Using this expression for relative coordinate, the author has succeeded in showing that Darwin's and Breit's approximately relativistic equations can be considered as expansions to order v^2/c^2 of our theory. The procedure is the same as that used in II.

DISCUSSION

In a paper where Dirac[4] generalizes the Hamiltonian dynamics to include cases when the momenta are not independent functions of the velocities, one finds that either in the instant form or the point form of relativistic dynamics one needs four Hamiltonians. In the customary constructions of dynamics in the instant form,[5] the role of Hamiltonian is played by the functions H, U, V, W; whereas in the point form,[6] the same role is played by the functions H, X, Y, Z. Our study has shown that in the instant form a suitable choice of coordinates on the instant plane transfers the role played by H, U, V, W to the four functions H, X, Y, Z. *Thus we obtain an instant form of dynamics which displays the symmetrical form of the point form.* Physically this choice can be justified on the grounds that, in relativity, the energy H and the momentum $\mathbf{R}$ $[=(X,Y,Z)]$ are components of the same four-vector and must be treated symmetrically; an interaction term introduced in H would call for similar terms in X, Y, Z. Besides, in relativity, when one introduces an energy of interaction between the particles of a system, one essentially increases its effective rest mass. This increase in mass will in turn contribute to increasing the total momentum of the system.

[4] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).
[5] See reference 2 and II of reference 1.
[6] See reference 2 and I of reference 1.

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Relativistic Model Field Theory with Finite Self-Masses

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A model field theory is invented in the following way: Dispersion relations in the energy are assumed to hold for all amplitudes. Unitarity gives the absorptive parts in the "physical" regions. If it is assumed that the absorptive parts are otherwise zero (in violation of crossing symmetry and the Mandelstam representation), then the dispersion relations and unitarity form an infinite set of coupled integral equations for all amplitudes. An exact solution (at least for the simplest amplitudes) to this set of equations can be found, in which all self-masses, etc., are finite. The solution is equivalent to summing a certain class of Feynman graphs, computed in the usual way. For a wide range of coupling constants, there are no "ghost" difficulties.

I

THERE are two quite distinct properties which must certainly be demanded of any acceptable physical theory. First of all, its consequences must agree with experimental observation, and secondly, the theory must be mathematically self-consistent.

One method of learning about the second property for quantum field theory, without facing up to its full difficulties, is to construct models, which violate enough of the assumptions of the complete field theory to be soluble, but (hopefully) few enough so that they are reasonable analogies.

A number of model field theories have been produced in the last few years.[1,2] Of these, several[1] have given a negative answer to the consistency question in that they contain for all values of the coupling constants unphysical singularities (ghosts) in scattering amplitudes, which correspond to the existence of states having imaginary energy and various associated and equally undesirable properties.

Whether or not the undesirable properties of these models occur in the complete field theory is of course a question, a question which is only safely answered by solving the real theory. Nevertheless, it may be of value to construct different kinds of models, violating different aspects of the actual theory, and see whether or not difficulties such as ghosts exist.

We should like to do this here. We shall construct a model field theory which is perhaps unusual in that it is not defined in terms of a Lagrangian or Hamiltonian; indeed we do not know if these functions even exist. Instead we shall assume the existence of a complete set of dispersion relations, which, together with unitarity, form a set of coupled integral equations for the transition amplitudes of the theory. These equations will be distorted by violating crossing symmetry; it is in this that our model deviates from full-scale field theory. We shall then obtain an exact solution to the distorted set of

[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954); R. E. Norton and W. K. R. Watson, Phys. Rev. **116**, 1597 (1959).
[2] W. Thirring, Ann. Phys. **9**, 91 (1958); Nuovo cimento **9**, 1007 (1958).

equations; this solution is our model theory. It will then be shown that the solution has a perturbation expansion, which may conveniently be expressed as a well-defined set of Feynman diagrams computed by the usual rules. Finally, we shall see that the model has finite mass and wave function renormalizations, and, for a wide range of values of the coupling constant, has no ghosts.

In Sec. II we construct the theory and explain the assumptions involved, in Sec. III solve it, in Sec. IV discuss its perturbation expansion, and in Sec. V sketch alternative ways of looking at it and mention various modifications.

