicz, Compt. rend. **246**, 893 (1958).

[16] For some classes of solutions, it can be shown that the field's energy is positive-definite: D. Brill, Ann. Phys. **7**, 466 (1959); R. Arnowitt, S. Deser, and C. W. Misner, Ann. Phys. **11**, 116 (1960).

[17] A. Lichnerowicz, *Theories Relativistes de la Gravitation et de l'Electromagnetisme* (Masson, Paris, 1955), p. 142.

[18] Actually, Lichnerowicz's theorem would have to be generalized to make this argument rigorous, since $g^S{}_{\mu\nu}$ does not rigorously satisfy the Einstein equations (the $h_{\mu\nu}$ acting as effective weak sources).

Another approach to radiation has been that of Petrov[19] and Pirani[20] in terms of the algebraic structure of the Riemann tensor, $R_{\mu\nu\alpha\beta}$. By analogy with Maxwell theory (where $\mathbf{E}\cdot\mathbf{B}=0=\mathbf{E}^2-\mathbf{B}^2$ are the characteristic algebraic properties of plane waves, and hold asymptotically for an outgoing radiation field), Pirani suggested that the form of $R_{\mu\nu\alpha\beta}$ representing radiation should be the case II-null of the Petrov classification. Thus, their approach would suggest that the curvature tensor represents free radiation in the wave zone only if its $1/r$ leading term has the II-null structure. This, in fact, agrees with the results obtained here. Thus, using the linearized form of the Riemann tensor and of the field equations (valid in the wave zone), one easily finds[21] that $R_{\mu\nu\alpha\beta}$ is type II-null to order $1/r$.

The wave zone metric provides the information concerning the behavior of radiation; it is also of interest to investigate the properties of the metric past the wave front. More generally, one may examine the truly asymptotic form of the metric whether or not radiation exists. This will be carried out in a following paper (IVc). Beyond the wave front, the dynamical variables $g_{ij}{}^{TT}$, $\pi_{ij}{}^{TT}$ are negligible, leaving only the Newtonian-like parts of the metric g^T and π^i and the coordinate-dependent components g_i and π^T. It is shown in Appendix C that g^T and π^i are determined entirely by the energy-momentum P^μ of the system; they are of order P^μ/r and their derivatives are $\sim P^\mu/r^2$ (see also IVc). Thus it is possible to choose coordinates yielding nonoscillating g_i and π^T (i.e., no coordinate waves). It would seem reasonable, therefore, to expect that one can choose frames in which Lichnerowicz's boundary conditions[17] $g_{\mu\nu,\alpha}\sim 1/r^2$ are satisfied even when radiation is present. Of course, they must be understood to hold in the truly asymptotic region beyond the wave front at each fixed time.

[19] A. Z. Petrov, Sci. Notices Kazan State Univ. **114**, 55 (1954).

[20] F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957); see also H. Bondi, F. A. E. Pirani, and I. Robinson, Proc. Roy. Soc. (London) **A251**, 529 (1959).

[21] By a suitable choice of reference frame at a point, the II-Null criterion states that $R_{1212}=R_{1220}=R_{3130}=R_{2020}=R_{1331}=R_{0330}$, with all other components not obtained by symmetry vanishing. This condition is satisfied by a wave travelling in the "1" direction, and polarized in the "2, 3" directions: $g_{22}{}^{TT}=-g_{33}{}^{TT}=fe^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}/r$, $(\omega=|\mathbf{k}|)$ all other $g_{ij}{}^{TT}=0$. Note that the II-Null criterion is satisfied by a superposition of spherical waves, provided they are purely outgoing (or purely incoming). However, it is conceivable that, from our wave zone definition, there may exist radiation propagating in more than one direction. In that case, the II-null criterion is satisfied by the separate contributions to the curvature from each propagation direction. Of course, at sufficiently large distances, all waves look spherical.

