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Continuation of Scattering Amplitudes and Form Factors through Two-Particle Branch Lines*†

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It is shown that scattering amplitudes and form factors have two-particle branch lines which connect two Riemann sheets. For partial wave amplitudes and form factors the dispersive parts and, except for square root factors, the absorptive parts are regular functions in the cut energy plane except for isolated poles, physical inelastic cuts and left-hand branch lines. In order to show this it is assumed that, for particles without composite structure, the amplitudes have only such singularities in the physical sheet which correspond to absorptive processes. The analytic properties of absorptive parts are used for a general discussion of structure singularities (anomalous thresholds). It is shown that these structure cuts are extensions of left-hand branch lines in the second Riemann sheet. An example is given of a dispersion relation on the Riemann surface in which the integral over the two-particle branch line is eliminated.

I. INTRODUCTION

RECENTLY, several authors[1,2] have considered the continuation of propagators and other amplitudes into second Riemann sheets, mainly with the aim to explore the connection between unstable particles and the isolated poles appearing in these sheets.[3] In this note we discuss the continuation of amplitudes through branch lines corresponding to two-particle intermediate states. We are interested in the character of these branch lines and in the analytic properties of the corresponding dispersive and absorptive parts. We explore the use of these analytic properties for the description of resonances, and we discuss their application to the treatment of structure singularities[4] (anomalous thresholds)[5] of form factors and production amplitudes.

In Sec. II we consider the partial wave projections of a scattering amplitude.[6] We assume analyticity in the cut plane of the physical sheet and find, as a simple consequence of the unitarity condition, that the two particle branch line connects just two Riemann sheets. The amplitudes can be written in the form

$$F(z)=d(z)+i\left(\frac{z-4\mu^2}{z}\right)^{\frac{1}{2}}a(z), \qquad (1.1)$$

where $d(z)$ and $a(z)$ are regular functions except for cuts $z=s$ with $s\leq 0$, $s\geq s_i$ [$s_i>4\mu^2$ is the threshold for inelastic processes] and isolated poles due to zeros of the S matrix

$$S=1+2i\left(\frac{z-4\mu^2}{z}\right)^{\frac{1}{2}}F(z). \qquad (1.2)$$

In the region $4\mu^2\leq s<s_i$ on the real axis the function $d(s)$ coincides with the dispersive part and $a(s)[(s-4\mu^2)/s]^{\frac{1}{2}}$ with the absorptive part of the amplitude the $F(s+i0)$. We use these properties of the amplitudes in Sec. III, where we give an example of a dispersion formula on the Riemann surface of $(z-4\mu^2)^{\frac{1}{2}}$. In this relation the explicite integral over the elastic region $4\mu^2\leq s<s_i$ has been eliminated in favor of the contributions from poles and branch lines in the second sheet.

Production amplitudes are discussed in Sec. IV along the same lines as the scattering amplitudes in Sec. II. At first we consider only particles which are sufficiently compact to have no composite structure.[4] Then we use the analytic properties of the absorptive part as a function of the energy variable in order to study the appearance of structure singularities in the physical sheet. These results are used in Sec. V, where we consider form factors of particles without and with composite structure. The analytic continuation of a form factor through the two-particle branch line makes it possible to give a clear description of the structure cut as an extension of the left-hand branch line in the second Riemann sheet.

II. ELASTIC SCATTERING AMPLITUDES

We consider the scattering of neutral pions as an example. The generalization to cases with spin and isotopic spins, or with unequal masses, is straightforward. Let us denote by $M(s,t)$ the covariant, causal amplitude, which has the Fourier representation

$$M(s,t)=2(p_0p_0')^{\frac{1}{2}}i\int d^4x\,\exp[-\tfrac{1}{2}i(k+k')\cdot x]\times\theta(x_0)\langle p'|[j(x/2),\,j(-x/2)]|p\rangle+\text{polynomial}. \qquad (2.1)$$

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† This work was completed during the summer of 1960 when the author was at the Brookhaven National Laboratory, Upton, New York.
[1] M. Lévy, Nuovo cimento **13**, 115 (1959). This paper contains further references.
[2] J. Gunson and J. G. Taylor, Phys. Rev. **119**, 1121 (1960).
[3] R. E. Peierls, *Proceedings of the Glasgow Conference on Nuclear and Meson Physics* (Pergamon Press, New York, 1954), p. 296.
[4] R. Oehme, Nuovo cimento **13**, 778 (1959). This paper contains further references.
[5] Y. Nambu, Nuovo cimento **9**, 1187 (1958); R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. **111**, 1187 (1958); R. Oehme, Phys. Rev. **111**, 1430 (1958).
[6] After this work was completed the author was informed that W. Zimmermann has obtained results similar to those described in Sec. II. Zimmermann's work is within the framework of the axiomatic approach and deals mainly with the analytic properties of $F(s,\cos\vartheta)$ in the second sheet. We would like to thank Professor H. Lehmann and Professor K. Symanzik for informing us about Zimmermann's work. See also K. Symanzik, J. Math. Phys. **1**, 249 (1960).

Here $k+p=k'+p'$, $s=-(k+p)^2$ and $t=-(k-k')^2$ are the usual variables, and the transition amplitude in the barycentric system is given by

$$T_{\mathrm{c.m.}}(s,\cos\vartheta)=\frac{1}{8\pi}s^{-\frac{1}{2}}M(s,t), \tag{2.2}$$

with $\cos\vartheta=1+2t/(s-4\mu^2)$. It is convenient to use the function

$$F(s,\cos\vartheta)=\frac{1}{16\pi}M(s,t)$$

and its partial-wave projection

$$F_l(s)=\frac{1}{2}\int_{-1}^{+1}d(\cos\vartheta)P_l(\cos\vartheta)F(s,\cos\vartheta). \tag{2.3}$$

In the physical region $s\geq 4\mu^2$, and below the first threshold $s_i=16\mu^2$ for inelastic processes, we may write

$$F_l(s)=\left(\frac{s}{s-4\mu^2}\right)^{\frac{1}{2}}\sin\delta_l(s)e^{i\delta_l(s)} \tag{2.4}$$

with real phases $\delta_l(s)$ and positive root.

The partial wave amplitude $F_l(s)$ is the boundary value of an analytic function $F_l(z)$, which is regular at least in some limited region R around the cut $z=s\geq 4\mu^2$. It is sensible and convenient to assume that $F_l(z)$ is actually regular in the complex z plane except for the absorptive branch lines ($z=s\geq 4\mu^2$ and $s\leq 0$ in our model), although analyticity in the domain R would in principle be sufficient for the continuation into the second sheet.