II

We shall concern ourselves with a world which contains two kinds of particles, labeled A and B. Both shall be spinless bosons, and A is to be its own antiparticle while $\bar{B}$, the antiparticle to B, is distinct from B. The A particle is scalar, so that the reaction $B+\bar{B}\rightarrow A$ is allowed with the $B\bar{B}$ pair in a relative S state. A and B have experimentally observed mass μ and M, respectively, with $2M>\mu$.

The first question which confronts us is how to construct a field theory of the interactions of A and B. Conventionally, field theories are defined by specifying a Lagrangian density, and from this obtaining field equations, perturbation expansions, and so on. It has been suggested,[3] however, that field theories can equally well be defined by writing down a set of dispersion relations which, when combined with unitarity, provide an infinite set of coupled integral equations from which all the transition amplitudes of the theory may be determined. This second approach has the virtue of not involving any unobservable quantities, such as bare masses or coupling constants, at any stage of the development. It has the fault, if it is a fault, that the connection with the Lagrangian formalism is quite obscure; for example, it is not known if the existence of the set of dispersion relations follows from the existence of a Lagrangian, or vice versa, or both.

We choose to define our theory of the A and B particles by writing down a set of dispersion relations, and we shall not concern ourselves about the existence or nonexistence of a Lagrangian.

To be specific, then, we shall assume an S matrix, and define

$$S_{ij}=\delta_{ij}-i(2\pi)^4\delta^4(P_i-P_j)\frac{T_{ij}}{N_iN_j},\qquad(1)$$

where P_i, P_j are the total 4-momenta of the states i and j, and where N_i and N_j are the usual normalization factors:

$$N_i=\prod_{\text{particles in } i}(2E).$$

[3] M. Gell-Mann, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, New York, 1956).

T_{ij} is then a function of whatever independent variables can be constructed from the momenta in the states i and j; in particular, it is a function of the total energy squared in the c.m. system, which we will call s. We then assume[4]

$$T_{ij}(s,\cdots)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im}T_{ij}(s',\cdots)}{s'-s-i\epsilon}ds'.\qquad(2)$$

Here the dots indicate that whatever other variables T_{ij} depends on are the same on both sides of Eq. (2). It is presumably true that Eq. (2) is not valid for an *arbitrary* choice of variables other than s; we assume that there *exists* a choice so that Eq. (2) holds; this choice is then supposed to be represented by the dots in the argument T_{ij}.

The set of dispersion relations, Eq. (2), are not complete until we specify the $\operatorname{Im}T_{ij}$'s, since only then do they become a system of coupled equations for the various T's. The requirement that S be unitary gives us the relation (for $P_i=P_j$)[5]

$$i(T_{ij}-T_{ji}^*)=\sum_n\frac{1}{N_n}T_{in}T_{jn}^*(2\pi)^4\delta^4(P_i-P_n).\qquad(3)$$

This equation must hold for s in the physical region, that is for

$$s\geq\max(M_i^2,M_j^2),$$

where $M_{i,j}$ is the total mass of state i or j. It says nothing about other values of s.

In order to find an expression for $\operatorname{Im}T_{ij}(s,\cdots)$ for unphysical values of s, an additional property is needed. In real field theory this is provided by crossing symmetry—or perhaps more precisely by the Mandelstam representation.[6] This asserts that a single function describes a number of related physical processes, each process being given by the function in certain ranges of its variables. Thus $T_{ij}(s,\cdots)$ for certain unphysical values of s represents some other reaction in its physical region; application of unitarity to this reaction then allows the evaluation of $\operatorname{Im}T_{ij}(s,\cdots)$ for these unphysical values of s.

It is, of course, too much to hope that we will be able to find any exact solutions to the set of dispersion relations for a real field theory, so some minimal mutilation of the equations is undoubtedly necessary. It is at this

[4] Note that we assume the dispersion relations in the unsubtracted form. It should also be remarked that as yet dispersion relations have not been written down in practice for more than four particle processes (two in and two out, for example). However, in view of the assumptions we shall make here, an explicit form for multiparticle processes is unnecessary, and the form of Eq. (2) suffices.

[5] An additional comment should be inserted here. On occasion we will also be interested in amplitudes involving virtual particles, specifically form factors and propagators. We assume that such amplitudes also satisfy dispersion relations of the type (2), where s is the four-momentum squared of the virtual particle, and that the unitarity condition (3) may be extended to obtain the absorptive parts of these virtual processes.