APPENDIX A

We state here a form of solution for the Poisson equation which is useful in making estimates of size of terms in text. In $\nabla^2\varphi=-4\pi\rho$, we expand φ in spherical harmonics:

$$\varphi(r)=\sum \chi_{lm}(r)r^l Y_{lm}. \tag{A.1}$$

One finds

$$\chi_{lm}(r)=\int_r^\infty \frac{M_{lm}(r')dr}{r'^{2l+2}}, \tag{A.2a}$$

with

$$M_{lm}(r)=\int_0^r d^3r'\,\rho r'^l Y_{lm}. \tag{A.2b}$$

The solution (A.1) has incorporated the boundary condition that $\varphi\to 0$ at infinity. Correspondingly, we have for $\varphi_{,i}$:

$$\varphi_{,i}=\sum \chi_{lm}(r^l Y_{lm})_{,i}-\sum M_{lm}(r)x^i r^{-l-3}Y_{lm}. \tag{A.3}$$

We will use these equations to obtain the asymptotic behavior of φ and $\varphi_{,i}$ both in the wave zone and beyond. We first establish the behavior of φ in a region where ρ has the oscillatory character $\rho\sim Y_{lm}e^{ikr}/r^n$. For our uses, the region will always extend over many wavelengths, and of course $kr\gg 1$. Then only the moment M_{lm} survives. By Eq. (A.2b),

$$M_{lm}=\left(\int_0^R+\int_R^r\right)d^3r(r^lY_{lm})\rho$$
$$\cong(ik)^{-1}r^{l-n+2}e^{ikr}[1+O(1/kr)]+c_{lm}. \tag{A.4}$$

Here R is a radius beyond which the asymptotic form for ρ is valid. The constant c_{lm} is determined by the interior behavior of ρ; it is the integral up to R plus the contribution from the lower limit (R) of the remaining integral. From Eq. (A.1), one finds

$$\varphi\sim(1/k^2)\frac{e^{ikr}}{r^n}Y_{lm}[1+O(1/kr)]+c_{lm}Y_{lm}\frac{1}{r^{l+1}}. \tag{A.5}$$

We shall be interested in text in the oscillatory part of φ. The c_{lm} term does not contribute to it; hence for high frequencies, ∇^2 can be inverted locally. This is true even if the c_{lm} term is numerically very large in the wave zone.

A second case of interest is one in which one knows only $\int_0^\infty d^3r|\rho|\equiv B$ is finite. Then

$$|M_{lm}|\leqslant Br^l, \tag{A.6a}$$

so that

$$r^l|\chi_{lm}|\leqslant B/r. \tag{A.6b}$$

For this situation, $\varphi\sim 1/r$ and $\varphi_{,i}\sim 1/r^2$.

APPENDIX B

In this appendix we consider the orthogonal decomposition of a symmetric tensor f_{ij}, and relate its asymptotic behavior with that of its orthogonal components. These components are defined by

$$f_{i,j}=(1/\nabla^2)\{f_{ik,kj}-\tfrac{1}{2}[(1/\nabla^2)f_{lm,lm}]_{,ij}\}, \tag{B.1a}$$

$$f_{ij}{}^T=\tfrac{1}{2}[f^T\delta_{ij}-(1/\nabla^2)f^T{}_{,ij}],$$
$$f^T=(1/\nabla^2)(f_{ll,mm}-f_{lm,lm}), \tag{B.1b}$$

$$f_{ij}{}^{TT}=f_{ij}-f_{ij}{}^T-(f_{i,j}+f_{j,i}), \tag{B.1c}$$

where $1/\nabla^2$ means that solution of the Poisson equation which vanishes at infinity. Note that only the operator $(1/\nabla^2)\partial_i\partial_j$ appears in Eqs. (B.1), and that the order of these operations ($\partial_i\partial_j$ and $1/\nabla^2$) can be important to insure that $1/\nabla^2$ exists. We restrict ourselves to a "static" asymptotic behavior for f_{ij} since oscillatory terms have been separately treated in Appendix A. Thus we assume