It follows from the well known reality properties of the covariant amplitude $M(s,t)$ that the partial wave amplitude $F_l(z)$ satisfies the condition

$$F_l^*(z^*)=F_l(z). \tag{2.5}$$

Furthermore, along the "elastic" cut $4\mu^2\leq s<s_i$ we have the unitarity relation

$$\mathrm{Im}F_l(s+i0)=[(s-4\mu^2/s)]^{\frac{1}{2}}|F_l(s+i0)|^2. \tag{2.6}$$

Let us now consider the analytic function

$$a_l(z)=\frac{F_l^2(z)}{1+2i[(z-4\mu^2)/z]^{\frac{1}{2}}F_l(z)}. \tag{2.7}$$

Here and in the following we define the roots $(z-4\mu^2)^{\frac{1}{2}}$ and $iz^{\frac{1}{2}}$ such that their imaginary parts are nonnegative in cut planes with the branch lines $z=s\geq 4\mu^2$ and $z=s\leq 0$, respectively. If we then introduce the abbreviation

$$\rho(z)=\left(\frac{z-4\mu^2}{z}\right)^{\frac{1}{2}}, \tag{2.8}$$

we have

$$\rho^*(z^*)=-\rho(z) \tag{2.9}$$

and

$$\rho(s+i0)=+\left(\frac{s-4\mu^2}{s}\right)^{\frac{1}{2}}\geq 0 \tag{2.10}$$

for $s\geq 4\mu^2$.

It follows from Eqs. (2.7) and (2.9) that $a_l(z)$ is a real analytic function which is regular in the cut plane except for the isolated poles due to possible zeros of the denominator. We note that this denominator cannot vanish for $z=s$ with $4\mu^2<s<s_i$, because this would require

$$\mathrm{Im}F_l^{-1}(s\pm i0)=\mp 2\rho(s+i0), \tag{2.11}$$

whereas the unitarity condition implies

$$\mathrm{Im}F_l^{-1}(s\pm i0)=\mp\rho(s+i0). \tag{2.12}$$

On the basis of Eq. (2.6), the function $a_l(z)$ has been constructed such that

$$a_l(s+i0)=a_l(s-i0)=|F_l(s+i0)|^2 \tag{2.13}$$

for $4\mu^2\leq s<s_i$. Hence $a_l(z)$ is a regular function along this section of the real axis, and its right-hand cut starts at $s=s_i$. It follows from Eqs. (2.6) and (2.13) that the absorptive part of $F_l(s+i0)$ is the boundary value of the analytic function

$$A_l(z)=\rho(z)a_l(z), \tag{2.14}$$

since $A_l(s\pm i0)=\mathrm{Im}F_l(s\pm i0)$ for $4\mu^2\leq s<s_i$. We may use the function $A_l(z)$ in order to continue the partial wave amplitude through the elastic cut into the second Riemann sheet. Writing[7]

$$F_l^{\mathrm{II}}(z)=F_l(z)-2iA_l(z)=\frac{F_l(z)}{1+2i\rho(z)F_l(z)}, \tag{2.15}$$

we find

$$F_l^{\mathrm{II}*}(z^*)=F_l^{\mathrm{II}}(z) \tag{2.16}$$

and

$$F_l^{\mathrm{II}}(s\pm i0)=F_l(s\mp i0) \tag{2.17}$$

for $4\mu^2\leq s<s_i$. We see that the elastic branch cut connects just two Riemann sheets. This feature may be exhibited by writing $F_l(z)$ in the form

$$F_l(z)=d_l(z)+i\left(\frac{z-4\mu^2}{z}\right)^{\frac{1}{2}}a_l(z), \tag{2.18}$$

where the root is given by $\rho(z)$ in the first sheet and by $\rho_{\mathrm{II}}(z)=-\rho(z)$ in the second sheet. The function $a_l(z)$ in Eq. (2.18) is defined by Eq. (2.7), and $d_l(z)$ is also a real analytic function which is regular except for the cuts $s\leq 0$, $s\geq s_i$ and the isolated poles due to zeros of the S matrix. We find

$$d_l(z)=F_l^{\mathrm{II}}(z)[1+i\rho(z)F_l(z)], \tag{2.19}$$

[7] *Note added in proof.* We note that Eq. (2.15) may be written in the form $S(q)=F(q)/F(-q)$, where $q=q(z)=(z/4-\mu^2)^{\frac{1}{2}}$ is the c.m. momentum. For potential scattering this relation is similar to the familiar expression of $S(q)$ in terms of Jost functions, but it is not the same [see, for instance, R. Blankenbecler, M. L. Goldberger, N. N. Khuri and S. B. Treiman, Ann. Phys. **10**, 62 (1960), Eq. (5.14)]. We would like to thank Professor R. Karplus for bringing these points to our attention.

and in the region $4\mu^2 \leq s < s_i$ we have

$$d_l(s+i0) = d_l(s-i0) = \operatorname{Re}F_l(s+i0)\,; \qquad (2.20)$$

$d_l(z)$ is the analytic continuation of the dispersive part of the partial wave amplitude. In the same way we can construct the function

$$C_l(z) = d_l(z)/a_l(z) = [1+i\rho(z)F_l(z)]F_l^{-1}(z)\,; \qquad (2.21)$$

it has no cut in the elastic scattering region, where it coincides with the familiar expression

$$C_l(s\pm i0) = \rho(s+i0)\,\operatorname{cotg}\delta_l(s). \qquad (2.22)$$

In the following sections we shall generally omit the subscript l.