[6] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

point that the mutilation will be made, and at which we will part company with real field theory. For we shall assume, in violation of crossing symmetry, that $\mathrm{Im}T_{ij}(s,\cdots)=0$ except as specified by Eq. (3). This leaves us with an infinite set of coupled integral equations for the T_{ij}'s, which are the same as those of a correct field theory except for the presence of integrals over negative values of s. No further distortion of the theory will be necessary; we will be able to find an exact nontrivial solution of the dispersion relations as they now stand.

Before we proceed to the construction of this solution, however, a few comments may be pertinent. We have chosen to use as the assumptions defining our field theory an infinite set of coupled integral equations rather than the existence of the usual Lagrangian. While there is nothing wrong with this, it is certainly unaesthetic to have such a complicated set of assumptions. In the real field theory a simple set of axioms has been formulated[7] from which it is hoped that the Mandelstam representation follows and, therefore, from which the infinite set of coupled dispersion relations also (hopefully) follows. Since in the real field theory the dispersion relations include the crossing terms, and since we have discarded the crossing terms, it is clear that our mutilated theory must violate one (or more) of these basic axioms. A little thought makes it clear (this will become obvious later) that the axiom which is violated is the statement that two field operators must commute at spacelike separations. This is the axiom which is usually associated with causality. However, since in our mutilated theory all amplitudes still satisfy dispersion relations, and, therefore, have no singularities in the upper half-energy plane, our theory is still causal at least in some sense. The commutability of two field operators for spacelike separations, then, seems to be a stronger statement than causality, and includes crossing symmetry as a consequence as well.

Equations (2) and (3), together with the statement that

$$\mathrm{Im}T_{ij}(s,\cdots)=0 \quad \text{if} \quad s<\max(M_i^2,M_j^2),$$

define the theory; or rather, they partially define the theory. Equations (2) and (3) provide an infinite set of coupled integral equations for all amplitudes; however, there is no guarantee that the solution of these equations is unique. In fact, all analyses which have been made of finite systems of such equations (either one or two in practice) have shown there are many solutions.[8] We may consequently expect that Eqs. (2) and (3) still actually encompass a number of different theories.

What we shall do is try to guess a solution of these dispersion relations. We should like our guess to be nontrivial—for example, one solution is all $T_{ij}=0$, but this is uninteresting—and still be sufficiently simple so that we can actually evaluate at least some of the T_{ij} exactly.

Therefore, we make the following guess: we look for a solution which is such that all $T_{ij}=0$ except those for which the states i and j differ only in that any number of A particles in i have been replaced by $B\bar{B}$ pairs in j and vice versa, and for which both i and j contain at least one A or one $B\bar{B}$ pair. It is evident that this guess is consistent with the unitarity relation (3). The nonzero amplitudes are now to be determined from the system of coupled integral equations which the dispersion relations become after we impose this guess, and, as we shall see below, this system of integral equations can in fact be solved.

To make it clear where we stand at this point let us write down the first few equations.

(1) $B\bar{B}$ scattering:

The relevant amplitude is $T_{p'\bar{p}',p\bar{p}}$ where p' and p stand for the final and initial B particle four-momenta, and $\bar{p}'$ and $\bar{p}$ are the corresponding $\bar{B}$ momenta. We may write

$$T_{p'\bar{p}',p\bar{p}}=T(s,t), \tag{4}$$

where $s=(p+\bar{p})^2$ and $t=(p'-p)^2$. From Eq. (3) we have

$$\begin{aligned}\mathrm{Im}T(s,t)&\\ =-\frac{1}{2}\int &\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}T_{p\bar{p}',q}T_{p\bar{p},q}{}^*(2\pi)^4\delta^4(q-p-\bar{p})\\ &-\frac{1}{2}\int\frac{d^3p''}{(2\pi)^3}\int\frac{d^3\bar{p}''}{(2\pi)^3}\frac{1}{2E_{p''}}\frac{1}{2E_{\bar{p}''}}\\ &\times T_{p'\bar{p}',p''\bar{p}''}T_{p\bar{p},p''\bar{p}''}{}^*(2\pi)^4\delta^4(p''+\bar{p}''-p-\bar{p}).\end{aligned} \tag{5}$$

Here we have used the fact that $T_{n,p\bar{p}}=0$ unless $n=A$ or $n=B\bar{B}$; also q denotes the 4-momentum of an A particle.

