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Relativistic Particle Dynamics*

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It is shown that an *instant form* of relativistic particle dynamics can be constructed which displays the symmetry properties inherently present in the *point form*. In this *new* instant form, interaction terms are introduced in the energy momentum four-vector of the system; physically this is more justifiable than the more customary method of having interaction terms appear in the energy and the three generators of the Lorentz group which give the three infinitesimal displacements in velocity.

INTRODUCTION

IN nonrelativistic mechanics, the transition from classical to quantum mechanics of point particles is most readily performed within the framework of the Hamiltonian formalism. In order to achieve the same in relativistic mechanics, one first needs to develop a Hamiltonian formalism for a system of relativistic particles. This is done by considering the coordinates, momenta, and spins of the individual particles as basic variables and constructing from them ten functions which act as generators of infinitesimal canonical transformations. These transformations correspond to the ten displacements involved in the inhomogeneous Lorentz Group. Using the notation of I and II,[1] the ten functions are the energy of the system H, the components of linear momentum X, Y, Z, those of angular momentum L, M, N, and three other functions U, V, W which correspond to ordinary Lorentz transformations for velocities in the x, y, z directions, respectively. The customary constructions of these functions is in terms of basic variables given at a *common time*. Dirac calls this the *instant form*.[2] He shows however, that other forms are also possible. For instance, in his *point form* the basic variables are defined on the positive branch of a spacelike hyperboloid rather than being given on an instant plane. When we compare the relative merits of these two forms we find that the point form is more symmetrical since it makes a clean separation between the translational and the rotational parts, in space-time, of the inhomogeneous Lorentz group. The hyperboloid is invariant under rotations (i.e., the homogeneous Lorentz transformations) but not so under translations. For this reason, one prefers to introduce interaction terms only in the generators H, X, Y, Z of translational displacements in order not to disturb the trivial invariance already present under rotational displacements. In the instant form the interaction terms are introduced in H, U, V, W which do not form an interesting subgroup of the Lorentz group. Yet, the instant form is more popular because of its familiarity. We therefore raise the question: Can one form a dynamics in the instant form which displays the same symmetry as that naturally present in the point form? We propose to show, in the next section, that this is indeed possible.

SYMMETRICAL INSTANT FORM

We adopt the point of view used in II. There, a contact transformation was performed on the basic variables comprising the position coordinates x_u, y_u, z_u forming the vector $\mathbf{r}_u$, their conjugate momenta X_u, Y_u, Z_u forming the vector $\mathbf{R}_u$, and the components of spin forming a vector $\boldsymbol{\omega}_u$, all defined on an instant plane and referring to the uth particle. The transformation yielded a new set of variables comprising the coordinates of the center of mass x, y, z forming vector $\mathbf{r}$, their conjugate momenta X, Y, Z representing the three components of the total linear momentum $\mathbf{R}$ of the system, new spin variables forming vector $\boldsymbol{\eta}_u$, and finally a set of internal variables giving relative positions and relative conjugate momenta of the particles. It was then shown how arbitrary interaction terms could be introduced in an invariant manner by making the mass m $(=H^2-\mathbf{R}^2)$ of the system an arbitrary function of scalar products of the internal variables. Since the ten generator functions are given by

$$\mathbf{R}=\mathbf{R}, \qquad H=(m^2+\mathbf{R}^2)^{\frac{1}{2}},$$
$$\boldsymbol{\Omega}=(\mathbf{r}\times\mathbf{R})+\boldsymbol{\eta}, \quad \mathbf{V}=\mathbf{r}H-\frac{(\boldsymbol{\eta}\times\mathbf{R})}{m+H}, \tag{1}$$

where

$$\mathbf{R}=\sum_u \mathbf{R}_u, \quad m=\sum(m_u{}^2+\mathbf{S}_u{}^2)^{\frac{1}{2}},$$
$$\boldsymbol{\eta}=\sum_u[\mathbf{s}_u\times\mathbf{S}_u]+\boldsymbol{\eta}_u,$$

one notes that interaction terms enter only in H, U, V, W in which m occurs.