III. REPRESENTATIONS ON THE RIEMANN SURFACE

The analytic continuation of $F(z)$ through the elastic branch line can be used in order to write dispersion formulas involving both sheets.[8] For instance, we may consider the contour integrals

$$\frac{1}{2\pi i}\oint dz'\frac{F(z')}{z'-z} \qquad (3.1)$$

and[9]

$$\frac{1}{2\pi i}\oint dz'\frac{F(z')}{[(z'-4\mu^2)(z'-s_i)]^{\frac{1}{2}}(z'-z)} \qquad (3.2)$$

on the Riemann surface with two sheets, which are analytically connected along the cut $z=s$ with $4\mu^2 \leq s < s_i$. Ignoring subtractions, we obtain from Eqs. (3.1) and (3.2) the following relations by deforming the contours on the Riemann surface:

$$F^{\mathrm{I}}(z)+F^{\mathrm{II}}(z) = \frac{1}{\pi}\left\{\int_{-\infty}^{0}+\int^{\infty}\right\}ds\frac{\operatorname{Im}F^{\mathrm{I}}(s+i0)+\operatorname{Im}F^{\mathrm{II}}(s+i0)}{s-z} + \sum_n\left\{\frac{b_n}{z_n-z}+\frac{b_n^*}{z_n^*-z}\right\}, \qquad (3.3)$$

and

$$\{F^{\mathrm{I}}(z)-F^{\mathrm{II}}(z)\}[(z-4\mu^2)(z-s_i)]_{\mathrm{I}}^{-\frac{1}{2}} = \frac{1}{\pi}\left\{\int_{-\infty}^{0}+\int_{s_i}^{\infty}\right\}ds\frac{\operatorname{Im}F^{\mathrm{I}}(s+i0)-\operatorname{Im}F^{\mathrm{II}}(s+i0)}{[(s-4\mu^2)(s-s_i)]_{\mathrm{I}}^{\frac{1}{2}}(s-z)} + \sum_n\left\{\frac{-b_n}{[(z_n-4\mu^2)(z_n-s_i)]_{\mathrm{I}}^{\frac{1}{2}}(z_n-z)} + \frac{-b_n^*}{[(z_n^*-4\mu^2)(z_n^*-s_i)]_{\mathrm{I}}^{\frac{1}{2}}(z_n^*-s)}\right\}. \qquad (3.4)$$

Here the root $[(z-4\mu^2)(z-s_i)]^{\frac{1}{2}}$ is defined in a plane with the cut $z=s$, $4\mu^2 \leq s < s_i$, and in the sheet indicated by the index I it is negative for $s<4\mu^2$. We can solve Eqs. (3.3) and (3.4) for F^{I} and F^{II}, and in this way we obtain a representation of $F(z)$ on the Riemann surface. It may be written in the form

$$F(z) = \frac{1}{\pi}\left\{\int_{-\infty}^{0}+\int_{s_i}^{\infty}\right\} \times ds\frac{P_+(z,s)\operatorname{Im}F^{\mathrm{I}}(s+i0)+P_-(z,s)\operatorname{Im}F^{\mathrm{II}}(s+i0)}{s-z} + \sum_n\left\{P_-(z,z_n)\frac{b_n}{z_n-z}+P_-(z,z_n^*)\frac{b_n^*}{z_n^*-z}\right\}, \qquad (3.5)$$

where the projection functions $P_\pm$ are defined by

$$P_\pm(z,z') = \frac{1}{2}\left\{1\pm\left[\frac{(z-4\mu^2)(z-s_i)}{(z'-4\mu^2)(z'-s_i)}\right]^{\frac{1}{2}}\right\}. \qquad (3.6)$$

They satisfy the reality condition

$$P_\pm^*(z^*,z'^*) = P_\pm(z,z'). \qquad (3.7)$$

In cases where stable one-particle states are possible, the corresponding poles appear only in the first (physical) sheet for $0<s<4\mu^2$.

We note that in the dispersion formula (3.5) the integral over the elastic cut $4\mu^2 \leq s < s_i$ has been eliminated in favor of the sum of the contributions from all singularities in the second sheet. These consist of the meromorphic parts due to the zeros of the S matrix in the denominator of Eq. (2.13) and the integrals over the inelastic cut as well as the left-hand branch line. The weight functions along these cuts in the second sheet are not determined by $\operatorname{Im}F^{\mathrm{I}}(s+i0)$ alone, but they involve also $\operatorname{Re}F^{\mathrm{I}}(s)$. For $s\leq 0$ and for $s\geq s_i$, we have the relation

$$\operatorname{Im}F^{\mathrm{II}}(s+i0) = \frac{\operatorname{Im}F^{\mathrm{I}}(s+i0)-2\rho(s+i0)|F^{\mathrm{I}}(s+i0)|^2}{1+4\rho(s+i0)\{\rho(s+i0)|F^{\mathrm{I}}(s+i0)|^2-\operatorname{Im}F^{\mathrm{I}}(s+i0)\}}. \qquad (3.8)$$

A dispersion formula like Eq. (3.5) could be useful for systems where the low-energy region is dominated by a pole term in the second sheet representing a resonance. In a first approximation it may then be possible to neglect the contribution from the cuts and to retain only the resonance term. The Breit-Wigner type formula obtained in this way could be used in connection with the crossing relations in order to compute correction terms.

In the N/D formulation of elastic π-π scattering,[10] the

[8] For similar considerations in Schrödinger theory with zero range potentials see: R. E. Peierls, Proc. Soc. (London) **253**, 16 (1959).

[9] This is, of course, a special choice. For instance, we could also use the expression $[(z-4\mu^2)/z]^{\frac{1}{2}}$ instead of the root in the denominator of Eq. (3.2).

[10] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

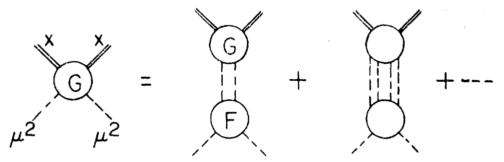

FIG. 1. Amplitude for the reaction $\mu^2+\mu^2 \rightarrow x+x$.

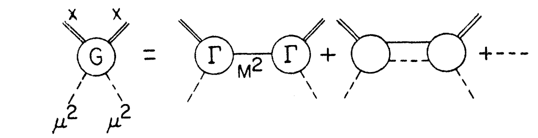

FIG. 2. The amplitude $\mu^2+x \rightarrow \mu^2+x$. We assume that the particles with masses M and $x^{\frac{1}{2}}$ are baryons. As an example one may identify them with Λ and Σ particles.

explicit integral over the two particle branch line has also been removed by use of the unitarity condition. We have, for instance in the case of p-wave scattering, the integral equation

$$D_1(z)=1+\frac{z-4\mu^2}{\pi}\int_{-\infty}^{0} ds(s-4\mu^2)^{-1} \times K(s,z)\,\mathrm{Im}F_1(s+i0)D_1(s),$$

where

$$\pi K(s,z)=\frac{1}{s-z}\{(s-4\mu^2)f(s)-(z-4\mu^2)f(z)\},$$

with

$$f_I(z)=\int_0^\infty ds\frac{1}{[s(s-4\mu^2)]^{\frac{1}{2}}}\frac{1}{s-z}$$
$$=\frac{1}{[z(z-4\mu^2)]^{\frac{1}{2}}}\ln\frac{4\mu^2-2z+2[z(z-4\mu^2)]^{\frac{1}{2}}}{4\mu^2}.$$