$$f_{ij}\sim\sum\frac{a_n(\theta,\varphi)}{r^n}, \tag{B.2}$$

and, of course, that f_{ij} is regular for some interior region where this asymptotic expansion is no longer valid. By linearity, it is sufficient to consider one term in f_{ij}, say $f_{ij}\equiv\psi\sim a_n/r^n$. To compute $(1/\nabla^2)\psi_{,ij}$, then, we need the moments M_{lm} of $\psi_{,ij}$. The monopole and dipole moments can be evaluated explicitly:

$$M_{00}=\int_0^r\psi_{,ij}d^3r=\oint\psi_{,i}dS_j\sim\langle a_n\rangle/r^{n-1}, \tag{B.3a}$$

$$M_{1k}=\int_0^r x^k\psi_{,ij}d^3r=\oint x^k\psi_{,i}dS_j-\oint\delta_j{}^k\psi dS_i$$
$$\sim\langle a_n\rangle/r^{n-2}. \tag{B.3b}$$

The symbol $\langle a_n\rangle$ is used generically to represent angle averages such as $\int d\Omega a_n Y_{lm}$. For the higher moments, we have

$$M_{lm}=\left(\int_0^R+\int_R^r\right)r^lY_{lm}\psi_{,ij}d^3r\sim\langle a_n\rangle r^{l-n+1}+c_{lm}. \tag{B.4}$$

Here R is a radius beyond which the asymptotic expansion for ψ is valid. The constant c_{lm} is determined by the interior behavior of ψ; that is, it is the integral up to R as well as the lower limit (R) contribution from the remaining integral. [Note that in Eq. (B.4), but not in Eq. (B.3), "r^0" can mean $\ln r$.] Then from Eqs. (A.1,2) we have

$$\frac{1}{\nabla^2}\psi_{,ij}\sim\frac{\langle a_n\rangle}{r^n}+\sum_{l=2}^\infty\frac{c_{lm}Y_{lm}}{r^{l+1}}. \tag{B.5}$$

Thus, for $n<3$, the leading term in (B.5) is determined by the asymptotic behavior of ψ (and no $\ln r$ factors appear), while for $n\geqslant 3$, the leading term is affected by the interior behavior of ψ (and the term with $l+1=n$ may also contain $r^{-n}\ln r$).

We summarize our results relating to the decomposition of f_{ij}: First, the orthogonal components are defined whenever f_{ij} vanishes as $1/r^\epsilon$ at infinity. Secondly, terms slower than $1/r^3$ in the asymptotic expansion of $f_{ij}{}^{TT}$, $f_{ij}{}^{T}$, and $f_{i,j}$ are determined by, and are comparable in magnitude to, the corresponding terms in f_{ij}. (By Appendix A, the identical statement holds for oscillatory terms.) Finally the $1/r^3$ and higher terms are influenced by the interior behavior of f_{ij}, so the magnitude of their coefficients cannot be estimated from the asymptotic behavior of f_{ij}. [These static "interior" terms in Eq. (B.5), which involve c_{lm}, can be numerically quite large if the system has enormously high interior multipole moments. In that case, they could, in some region, be the dominant terms of the orthogonal components of f_{ij} even though they may not be the leading terms in powers of $1/r$. For the cases $n<3$, however, they can always be made negligible by taking r large enough. This procedure will be used in Appendix C, but is not necessary for the wave zone derivation in text.]

APPENDIX C

We give an alternate derivation of Eqs. (2.9), that the rigorous dynamical modes obey the linearized equations in the wave zone. The proof is performed directly on the asymptotic form of the field equations without applying the orthogonal decomposition operator on them. This can be done by use of coordinate conditions (2.2), and it will then be seen that the results hold in any asymptotically rectangular frame.