We wish to show that a further contact transformation *in the instant plane*, removes m from U, V, W only to make it reappear in X, Y, Z. As a result interaction terms, still introduced through m as before, now affect only H, X, Y, Z thus resembling the point form[2,3] of dynamics. We shall use the transformation

$$\mathbf{R}=m\mathbf{U} \quad \text{and} \quad \mathbf{r}=(1/m)\mathbf{u}, \tag{2}$$

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[1] This paper should be considered as a continuation of two preceding papers: L. H. Thomas, Phys. Rev. **85**, 868 (1952) and B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953) hereafter called I and II, respectively.

[2] The terminology is Dirac's: P. A. M. Dirac, Revs. Modern Phys. **21**, 392 (1949).

[3] See I of reference 1.

defining new variables $\mathbf{U}$ and $\mathbf{u}$. This is clearly a contact transformation since m has zero Poisson brackets with all of the basic variables. Substituting in Eq. (1) we get:

$$\mathbf{R}=m\mathbf{U}, \qquad H=m(1+\mathbf{U}^2)^{\frac{1}{2}},$$

$$\boldsymbol{\Omega}=[\mathbf{u}\times\mathbf{U}]+\boldsymbol{\eta}, \quad \mathbf{V}=\mathbf{u}(1+\mathbf{U}^2)-\frac{[\boldsymbol{\eta}\times\mathbf{U}]}{1+(1+\mathbf{U}^2)^{\frac{1}{2}}}, \tag{3}$$

where

$$[u_xU_x]=1; \quad [u_xU_y]=0, \quad \text{etc.}, \quad \cdots,$$

and our purpose has been achieved: m appears only in H, X, Y, Z.

MODIFIED RELATIVE COORDINATES

In II we had shown detailed calculations leading to the evaluation of the relative coordinates and momenta. The same procedure would yield in our present case a different set of relative coordinates. Instead of repeating equivalent calculations, we shall rewrite some results obtained in II and show how they are modified under the transformation given by Eq. (2). The contact transformation in II was of the form given by Eq. (4.5) of that reference:

$$\sum_u \mathbf{r}_u\cdot d\mathbf{R}_u+\sum_u\boldsymbol{\omega}_u\cdot d\boldsymbol{\pi}_u =\mathbf{r}\cdot d\mathbf{R}+\sum_u \mathbf{s}_u\cdot d\mathbf{S}_u-\sum_u\boldsymbol{\eta}_u\cdot d\boldsymbol{\theta}_u, \tag{4}$$

where $\mathbf{S}_u$ are relative momenta satisfying the constraint equation $\sum\mathbf{S}_u=0$, and $\mathbf{s}_u$ are relative coordinates. The introduction of a Lagrange multiplier had led us to Eq. (4) of Appendix II which we reproduce here:

$$\begin{aligned}\mathbf{s}_u+\boldsymbol{\lambda}=\mathbf{r}_u+(\mathbf{r}_u\cdot\mathbf{R})\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m(m+H)K_u}\right\} -\sum_v(\mathbf{r}_v\cdot\mathbf{R})H_v\frac{\mathbf{S}_u}{mHK_u}+\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m_u+H_u)}\\ +\frac{[\boldsymbol{\omega}_u\times\mathbf{S}_u](H-m)}{m(m_u+K_u)(m_u+H_u)}+\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{S}_u}{mK_u(m_u+H_u)(m_u+K_u)} +\frac{2(\boldsymbol{\omega}_u\cdot\mathbf{R})(\mathbf{S}_u\times\mathbf{R})}{m(m+H)(m_u+K_u)(m_u+H_u)}\\ +\sum_v\frac{(\boldsymbol{\omega}_v\cdot[\mathbf{S}_v\times\mathbf{R}])}{m_v+H_v}\frac{\mathbf{S}_u}{mHK_u}.\end{aligned} \tag{5}$$

It was further shown in II that to obtain the transformation from $\mathbf{r}_u$ to $\mathbf{s}_u$ and $\mathbf{r}$, one could replace our Eq. (4) by Eq. (4.7) of II:

$$\sum_u \mathbf{r}_u\cdot d\mathbf{R}_u=\mathbf{r}\cdot d\mathbf{R}+\sum_u \mathbf{s}_u\cdot d\mathbf{S}_u-\left(\sum_u \boldsymbol{\eta}_u\cdot d\boldsymbol{\sigma}_u\right), \tag{6}$$