In the physical sheet (I), $f(z)$ has only the branch line $s\geq 4\mu^2$, which connects just two Riemann sheets. In all other sheets there is in addition a left-hand cut for $s\leq 0$, which also connects two sheets only. We obtain the complete surface by continuing alternatively through right- and left-hand branch lines. The Riemann surface of

$$F_1(z)=N_1(z)/D_1(z)$$

does not in general have the simple structure described above. In the first sheet there is a left-hand branch line for $s\leq 0$, and the character of the left-hand cut in the second sheet depends upon the properties of $\mathrm{Im}F_1{}^I(s+i0)$ for $s<0$. Only in the pole approximation do we have a structure corresponding to that of $f(z)$. We note that the poles for $z=s_n<0$ are present only in the first sheet.

IV. PRODUCTION AMPLITUDES

Let us discuss the partial wave projection $G(s)$ of a production amplitude $G(s,t)$ corresponding to the graph in Fig. 1. It is reasonable to assume that $G(s)$ is the boundary value of an analytic function $G(z)$, which is regular except for cuts along the real axis. From the reality properties of $G(s,t)$ we find then that $G^*(z^*)=G(z)$. As long as the mass variable x is sufficiently small[4] $[x<(M^2+\mu^2)$ for the amplitude described in Figs. 1 and 2], the corresponding particle has no composite structure, and we have only the branch lines due to absorptive processes, namely for $z=s\geq 4\mu^2$ and for $z=s\leq g(x)$. The function $g(x)$ is determined by the lowest mass intermediate state of the scattering process $\mu^2+x \rightarrow \mu^2+x$ (we indicate particles by their mass variables); it is given by[4,11]

$$g(x)=-\frac{1}{M^2}[x-(M+\mu)^2][x-(M-\mu)^2]. \quad (4.1)$$

Note that $g(x)$ attains its maximum value of $4\mu^2$ at $x=M^2+\mu^2$. The next branch point on the left-hand side is at $s=g_1(x)$, where

$$g_1(x)=-\frac{1}{(M+\mu)^2}[x-(M+2\mu)^2][x-M^2]. \quad (4.2)$$

On the right-hand side we have an unphysical region $4\mu^2\leq s<4x$, where the absorptive part is due to intermediate μ^2-particle states. We are mainly interested in the interval $4\mu^2\leq s<s_i$ of the two-particle branch line. For $x<M^2+\mu^2$ and $x\gtrsim M^2+\mu^2$ we have $s_i=16\mu^2$; for larger values of x there will appear structure singularities associated with the inelastic threshold at $16\mu^2$, and hence $s_i<16\mu^2$. In this paper we shall not consider these higher order structure effects. At least for $x<M^2+\mu^2$, we assume that the absorptive part of $G(z)$ is given in this region by the unitarity relation[12]

$$\mathrm{Im}G(s+i0)=\rho(s+i0)G(s+i0)F(s-i0), \quad (4.3)$$

where $F(s)$ is a scattering amplitude of the type we have discussed in Sec. II and $\rho(z)$ is given by Eq. (2.8). It follows from Eqs. (4.3) and (2.13) that

$$G(s+i0)F(s-i0)=G(s-i0)F(s+i0),$$

for $4\mu^2\leq s<s_i$. This relation assures the reality of $\mathrm{Im}G(s+i0)$. Furthermore, we find, on the basis of these relations, that $G(z)$ may be written in the form

$$G(z)=d_G(z)+i[(z-4\mu^2/z)]^{\frac{1}{2}}a_G(z), \quad (4.4)$$

where $a_G(z)$ and $d_G(z)$ are real analytic functions which are regular except for the cuts $z=s\leq g(x)$, $z=s\geq s_i$, and poles at points where the S matrix,

$$S(z)=1+2i\rho(z)F(z),$$

vanishes. The functions a_G and d_G may be expressed in

[11] S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

[12] A proof would require analytic continuation of the amplitude G in the mass variable x from $x<\mu^2$ toward larger values. This can be done at present only for some simple cases within the framework of perturbation theory.[4,11]

terms of the amplitudes G and F. We find

$$a_G(z)=\frac{G(z)F(z)}{1+2i\rho(z)F(z)}, \tag{4.5}$$

$$d_G(z)=\frac{G(z)[1+i\rho(z)F(z)]}{1+2i\rho(z)F(z)}, \tag{4.6}$$

and in the interval $4\mu^2\leq s<s_i$ on the real axis we have

$$\begin{aligned} a_G(s)&=\rho^{-1}(s+i0)\,\mathrm{Im}G(s+i0),\\ d_G(s)&=\mathrm{Re}G(s). \end{aligned} \tag{4.7}$$

The continuation of $G(z)$ through the two-particle branch line into the second Riemann sheet may be written in the form

$$G^{\mathrm{II}}(z)=\frac{G(z)}{1+2i\rho(z)F(z)}. \tag{4.8}$$

So far we have assumed that $x<(M^2+\mu^2)$, but now we want to consider larger values of x in order to see how the structure singularities (anomalous thresholds) emerge. Ignoring subtractions, which may be required, we represent $G(z)$ for $x<(M^2+\mu^2)$ by the dispersion formula

$$G(z)=\frac{1}{\pi}\int_{-\infty}^{g(x)}ds\frac{\mathrm{Im}G(s+i0)}{s-z}+\frac{1}{\pi}\int_{4\mu^2}^{s_i}ds\frac{\rho(s+i0)a_G(s)}{s-z}+\frac{1}{\pi}\int_{s_i}^{\infty}ds\frac{\mathrm{Im}G(s+i0)}{s-z}. \tag{4.9}$$

Similarly the function $a_G(z)$ may be expressed in the form

$$a_G(z)=\frac{1}{\pi}\int_{-\infty}^{g(x)}ds\frac{\mathrm{Im}a_G(s+i0)}{s-z}+\frac{1}{\pi}\int_{s_i}^{\infty}ds\frac{\mathrm{Im}a_G(s+i0)}{s-z}+\sum_n\left\{\frac{c_n}{z_n-z}+\frac{c_n^*}{z_n^*-z}\right\}. \tag{4.10}$$