Equation (5) may be simplified directly: We note that $T_{p\bar{p},q}$ is just a number; there are no independent variables to be formed from p, $\bar{p}$, and q due to the restrictions $q^2=\mu^2$, $p^2=\bar{p}^2=M^2$. Thus, we may define

$$T_{p\bar{p},q}=T_{p'\bar{p}',q}=g. \tag{6}$$

Anticipating slightly, we note that $T(s,t)$ clearly need not depend on t, so that we get from (5)

$$\mathrm{Im}T(s)=-g^2\delta(s-\mu^2)-\frac{1}{16\pi}\left(\frac{s-4M^2}{s}\right)^{\frac{1}{2}}|T(s)|^2. \tag{7}$$

Inserting this into Eq. (2) gives the integral equation,

$$T(s)=\frac{g^2}{s-\mu^2}-\frac{1}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\frac{|T(s')|^2}{s'-s-i\epsilon}ds'. \tag{8}$$

We shall not now solve this equation; suffice it to say that it is clearly soluble, so that $T(s)$ can be determined

[7] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 1425 (1955).

[8] F. J. Dyson, R. H. Dalitz, and L. Castillejo, Phys. Rev. **101**, 453 (1956); M. Baker and F. Zachariasen, Phys. Rev. **119**, 438 (1960).

from it. For this process, notice, there is no coupling of amplitudes—Eq. (8) is an equation involving $T(s)$, i.e., $B\bar{B}$ scattering, alone.

(2) BB and $\bar{B}\bar{B}$ scattering:

Both these processes are zero according to our guessed solution; evidently this is consistent with Eq. (3).

(3) AB scattering:

Here the relevant amplitude is $T_{p'q',pq}$. We shall use the notation

$$U(s,t)=T_{p'q',pq}, \tag{9}$$

where $s=(p+q)^2$, $t=(p-p')^2$. From the unitarity condition (3) only states $n=AB$ or $n=BB\bar{B}$ contribute in the sum, and we have

$$\begin{aligned}\operatorname{Im}U(s,t)&\\ =-\frac{1}{2}&\int\frac{d^3p''}{(2\pi)^3}\int\frac{d^3q''}{(2\pi)^3}\frac{1}{2E_{p''}}\frac{1}{2\omega_{q''}}\\ &\times U(s,t')U(s,t'')^*(2\pi)^4\delta^4(p''+q''-p-q)\\ &-\frac{1}{2}\int\frac{d^3p''}{(2\pi)^3}\int\frac{d^3p'''}{(2\pi)^3}\int\frac{d^3\bar{p}''}{(2\pi)^3}\frac{1}{2E_{p''}}\frac{1}{2E_{p'''}}\frac{1}{2E_{\bar{p}''}}\\ &\times T_{p'q',p''p'''\bar{p}''}T_{pq,p''p'''\bar{p}''}{}^*\\ &\qquad\times(2\pi)^4\delta^4(p''+p'''+\bar{p}''-p-q).\end{aligned} \tag{10}$$

Using this in the dispersion relation for $U(s,t)$ thus leads to an integral equation coupling AB scattering to itself and the process $A+B\rightarrow B+B+\bar{B}$. Similarly, the dispersion relation for $AB\rightarrow BB\bar{B}$ will involve $AB\rightarrow AB$, $AB\rightarrow BB\bar{B}$ and $BB\bar{B}\rightarrow BB\bar{B}$. Thus the AB scattering process does *not* lead to a single uncoupled equation, but is one of three coupled equations for the three amplitudes describing the processes

$$\begin{aligned}A+B&\leftrightarrow A+B,\\ A+B&\leftrightarrow B+B+\bar{B},\\ B+B+\bar{B}&\leftrightarrow B+B+\bar{B}.\end{aligned}$$

For our purposes the solutions to these equations are not necessary, so we shall not attempt to obtain them explicitly.

In general we can see that the set of dispersion relations breaks up into finite sets of coupled equations for finite subgroups of the amplitudes T_{ij}. In practice, of course, it may be difficult to solve some of these sets of integral equations once we get to sufficiently many particles going in and coming out; however, the simpler T_{ij}'s can be evaluated.