where

$$d\boldsymbol{\sigma}=d\boldsymbol{\pi}-d\boldsymbol{\theta}.$$

Now Eq. (2) suggests that we replace Eq. (6) by

$$\sum_u\mathbf{r}_u\cdot d\mathbf{R}_u=\mathbf{u}\cdot d\mathbf{U}+\sum\mathbf{s}_u'\cdot d\mathbf{S}_u'-\left(\sum\boldsymbol{\eta}_u\cdot d\boldsymbol{\sigma}_u\right), \tag{6a}$$

where the primes indicate possible modified values of the relative coordinates and momenta. Since Eq. (2) does not specify how the $\mathbf{S}_u$ transform, we can set $\mathbf{S}_u'=\mathbf{S}_u$ and thus assume that only $\mathbf{s}_u$ is affected by the transformation. Note that the third term in (6) is unaffected by the transformation. Now

$$\mathbf{u}\cdot d\mathbf{U}=\mathbf{u}\cdot d\left(\frac{\mathbf{R}}{m}\right)=\left(\mathbf{u}\cdot\frac{1}{m}d\mathbf{R}-\mathbf{R}\frac{dm}{m^2}\right) =\mathbf{r}\cdot d\mathbf{R}-(\mathbf{r}\cdot\mathbf{R})\frac{dm}{m}.$$

Therefore Eq. (6a) becomes

$$\sum_u \mathbf{r}_u\cdot d\mathbf{R}_u=\mathbf{r}\cdot d\mathbf{R}-(\mathbf{r}\cdot\mathbf{R})\frac{dm}{m} +\sum(\mathbf{s}_u'\cdot d\mathbf{S}_u)-\left(\sum_u \boldsymbol{\eta}_u\cdot d\boldsymbol{\sigma}_u\right). \tag{7}$$

Comparing Eqs. (6a) and (7), we find

$$\sum \mathbf{s}_u'\cdot d\mathbf{S}_u=\sum \mathbf{s}_u\cdot d\mathbf{S}_u+(\mathbf{r}\cdot\mathbf{R})\frac{dm}{m}$$

and remembering that

$$dm=\sum_u\frac{\mathbf{S}_u\cdot d\mathbf{S}_u}{K_u},$$

we now see that when $\mathbf{s}_u'+\boldsymbol{\lambda}$ is evaluated as before, it contains the extra term

$$(\mathbf{r}\cdot\mathbf{R})\frac{\mathbf{S}_u}{mK_u}=\left\{\sum_v(\mathbf{r}_v\cdot\mathbf{R})\frac{H_v}{H}+\sum_v\frac{(\boldsymbol{\omega}_v\cdot[\mathbf{R}\times\mathbf{R}_v])}{H(m_v+H_v)}\right\}\frac{\mathbf{S}_u}{mK_u},$$

which just cancels the summed terms in Eq. (5) thus giving a simpler expression for $\mathbf{s}_u'$. Thus

$$\begin{aligned}\mathbf{s}_u'+\boldsymbol{\lambda}=\mathbf{r}_u+(\mathbf{r}_u\cdot\mathbf{R})\left\{\frac{\mathbf{S}_u(m+H)+\mathbf{R}K_u}{m(m+H)K_u}\right\}+\frac{[\boldsymbol{\omega}_u\times\mathbf{R}]}{m(m_u+H_u)}\\ +\frac{[\boldsymbol{\omega}_u\times\mathbf{S}_u](H-m)}{m(m_u+K_u)(m_u+H_u)}+\frac{(\boldsymbol{\omega}_u\cdot[\mathbf{S}_u\times\mathbf{R}])\mathbf{S}_u}{mK_u(m_u+H_u)(m_u+K_u)}\\ +\frac{2(\boldsymbol{\omega}_u\cdot\mathbf{R})[\mathbf{S}_u\times\mathbf{R}]}{m(m+H)(m_u+K_u)(m_u+H_u)}.\end{aligned}$$