Since $F^{\mathrm{II}}(s)$ is real for $0<s<4\mu^2$, we have in this interval

$$\mathrm{Im}a_G(s+i0)=\mathrm{Im}G(s+i0)F^{\mathrm{II}}(s), \tag{4.11}$$

except for δ-function contributions due to possible poles of $F^{\mathrm{II}}(s)$. The resulting real poles in Eq. (4.10) are represented explicitly by the last term. In the interval $g_1(x)<s\leq g(x)$, the absorptive part of $G(s)$ is given by the partial wave projection of the one-particle terms in $G(s,t)$. If these terms are of the form

$$G(s,t)=\frac{\Gamma^2}{M^2-t}+\frac{\Gamma^2}{M^2-\bar{s}}+\cdots,\quad (\bar{s}=2\mu^2+2x-s-t),$$

we find, for instance for the s-wave amplitude,[13]

$$\mathrm{Im}G(s+i0)=A_0(s)=2\pi\Gamma^2(x)/[(s-4\mu^2)(s-4x)]^{\frac{1}{2}}. \tag{4.12}$$

The root in Eq. (4.12) is defined such that it is negative for $s<4\mu^2$ and has a cut for $4\mu^2\leq s\leq 4x$.

Let us now continue Eq. (4.9) in the mass variable x from $x<(M^2+\mu^2)$ to $x>(M^2+\mu^2)$ [but $x<(M+\mu)^2$]. We assume that this continuation is possible and straightforward for the integrals from s_i to ∞ and from $-\infty$ to $g_1(x)$. Note that $g_1(x)<4\mu^2$ because of the stability condition $x<(M+\mu)^2$. In order to continue the remaining integrals in Eqs. (4.10) and (4.9) we take the mass variable slightly off the real axis. Then we have

$$g(x+iy)=[g(x)+y^2/M^2]-2i(y/M^2)(x-M^2-\mu^2), \tag{4.13}$$

and $g(x)$ describes a curve in the complex z plane which encircles the point $z=4\mu^2$ as x passes through $x=M^2+\mu^2$.

The relevant portion of the first integral in Eq. (4.9) is

$$\frac{1}{\pi}\int_{g_1(x)}^{g(x)}dz'\frac{A(z')}{z'-z}, \tag{4.14}$$

where $A(z)$ is given by Eq. (4.12). As $g(x+iy)$ moves around the point $z=4\mu^2$ the path of integration in Eq. (4.14) dives into the second sheet of the root in the denominator of $A(z)$. We finally obtain for $x>(M^2+\mu^2)$ the contributions

$$\frac{1}{\pi}\int_{g_1(x)}^{4\mu^2}ds\frac{A(s)}{s-z}+\frac{1}{\pi}\int_{4\mu^2}^{g(x)}ds\frac{A^{\mathrm{II}}(s)}{s-z}, \tag{4.15}$$

where $g(x)<4\mu^2$ and $A^{\mathrm{II}}(s)=-A(s)$. In a completely analogous way the integral

$$\frac{1}{\pi}\int_{g_1(x)}^{g(x)}dz'\frac{A(z')F^{\mathrm{II}}(z')}{z'-z} \tag{4.16}$$

in the representation (4.10) for $a_G(z)$ gives rise to the expression

$$\frac{1}{\pi}\int_{g_1(x)}^{4\mu^2}ds\frac{A(s)F^{\mathrm{II}}(s)}{s-z}+\frac{1}{\pi}\int_{4\mu^2}^{g(x)}ds\frac{A^{\mathrm{II}}(s)F(s)}{s-z} \tag{4.17}$$

for $x>(M^2+\mu^2)$.

Finally we consider the second integral in Eq. (4.9). The function $a_G(s)$ in the integrand is given by Eq. (4.10), and it is regular in the neighborhood of the point $s=4\mu^2$ as long as $x<(M^2+\mu^2)$. There could, of course, be isolated poles due to resonances or virtual states, but these can always be avoided in the following deforma-

[13] It can be proven that the vertex function $\Gamma(\xi)$ is a real analytic function which is regular for $\xi<(M+\mu)^2$, if we have on the mass shell $x=m^2<M^2$. Such a proof can be given using the methods of R. Oehme, Nuovo cimento 4, 1316 (1956); see K. Symanzik, Phys. Rev. **105**, 743 (1957), also A. M. Bincer, Phys. Rev. **118**, 855 (1960).

tion of integration paths. Note that the position of these poles is independent of the mass variable x. With increasing x the point $g(x+iy)$ circles the point $s=4\mu^2$ and then moves back on the opposite side of the real axis. In order to avoid the branch line ending in $g(x+iy)$ we have to deform the contour of the integral in Eq. (4.9), and, in the limit as $y\to 0$, we find the expression

$$\frac{1}{\pi}\int_{g(x)}^{4\mu^2} ds\frac{\pm\rho(s)\{\pm h(s+i0)\mp h(s-i0)\}}{s-z}+\frac{1}{\pi}\int_{4\mu^2}^{s_i} ds\frac{\rho(s+i0)a_G(s)}{s-z}, \tag{4.18}$$

where

$$h(z)=\frac{1}{\pi}\int_{4\mu^2}^{g(x)} ds\frac{A^{\mathrm{II}}(s)F(s)}{s-z}$$

and hence

$$h(s+i0)-h(s-i0)=-2iA^{\mathrm{II}}(s)F(s) \tag{4.19}$$

for $g(x)\leq s\leq 4\mu^2$. The two sign combinations in the first integral of Eq. (4.18) correspond to the choice $y<0$ or $y>0$, respectively [see Eq. (4.13)]. We note that for $y<0$ the deformed path of the integral in Eq. (4.9) remains in the first sheet of the root $(s-4\mu^2)^{\frac{1}{2}}$, whereas for $y>0$ the contour is dragged into the second sheet through the branch line $s\geq 4\mu^2$. The deformation of the contour is possible because of the analytic properties of the function $a_G(z)$, which have been exhibited in Eqs. (4.10) and (4.17).