There are some further types of amplitudes which will be of interest, the first of which is the form factor for an A particle. This is the continuation of the $AB\bar{B}$ vertex off the mass shell for the A particle—symbolically,

$$T_{p\bar{p},q}|_{q^2=s}=F(s),\quad\text{so}\quad F(\mu^2)=g. \tag{11}$$

From Eq. (3), then, we immediately have

$$\begin{aligned}\operatorname{Im}F(s)&=-\frac{1}{2}\int\frac{d^3p'}{(2\pi)^3}\int\frac{d^3\bar{p}'}{(2\pi)^3}\frac{1}{2E_{p'}}\frac{1}{2E_{\bar{p}'}}\\ &\qquad\times T_{p\bar{p},p'\bar{p}'}F(s)^*(2\pi)^4\delta^4(p'+\bar{p}'-q)\\ &=-T(s)F(s)^*\left[\left(\frac{s-4M^2}{s}\right)^{\frac{1}{2}}\frac{1}{16\pi}\right].\end{aligned} \tag{12}$$

Thus the dispersion relation for $F(s)$ is, as usual,[9]

$$\begin{aligned}F(s)&=\frac{1}{\pi}\int\frac{\operatorname{Im}F(s')}{s'-s-i\epsilon}ds'\\ &=\frac{1}{\pi}\int_{4M^2}^{\infty}\frac{\sin\delta(s')e^{i\delta(s')}F(s')^*}{s'-s-i\epsilon}ds',\end{aligned} \tag{13}$$

where we write

$$T(s)\equiv-16\pi\left(\frac{s}{s-4M^2}\right)^{\frac{1}{2}}\sin\delta(s)e^{i\delta(s)}. \tag{14}$$

Hence, once the equation for $T(s)$ is solved, we will know $F(s)$ by solving Eq. (13).

We are also interested in the propagator for an A particle. Its dispersion relation is[10]

$$\Delta(s)=\frac{1}{s-\mu^2}+\int\frac{\rho(s')}{s'-s-i\epsilon}ds', \tag{15}$$

where

$$\rho(s)=\textstyle\sum_n(2\pi)^3\delta^3(\mathbf{P}_n)(s-\mu^2)^{-2}\delta(E_n{}^2-s)\,|T_{n,q}|^2{}_{q^2=s}, \tag{16}$$

so that it again involves the continuation of processes of the form $A\rightarrow n$ off the mass shell of the initial A. By our rule, only $n=B\bar{B}$ appears in the sum, since $T_{n,A}=0$ except if $n=B\bar{B}$. From this, it is easy to deduce that

$$\rho(s)=\frac{1}{16\pi^2}|F(s)|^2\left(\frac{1}{s-\mu^2}\right)^2\left(\frac{s-4M^2}{s}\right)^{\frac{1}{2}}. \tag{17}$$

The virtue of the propagator is that the self-mass and the A-particle wave-function renormalization may be obtained from it,[10] if we happen to be interested in these quantities. It is only in order to discuss the renormalizations that it is necessary to concern ourselves with objects such as $F(s)$ and $\Delta(s)$ which do not actually represent physical processes.

Let us summarize where we are. We have formulated a theory of the A and B particles which satisfies relativity, unitarity, and causality (in the limited sense that ordinary dispersion relations hold). It has the correct spectrum of eigenstates of the energy-momentum operator. It is soluble exactly (at least partially; that is, the simpler amplitudes can be evaluated in practice), and

[9] P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958).

[10] H. Lehmann, Nuovo cimento **11**, 342 (1954).

the solution is nontrivial (i.e., there exists scattering, etc.) The theory violates crossing (it seems that it has to violate something if it is to be soluble) and therefore the Mandelstam representation and the usual commutation rules.

We have formulated the theory by giving a rule for writing down the infinite set of coupled dispersion relations for it; however, we are unable to find a Lagrangian explicitly. Whether or not one exists we do not know.

In the following we shall show that the theory is in fact not even yet unique, and that some of the theories encompassed have finite mass and wave function renormalizations. We shall also relate the model to its Feynman graphs—i.e., its perturbation expansion, and observe that summing the perturbation series gives the correct expression for all physically observable quantities, but that the perturbation expansion does not exist for unobservable quantities, such as the self-mass, which are infinite in perturbation theory.