In particular, for two particles, the relative coordinate evaluated to order v^2/c^2 as before is

$$\begin{aligned}\boldsymbol{\varrho}'=(\mathbf{s}_1'+\boldsymbol{\lambda})-(\mathbf{s}_2'+\boldsymbol{\lambda})=\mathbf{s}_1'-\mathbf{s}_2'\cong\mathbf{r}_1-\mathbf{r}_2\\ +(\mathbf{r}_1-\mathbf{r}_2\cdot\mathbf{R})\frac{\mathbf{R}}{2m^2}-(\mathbf{r}_1-\mathbf{r}_2\cdot\mathbf{R})\frac{\mathbf{P}}{m}\frac{m_1-m_2}{m_1m_2}\\ +(\mathbf{r}\cdot\mathbf{R})\frac{\mathbf{P}}{m}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+\left[\frac{\boldsymbol{\omega}_1}{m_1}-\frac{\boldsymbol{\omega}_2}{m_2}\times\frac{\mathbf{R}}{2m^2}\right],\end{aligned}$$

where we have used the approximate relation

$$\mathbf{r}\cong(\mathbf{m}_1\mathbf{r}_1+\mathbf{m}_2\mathbf{r}_2)/(m_1+m_2).$$

Using this expression for relative coordinate, the author has succeeded in showing that Darwin's and Breit's approximately relativistic equations can be considered as expansions to order v^2/c^2 of our theory. The procedure is the same as that used in II.

DISCUSSION

In a paper where Dirac[4] generalizes the Hamiltonian dynamics to include cases when the momenta are not independent functions of the velocities, one finds that either in the instant form or the point form of relativistic dynamics one needs four Hamiltonians. In the customary constructions of dynamics in the instant form,[5] the role of Hamiltonian is played by the functions H, U, V, W; whereas in the point form,[6] the same role is played by the functions H, X, Y, Z. Our study has shown that in the instant form a suitable choice of coordinates on the instant plane transfers the role played by H, U, V, W to the four functions H, X, Y, Z. *Thus we obtain an instant form of dynamics which displays the symmetrical form of the point form.* Physically this choice can be justified on the grounds that, in relativity, the energy H and the momentum $\mathbf{R}$ $[=(X,Y,Z)]$ are components of the same four-vector and must be treated symmetrically; an interaction term introduced in H would call for similar terms in X, Y, Z. Besides, in relativity, when one introduces an energy of interaction between the particles of a system, one essentially increases its effective rest mass. This increase in mass will in turn contribute to increasing the total momentum of the system.

[4] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).
[5] See reference 2 and II of reference 1.
[6] See reference 2 and I of reference 1.

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Relativistic Model Field Theory with Finite Self-Masses

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A model field theory is invented in the following way: Dispersion relations in the energy are assumed to hold for all amplitudes. Unitarity gives the absorptive parts in the "physical" regions. If it is assumed that the absorptive parts are otherwise zero (in violation of crossing symmetry and the Mandelstam representation), then the dispersion relations and unitarity form an infinite set of coupled integral equations for all amplitudes. An exact solution (at least for the simplest amplitudes) to this set of equations can be found, in which all self-masses, etc., are finite. The solution is equivalent to summing a certain class of Feynman graphs, computed in the usual way. For a wide range of coupling constants, there are no "ghost" difficulties.

I

THERE are two quite distinct properties which must certainly be demanded of any acceptable physical theory. First of all, its consequences must agree with experimental observation, and secondly, the theory must be mathematically self-consistent.

One method of learning about the second property for quantum field theory, without facing up to its full difficulties, is to construct models, which violate enough of the assumptions of the complete field theory to be soluble, but (hopefully) few enough so that they are reasonable analogies.

A number of model field theories have been produced in the last few years.[1,2] Of these, several[1] have given a negative answer to the consistency question in that they contain for all values of the coupling constants unphysical singularities (ghosts) in scattering amplitudes, which correspond to the existence of states having imaginary energy and various associated and equally undesirable properties.

Whether or not the undesirable properties of these models occur in the complete field theory is of course a question, a question which is only safely answered by solving the real theory. Nevertheless, it may be of value to construct different kinds of models, violating different aspects of the actual theory, and see whether or not difficulties such as ghosts exist.

We should like to do this here. We shall construct a model field theory which is perhaps unusual in that it is not defined in terms of a Lagrangian or Hamiltonian; indeed we do not know if these functions even exist. Instead we shall assume the existence of a complete set of dispersion relations, which, together with unitarity, form a set of coupled integral equations for the transition amplitudes of the theory. These equations will be distorted by violating crossing symmetry; it is in this that our model deviates from full-scale field theory. We shall then obtain an exact solution to the distorted set of

[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954); R. E. Norton and W. K. R. Watson, Phys. Rev. **116**, 1597 (1959).
[2] W. Thirring, Ann. Phys. **9**, 91 (1958); Nuovo cimento **9**, 1007 (1958).