The usefulness of the frame (2.2) lies in the fact that for it, $g_{i,j}=0=\pi^{ijT}$. Thus by substituting for g_{ij} and π^{ij} their decomposition (2.1) into Eqs. (2.7), the latter may be written as

$$\begin{aligned}\partial_0 g_{ij}{}^{TT}+\partial_0 g_{ij}{}^{T} &= 2\pi^{ijTT}+(2\pi^i+\eta_i)_{,j}+(2\pi^j+\eta_j)_{,i} \\ &\quad -2\delta_{ij}\pi_{j,l}+O(1/r^2), \qquad \text{(C.1a)}\end{aligned}$$

$$\begin{aligned}\partial_0\pi^{ijTT}+\partial_0(\pi^i{}_{,j}+\pi^j{}_{,i}) &= \tfrac{1}{2}\nabla^2 g_{ij}{}^{TT}-\tfrac{1}{2}[\delta_{ij}\nabla^2(2N+\tfrac{1}{2}g^T) \\ &\quad -(2N+\tfrac{1}{2}g^T)_{,ij}]+O(1/r^2), \qquad \text{(C.1b)}\end{aligned}$$

where the $O(1/r^2)$ terms are negligible by the two conditions on the wave zone which eliminate nonlinear terms. We will derive Eqs. (2.9) by showing that all terms except those involving $g_{ij}{}^{TT}$ and π^{ijTT} are negligible. We first show that $\pi^i{}_{,j}\sim 1/r^2$ and is negligible. (Note that $1/r^2$ is *not a priori* negligible with respect to $1/r$, since the ratio of such terms might be a large number such as R/r where R is a length of the order of the wave front radius.) From the constraint equation (2.3b), we have

$$\nabla^2\pi^i{}_{,i}=-\tfrac{1}{2}(\pi^{lm}\Gamma^i{}_{lm})_{,i}\equiv-\tfrac{1}{2}(\pi\Gamma^i)_{,i}. \qquad \text{(C.2)}$$

By Appendix A, the oscillatory parts of $\pi^i{}_{,i}$ arise entirely from the oscillatory parts of $(\pi\Gamma^i)_{,i}$ in the wave zone. These behave as $\sim k^3a^2e^{ikx}/r^2$, and, by Eq. (A.5), contribute to $\pi^i{}_{,i}$ there a term $\sim ka^2e^{ikx}/r^2$, which is negligible. We may therefore assume in the remainder of the derivation of $\pi^i{}_{,j}$ that $(\pi\Gamma^i)$ is nonoscillatory in the wave zone. The moment (see Appendix A) associated with the source $(\pi\Gamma^i)_{,i}$ is

$$\begin{aligned}M_{lm} &= \int_0^r r'^l Y_{lm}'(\pi\Gamma^i)_{,i}d^3r' \\ &= \oint_r (r^l Y_{lm}')(\pi\Gamma^i)dS_i' \\ &\quad -\int_0^r (r'^l Y_{lm}')_{,i}(\pi\Gamma^i)d^3r'. \qquad \text{(C.3)}\end{aligned}$$

We estimate M_{lm} as follows:

$$M_{lm}\lesssim r^{l+2}\langle|\pi\Gamma^i|\rangle+r^{l-1}\int_0^r|\pi\Gamma^i|d^3r', \qquad \text{(C.4)}$$

where the symbol $\langle\ \rangle$ represents the angular average. Hence from Eqs. (A.1), (A.2a),

$$r^l\chi_{lm}\lesssim r^{-2}\int_r^\infty\langle|\pi\Gamma^i|\rangle d^3r'+\int_r^\infty\frac{dr'}{r'^3}\int_0^{r'}d^3r''|\pi\Gamma^i|. \qquad \text{(C.5)}$$