Taking all pieces together we have for $x>(M^2+\mu^2)$, instead of Eq. (4.9), the dispersion formula[14]

$$G(z)=\frac{1}{\pi}\int_{-\infty}^{g_1(x)} ds\frac{\mathrm{Im}G(s+i0)}{s-z}+\frac{1}{\pi}\int_{g_1(x)}^{4\mu^2} ds\frac{A(s)}{s-z}+\frac{1}{\pi}\int_{g(x)}^{4\mu^2} ds\frac{-A^{\mathrm{II}}(s)[1+2i\rho(s)F(s)]}{s-z}+\frac{1}{\pi}\int_{4\mu^2}^{s_i} ds\frac{\rho(s+i0)G(s+i0)F^{\mathrm{II}}(s+i0)}{s-z}+\frac{1}{\pi}\int_{s_i}^{\infty} ds\frac{\mathrm{Im}G(s+i0)}{s-z}, \tag{4.20}$$

where $A(s)$ is given by Eq. (4.12) for the case of s-wave amplitudes.[15] The limits $g(x)$ and $g_1(x)$ have been defined earlier. We see that the absorptive part of $G(z)$ for $4\mu^2\leq s<s_i$ is again given by

$$\mathrm{Im}G(s+i0)=\rho(s+i0)a_G(s),$$

[14] If only two-particle singularities are retained and if $\mathrm{Im}G(s+i0)$ is replaced by $A(s)$ [as given in Eq. (4.12)] for $s<g_1(x)$, then Eq. (4.20) agrees with the result of Mandelstam (reference 9). See also R. Blankenbecler and Y. Nambu, Nuovo cimento (to be published); R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960); Phys. Rev. Letters **4**, 532 (1960).

[15] For higher partial waves the relevant features of $A(s)$ are the same as for s waves.

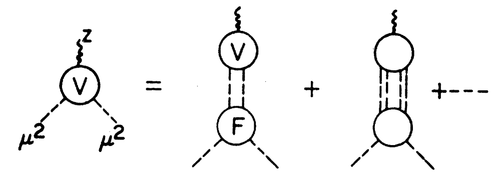

FIG. 3. Vertex for μ^2 particles (mesons).

where

$$a_G(z)=G(z)F^{\mathrm{II}}(z)$$

is regular along this stretch of the real axis.

It is of interest to compare Eq. (4.20) with the corresponding relation for $G^{\mathrm{II}}(z)$. We may write

$$G^{\mathrm{II}}(z)=G(z)\{1-2i\rho(z)F^{\mathrm{II}}(z)\}, \tag{4.21}$$

and, for $x<(M^2+\mu^2)$, we have a representation corresponding to Eq. (4.9) with G replaced by G^{II} and $a_G(s)$ in the second integral by $-a_G(s)$. The continuation to $x>(M^2+\mu^2)$ is analogous to the one leading to Eq. (4.20). We find

$$G^{\mathrm{II}}(z)=\frac{1}{\pi}\int_{-\infty}^{g_1(x)} ds\frac{\mathrm{Im}G^{\mathrm{II}}(s+i0)}{s-z}+\frac{1}{\pi}\int_{g_1(x)}^{4\mu^2} ds\frac{A(s)[1-2i\rho(s)F^{\mathrm{II}}(s)]}{s-z}+\frac{1}{\pi}\int_{g(x)}^{4\mu^2} ds\frac{-A^{\mathrm{II}}(s)}{s-z}+\frac{1}{\pi}\int_{4\mu^2}^{s_i} ds\frac{-\rho(s+i0)a_G(s)}{s-z}+\frac{1}{\pi}\int_{s_i}^{\infty} ds\frac{\mathrm{Im}G^{\mathrm{II}}(s+i0)}{s-z}+\text{pole terms}. \tag{4.22}$$

In the interval $g(x)\leq s<4\mu^2$ the production amplitude G has the absorptive part

$$A(s)-A^{\mathrm{II}}(s)[1+2i\rho(s)F(s)] \tag{4.23}$$

in the first sheet [for $x>(M^2+\mu^2)$], whereas in the second sheet we find from Eq. (4.22) for the corresponding interval

$$-A^{\mathrm{II}}(s)+A(s)[1-2i\rho(s)F^{\mathrm{II}}(s)]. \tag{4.24}$$

Note that, except for a change in sign, Eq. (4.24) is just the continuation of Eq. (4.23) through the branch cut $s\geq 4\mu^2$ of the root $(s-4\mu^2)^{\frac{1}{2}}$.

V. VERTEX FUNCTIONS

The vertex function or form factor of a μ^2 particle (see Fig. 3) satisfies a dispersion relation of the form

$$V(z)=1+\frac{z}{\pi}\int_{4\mu^2}^{\infty} ds\frac{\mathrm{Im}V(s+i0)}{s(s-z)}. \tag{5.1}$$

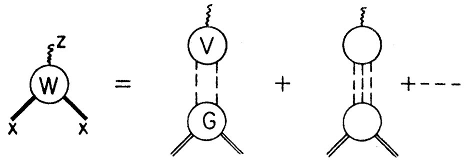

FIG. 4. Vertex for x particles (baryons).

Along the two-particle cut the absorptive part is given by

$$\operatorname{Im} V(s+i0)=\rho(s+i0)a_V(s), \tag{5.2}$$

where

$$a_V(z)=V(z)F^{\mathrm{II}}(z) \tag{5.3}$$

is again regular for $4\mu^2 \leq s < s_i$. The continuation of $V(z)$ into the second Riemann sheet is

$$V^{\mathrm{II}}(z)=\frac{V(z)}{1+2i\rho(z)F(z)}, \tag{5.4}$$

and, with

$$d_V(z)=V^{\mathrm{II}}(z)[1+i\rho(z)F(z)], \tag{5.5}$$

we can write

$$V(z)=d_V(z)+i\left(\frac{z-4\mu^2}{z}\right)^{\frac{1}{2}}a_V(z). \tag{5.6}$$

We may use these properties of the vertex $V(s)$ in order to discuss the form factor of x particles (see Fig. 4):

$$W(s)=2(p_0p_0')^{\frac{1}{2}}\langle p'|J(0)|p\rangle, \tag{5.7}$$

where J is a "scalar" photon current operator and $p'^2=p^2=-x$, $(p'-p)^2=-s$.

We take first $x<(M^2+\mu^2)$ and assume that $W(s)$ is the boundary value of an analytic function, which, except for subtractions, may be represented in the form

$$W(z)=\frac{1}{\pi}\int_{4\mu^2}^{\infty} ds\frac{\operatorname{Im}W(s+i0)}{s-z}, \tag{5.8}$$

where

$$\operatorname{Im}W(s+i0)=\rho(s+i0)G(s+i0)V(s-i0) \tag{5.9}$$

for $4\mu^2 \leq s < s_i$. Using Eq. (5.4) we construct the real analytic function

$$a_W(z)=G(z)V^{\mathrm{II}}(z), \tag{5.10}$$

which is easily seen to satisfy the relation

$$\begin{aligned} a_W(s+i0)=a_W(s-i0)&=G(s+i0)V(s-i0)\\ &=G(s-i0)V(s+i0). \end{aligned} \tag{5.11}$$

As before, the two-particle cut connects two Riemann sheets. The continuation into the second sheet is here

$$W^{\mathrm{II}}(z)=W(z)-2i\rho(z)G(z)V^{\mathrm{II}}(z), \tag{5.12}$$

and we see that $W^{\mathrm{II}}(z)$ has a left-hand branch line for $s\leq g(x)$, where $g(x)$ is given by Eq. (4.1). Let us now write Eq. (5.8) in the form

$$W(z)=\frac{1}{\pi}\int_{4\mu^2}^{s_i} ds\frac{\rho(s+i0)a_W(s)}{s-z} +\frac{1}{\pi}\int_{s_i}^{\infty} ds\frac{\operatorname{Im}W(s+i0)}{s-z}, \tag{5.13}$$

where we can express the function $a_W(z)$ by the dispersion formula

$$a_W(z)=\frac{1}{\pi}\int_{-\infty}^{g(x)} ds\frac{\operatorname{Im}a_W(s+i0)}{s-z}+\frac{1}{\pi}\int_{s_i}^{\infty} ds\frac{\operatorname{Im}a_W(s+i0)}{s-z} +\sum_n\left\{\frac{b_n}{z_n-z}+\frac{b_n^*}{z_n^*-z}\right\}. \tag{5.14}$$

According to Eqs. (5.10) and (5.4) we have, except for δ functions,

$$\operatorname{Im}a_W(s+i0)=\operatorname{Im}G(s+i0)V^{\mathrm{II}}(s) \tag{5.15}$$

for $0<s<4\mu^2$, and from Sec. IV we know that for $g_1(x)<s\leq g(x)$ $\operatorname{Im}G(s+i0)=A(s)$; for s-waves $A(s)$ is given by Eq. (4.12). With the help of Eqs. (4.18), (4.13) etc. the first integral in Eq. (5.13) may now be continued to values of x above $M^2+\mu^2$, but below $(M+\mu)^2$.[16] The situation is completely analogous to the one encountered in Sec. IV. We obtain the relation

$$W(z)=\frac{1}{\pi}\int_{g(x)}^{4\mu^2} ds\frac{-2i\rho(s)A^{\mathrm{II}}(s)V(s)}{s-z} +\frac{1}{\pi}\int_{4\mu^2}^{s_i} ds\frac{\rho(s+i0)G(s+i0)V^{\mathrm{II}}(s+i0)}{s-z} +\frac{1}{\pi}\int_{s_i}^{\infty} ds\frac{\operatorname{Im}W(s+i0)}{s-z}, \tag{5.16}$$

and $a_W(z)$ is now given by a relation corresponding to Eq. (5.14) with the first integral replaced by

$$\frac{1}{\pi}\int_{-\infty}^{g_1(x)} ds\frac{\operatorname{Im}a_W(s+i0)}{s-z}+\frac{1}{\pi}\int_{g_1(x)}^{4\mu^2} ds\frac{A(s)V^{\mathrm{II}}(s)}{s-z} +\frac{1}{\pi}\int_{g(x)}^{4\mu^2} ds\frac{-A^{\mathrm{II}}(s)V(s)}{s-z}. \tag{5.17}$$

[16] In an earlier paper (see reference 4) we have performed this continuation explicitly in the approximation $\operatorname{Im}a_W(s)=A_0(s)$. We found there that the correct continuation leads to a function $a_W(s)[(s-4\mu^2)(s-4x)]^{\frac{1}{2}}$ which is regular along the branch cut $s\geq 4\mu^2$ and which does not vanish for $s\rightarrow 4\mu^2+$ if $x>(M^2+\mu^2)$. Another branch of this function, which vanishes at $s=4\mu^2$, has a jump at $s=2(x-M^2+\mu^2)$, and in the interval $4\mu^2\leq s<2(x-M^2+\mu^2)$ it is displaced by a constant term from the regular branch. The constant is, of course, given by the value of the regular branch for $s\rightarrow 4\mu^2+$. See also Blankenbecler and Nambu, reference 14. These authors consider the form factor in the approximation $\operatorname{Im}a_W(s)=A_0(s)V^{\mathrm{II}}(s)$ for $s<0$.

It is of interest to see how the branch line $g(x)\leq s\leq 4\mu^2$, which appears for $x>(M^2+\mu^2)$ in the physical sheet of the form factor $W(z)$, continues into the left-hand branch line for $s<4\mu^2$ in the second sheet of the root $(z-4\mu^2)^{\frac{1}{2}}$. We consider the function $W^{\mathrm{II}}(z)$ given by Eq. (5.12), which, for $x<(M^2+\mu^2)$, may be written in the form

$$W^{\mathrm{II}}(z)=\frac{1}{\pi}\int_{-\infty}^{g_1(x)} ds\frac{\mathrm{Im}W^{\mathrm{II}}(s+i0)}{s-z}+\frac{1}{\pi}\int_{g_1(x)}^{g(x)} ds\frac{-2i\rho(s)A(s)V^{\mathrm{II}}(s)}{s-z}+\frac{1}{\pi}\int_{4\mu^2}^{s_i} ds\frac{-\rho(s+i0)a_W(s)}{s-z}+\frac{1}{\pi}\int_{s_i}^{\infty} ds\frac{\mathrm{Im}W^{\mathrm{II}}(s+i0)}{s-z}+\text{pole terms.} \tag{5.18}$$

If we now increase x above $x=M^2+\mu^2$, the second integral is replaced by an integral over the same expression but with the limits $g_1(x)$ and $4\mu^2$, plus the term

$$\frac{1}{\pi}\int_{g(x)}^{4\mu^2} ds\frac{-2i\rho(s)A^{\mathrm{II}}(s)V(s)}{s-z}. \tag{5.19}$$

The continuation of the third integral in Eq. (5.18) has been discussed before [see Eqs. (5.13) and (5.16)]; it produces an additional expression which is just the opposite of Eq. (5.19) and hence cancels this term.