We note first that, since $2(\pi\Gamma^i)$ is just the momentum density of the field, $2\int_0^\infty d^3r(\pi\Gamma^i)=P^i\lessgtr P^0$. In the inequality (C.5), there appear the absolute values of the density. It can be shown (see end of this Appendix) that π^{ij} and $\Gamma^i{}_{jk}$ each go as $\sim 1/r^{\frac{3}{2}+\epsilon}$ beyond the wave zone, so that $|\pi\Gamma^i|\sim 1/r^{3+\epsilon}$ there; hence these integrals exist (assuming, as always, the regularity of the field throughout the interior). Setting $\int_0^\infty d^3r|\pi\Gamma^i|\equiv P^{i\prime}$, one sees that $|\pi^i{}_{,i}|\lesssim P^{i\prime}/r^2$. For most physical situations one expects $P^{i\prime}\simeq P^i\lesssim P^0$; it is conceivable that there can exist situations with portions of the interior field carrying enormous momenta ($P^{i\prime}\sim R$, the radius of the wave zone) such that these cancel to the small number $P^i(\ll R)$. Here, one would have to redefine[22] the wave zone to be a region of larger r such that $P^{i\prime}$ again becomes $\ll r$ (and wait for the waves to reach the larger boundary). For simplicity, then, we shall assume that $P^{i\prime}\ll r$, so that

$$\pi^i{}_{,i}\sim(P^{i\prime}/r)(1/r), \qquad \text{(C.6)}$$

and is negligible in (C.1). To establish that the full $\pi^i{}_{,j}$ is also negligible one again uses the methods of Appendix A, since

$$\nabla^2\pi^i{}_{,j}=-(\pi\Gamma^i)_{,j}-\partial_i\partial_j\pi^p{}_{,p}. \qquad \text{(C.7)}$$

The first source term, $(\pi\Gamma^i)_{,j}$, may be handled as before, while the $\partial_i\partial_j\pi^p{}_{,p}$ source can be shown by Appendix B

[22] An identical situation holds in electrodynamics in the presence of a current j_μ. One usually requires that $\int|j_\mu|d^3r$ is comparable in size to $\int j_\mu d^3r$. For anomalous situations, the wave zone must be redefined as in the gravitational case.

to contribute $\sim(P^{i\prime}/r)1/r$, using the previous estimates (C.4, C.5) for $\pi^{p}{}_{,p}$.

We next examine the behavior of $g^{T}{}_{,ij}$; this is obtained from Eq. (2.3a). The linear part of ${}^{3}R$ is clearly $-\nabla^2 g^T$ and so the equation may be rewritten as

$$-\nabla^2 g^T = \mathcal{P}^0(g_{ij},\pi^{ij}). \qquad \text{(C.8)}$$

Since $\int \mathcal{P}^0 d^3r = -\int \nabla^2 g^T d^3r = P^0$, one may think of $\mathcal{P}^0$ as an energy density. {Note, however, that $\mathcal{P}^0$ is not the Hamiltonian density $-\mathcal{T}^0{}_0[g_{ij}{}^{TT},\pi^{ijTT}]$, as g^T and π^i have not been eliminated in terms of canonical variables in $\mathcal{P}^0$ (see III).} From the wave zone conditions, it is clear that $\mathcal{P}^0 \sim 1/r^2$ *and* is negligible, so that $\nabla^2 g^T$ is to be discarded in Eq. (C.1b). To see that $g^T{}_{,ij}$ is also negligible in this equation, we use the multipole expansion of Appendix B. The source term is then $\mathcal{P}^0{}_{,ij}$. Again the oscillatory terms of $\mathcal{P}^0$ can be treated separately, and shown to be negligible, i.e., to contribute to $g^T{}_{,ij}$ a term $\sim k^2 f^2 e^{ikx}/r^2$. From the remaining part of $\mathcal{P}^0$, which is nonoscillatory in the wave zone, one then finds a contribution to $g^T{}_{,ij} \sim P^{0\prime}/r^2$, where $P^{0\prime} = \int_0^\infty d^3r |\mathcal{P}^0|$. One can show $P^{0\prime}$ to be finite, since $\mathcal{P}^0 \sim 1/r^{3+\epsilon}$ beyond the wave zone; its magnitude is expected in general to be $\sim P^0 = \int d^3r \mathcal{P}^0$. Again, if there exist situations of large positive and negative energy contributions to the total P^0, each much greater than P^0, then $P^{0\prime}$ would greatly exceed P^0, and one would redefine the wave zone appropriately further out.