So we are left, for $x>(M^2+\mu^2)$ [but $x<(M+\mu)^2$], with an expression for $W^{\mathrm{II}}(z)$ which is the same as the one given in Eq. (5.18), but with $g(x)$ replaced by $4\mu^2$. We see that the branch line associated with the composite structure of the x particle extends from $g(x)$ in the physical sheet to $4\mu^2$, there it dives into the second sheet and extends from $4\mu^2$ to $-\infty$. The "anomalous" portion of the cut in the first sheet appears just as an extension of the left-hand cut in the second sheet, which, at least for $s>0$, is due to the left-hand branch line of the production amplitude $G(s)$. We note that the appearance of the structure cut in the physical sheet for $s<4\mu^2$ does not change the analyticity of $a_W(s)$ in the neighborhood of the internal $4\mu^2\leq s<s_i$, where we always have the relation $a_W(z)=G(z)V^{\mathrm{II}}(z)$.

VI. CONCLUDING REMARKS

In the preceding sections we have seen that partial wave amplitudes and form factors have two-particle branch lines which are simply described by the square roots corresponding to the related c.m. momenta. The same properties may be inferred for the complete amplitudes using partial wave expansions. So we have, for instance, for the production amplitude $G(z, \cos\vartheta)$

$$G(z,\cos\vartheta)=d_G(z,\cos\vartheta)+i\left(\frac{z-4\mu^2}{z}\right)^{\frac{1}{2}}a_G(z,\cos\vartheta), \tag{6.1}$$

where

$$a_G(z,\cos\vartheta)=\sum_l(2l+1)G_l(z)F_l^{\mathrm{II}}(z)P_l(\cos\vartheta) \tag{6.2}$$

in the region of convergence.

The considerations of this note may also be generalized to more complicated amplitudes. We may consider reactions with two incoming and an arbitrary number of outgoing particles. If one is willing to make assumptions about the relevant analytic properties of these many-particle Green's functions in the physical sheet of the energy variable, then one can obtain properties of the two-particle branch lines which are analogous to those discussed in this paper for scattering amplitudes.

Let us finally add some remarks concerning the physical and mathematical aspects of structure singularities, which we have already discussed in an earlier publication.[4] We have seen in Sec. IV and Sec. V that the structure cuts are extensions of the left-hand branch lines of the amplitudes and their continuations into the second Riemann sheet of $(z-4\mu^2)^{\frac{1}{2}}$. So we find in the case of production amplitudes that the structure branch line with the weight

$$-A^{\mathrm{II}}(s)[1+2i\rho(s)F(s)] \tag{6.3}$$

for $g(x)\leq s<4\mu^2$ in sheet I is an extension of the line with weight

$$A(s)[1-2i\rho(s)F^{\mathrm{II}}(s)] \tag{6.4}$$

for $s\geq g_1(x)$ in sheet II. In the same way the branch line with weight $-A^{\mathrm{II}}(s)$ in sheet II is an extension of the cut with weight $A(s)$ in sheet I. The continuous transition of branch lines from one sheet into the other may be seen more explicitly if we map both Riemann sheets onto the complex plane of $q(z)=\frac{1}{2}(z-4\mu^2)^{\frac{1}{2}}$ [see Fig. 5].

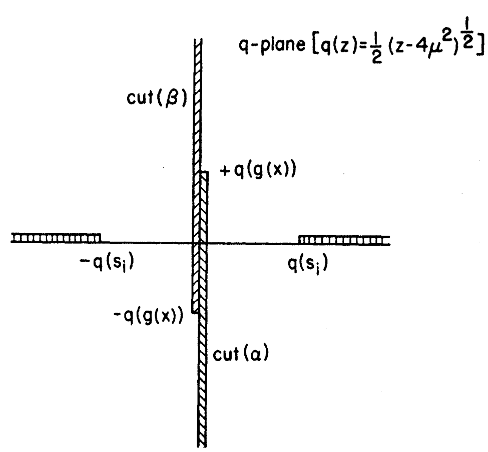

FIG. 5. Singularities of the production amplitude in the complex q plane. There can also be poles in the lower half plane. The "anomalous threshold" appears simply as the endpoint of the cut along the imaginary axis from $-i\infty$ to $+q[g(x)]$.

On the real axis we have the inelastic cuts for $q \geq q(s_i)$ and $q \leq -q(s_i)$. The first branch line discussed above maps into the cut from $+q[g(x)]$ to $-i\infty$ on the imaginary axis [denoted by α in Fig. 5], and the second branch line [β] extends from $-q[g(x)]$ to $+i\infty$. Both lines overlap in the interval $-q[g(x)] \leq q \leq +q[g(x)]$ provided $x > (M^2+\mu^2)$. For $x < (M^2+\mu^2)$ we have a gap in this interval. Note that there can also be poles in the lower half q plane.

For the baryon form factor $W(z)$ the situation is somewhat simpler because there is no left-hand cut in the physical sheet. The "anomalous" branch line in sheet I of the z plane is just an extension of the left-hand cut in sheet II. In the q plane we have only a cut corresponding to α in Fig. 5, the usual inelastic branch lines and the poles for $\mathrm{Im}q < 0$.

It is true that the appearance of a structure cut for $x > (M^2+\mu^2)$ in the physical sheet of the z plane produces an explicit extension of the maximal range of the charge or moment distribution. However, as we have already mentioned in reference 4, for $x \lesssim (M^2+\mu^2)$ the left-hand cut in the second sheet starts near $s = 4\mu^2$, and it can have an essential influence on the outer parts of the charge distribution via the absorptive part $\rho(s+i0)a_W(s)$. This may be seen from the dispersion representation of the analytic function $a_W(z)$, which is of the form [see Sec. V]

$$a_W(z) = \frac{1}{\pi}\int_{-\infty}^{g(x)} ds \frac{\mathrm{Im}a_W(s+i0)}{s-z} + \frac{1}{\pi}\int_{s_i}^{\infty} ds \frac{\mathrm{Im}a_W(s+i0)}{s-z} + \text{pole terms}, \quad (6.5)$$

for $x < M^2+\mu^2$. There may also be resonance poles near the physical region $z = s \geq 4\mu^2$, which then could dominate $a_W(s)$. In practical calculations of the nuclear form factors it is just the first integral in Eq. (6.5) [with $g(m^2) = 4\mu^2(1-\mu^2/4m^2)$] which is evaluated approximately; a π-π resonance would give rise to a pole term in this equation.

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