The remaining terms to be proven negligible in Eqs. (C.1) (now that $g^T{}_{,ij}$, $\pi^i{}_{,j}$ and hence $\partial_0 \pi^i{}_{,j}$ are known to be negligible) are $\eta_{i,j}$ and $N_{,ij}$ as well as $\partial_0 g^T{}_{ij}$. The η_i and N terms are determined by the coordinate conditions, Eqs. (2.2): $g_{ij,j} = 0 = \pi^T$. Thus the divergence of Eq. (2.6a) provides an equation from which η_i may be estimated by the techniques of Appendix A. Similarly, the trace of Eq. (2.6b) allows N to be estimated. These estimates show that $\eta_{i,j}$ and $N_{,ij}$ go as $\sim 1/r^2$ *and* are negligible, by use of the absolute value bounds on multipole integrals. (Note also that η_i and N go as $\sim 1/r$.) Thus Eqs. (2.9) have been derived; at the same time, one sees that Eqs. (2.11, 2.12) are also valid in the frame (2.2).

To see that Eqs. (2.9) hold in any asymptotically rectangular frame, one uses the invariance of $g_{ij}{}^{TT}$ and π^{ijTT} to $O(1/r)$ and the fact that the transformation of the explicit derivatives in these equations only introduces higher order terms. Equations (2.11) are similarly invariant, while (2.12) are clearly the transform of the above results to an arbitrary frame, due to the transformation properties of N and η_i (i.e., $g_{0\mu}$). The above analysis has therefore shown that, in the wave zone, the full set of linearized theory equations are valid at *all* frequencies.

We conclude this Appendix with a brief discussion of the behavior of the metric beyond the wave front. A more complete analysis of this domain will be found in IVc. The canonical modes $g_{ij}{}^{TT}$, π^{ijTT} must fall off at least as $\sim 1/r^{\frac{3}{2}+\epsilon}$ in order that the energy in the asymptotic region, $\int^\infty d^3r[\frac{1}{4}(g_{ij}{}^{TT}{}_{,k})^2 + (\pi^{ijTT})^2]$ be finite. In the frame (2.2), $g_i = 0 = \pi^T$ while the quantities g^T and π^i are determined by the constraint equations (2.3). As shown in III, these equations may be rewritten as

$$g^T{}_{,ii} = \mathcal{T}^0{}_0, \qquad \text{(C.9a)}$$

$$-2\pi^{ij}{}_{,j} \equiv -2(\pi^i{}_{,jj} + \pi^j{}_{,ij}) = \mathcal{T}^0{}_i, \qquad \text{(C.9b)}$$

where $\mathcal{T}^0{}_\mu$ is the energy-momentum density in the frame (2.2). Since these are Poisson-like equations, the leading $1/r$ terms in their solutions depend only on the monopole moments of the sources, $P_\mu \equiv \int \mathcal{T}^0{}_\mu d^3r$. One has then[23]

$$g^T \sim P_0/r, \qquad \text{(C.10a)}$$

$$\pi^i \sim P_i/r. \qquad \text{(C.10b)}$$

Thus $g^T{}_{,i}$ and $\pi^i{}_{,j}$ go as $1/r^2$, and by conservation of P_μ their time derivatives go faster than $1/r$. Consequently, $(\pi\Gamma^i)$ goes faster than $1/r^3$ and $\int d^3r|\pi\Gamma^i|$ exists. A similar derivation shows that $\int d^3r|\mathcal{P}^0|$ exists.

[23] In distinction to the wave zone analysis, one need not estimate here the coefficients of terms going faster than $1/r$ since r may be taken arbitrarily large *beyond* the wave front.