III

As derived in Sec. II, the equation determining $B\bar{B}$ scattering is

$$T(s)=\frac{g^2}{s-\mu^2}-\frac{1}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\frac{|T(s')|^2}{s'-s-i\epsilon}. \tag{18}$$

A solution of this equation is readily obtained: Define

$$D(s)=\frac{(g^2/s-\mu^2)}{T(s)}; \tag{19}$$

then the analytic properties of $T(s)$ which follow from Eq. (18), together with the assumption that $T(s)$ has no zeros, allow us to deduce analytic properties of $D(s)$ from which we may immediately infer that[11]

$$D(s)=1+\frac{s-\mu^2}{16\pi^2}g^2\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\times\frac{ds'}{(s'-\mu^2)^2(s'-s-i\epsilon)}. \tag{20}$$

This result for $D(s)$ clearly has no poles, so that the assumption that $T(s)$ had no zeros is consistent. It is also worth noting that as $s\to\pm\infty$,

$$D(s)\to 1-\frac{g^2}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\frac{ds'}{(s'-\mu^2)^2}.$$

Thus if g^2 is too large, $D(s)$ will have a zero for some $s<\mu^2$, which implies a pole of $T(s)$ which was not present in the original integral equation. Equations (19) and (20) therefore constitute a solution to Eq. (18) only for sufficiently small values of g^2.

[11] This method for solving equations such as (18) is well known. For details see for example G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

It is easy to construct more solutions to Eq. (18): Let us define a new D by

$$D(s)=\frac{\lambda+(g^2/s-\mu^2)}{T(s)}, \tag{21}$$

instead of by Eq. (18). At the moment, λ is to be considered to be an arbitrary number; its range will be restricted later. Again we may derive analytic properties of $D(s)$ [still on the assumption that $T(s)$ has no zeros, except possibly at the point $s_0=\mu^2-g^2/\lambda$ where the numerator of Eq. (21) vanishes; a zero of T here does not imply a pole in D], from which, in analogy to Eq. (20), we find

$$D(s)=1+\frac{s-\mu^2}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\times\left(\lambda+\frac{g^2}{s'-\mu^2}\right)\frac{ds'}{(s'-\mu^2)(s'-s-i\epsilon)}. \tag{22}$$

This is an explicit form for $D(s)$; reversing the roles of D and T in Eq. (21) thus constitutes a more general solution of Eq. (18), from which the earlier case is obtained by setting $\lambda=0$. More precisely, Eq. (22) is a solution of Eq. (18) for such values of λ that $D(s)$ has no zeros, and hence so that $T(s)$ has no poles not allowed by Eq. (18). This requirement restricts the range of λ.

$D(s)$, as given by (22), has no poles, so $T(s)$ has no zeros, except possibly at s_0. It is easily seen from Eq. (18) that the only place where a zero could occur in T is on the real axis above μ^2. Therefore, if s_0 is to be a zero of T, we must have

$$s_0=\mu^2-g^2/\lambda>\mu^2, \tag{23}$$

from which we conclude

$$\lambda<0. \tag{24}$$

If $T(s_0)\neq 0$, on the other hand, then we must have $D(s_0)=0$. It is easily found from Eq. (22) by straightforward algebra that if $D(s_0)=0$, then

$$g^{-2}=\frac{1}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\frac{ds'}{(s'-\mu^2)^2}. \tag{25}$$

λ may be related to the expansion of $T(s)$ around the point $s=\mu^2$, as follows: From (21) and (22), it is clear that near $s=\mu^2$,

$$T(s)=\frac{g^2}{s-\mu^2}+\lambda-\frac{g^2}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\times\left(\lambda+\frac{g^2}{s'-\mu^2}\right)\frac{ds'}{(s'-\mu^2)^2}+O(s-\mu^2). \tag{26}$$

By also expanding Eq. (18) around the point $s=\mu^2$, we find

$$\lambda-\frac{g^2}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\left(\lambda+\frac{g^2}{s'-\mu^2}\right)\frac{ds'}{(s'-\mu^2)}$$
$$\equiv\bar{\lambda}=-\frac{1}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\left(\frac{|T(s')|^2}{s'-\mu^2}\right)ds'. \quad (27)$$

Note that $\bar{\lambda}<0$, and $\lambda>\bar{\lambda}$. The range of λ therefore cannot be outside the interval $\bar{\lambda}$ to 0.

We have already noted that for $\lambda=0$ this solution reduces to that expressed by Eqs. (19) and (20). There we found $D(s)\to$ const as $s\to\infty$. However, if $\lambda\neq0$, Eq. (22) shows us that as $s\to\infty$,

$$D(s)\to-(\lambda/16\pi^2)\ln|s|.$$

From (21), then, we see $T(s)\to-16\pi^2/\ln|s|$ as $s\to\infty$, independent of the sign or value of λ. This, it may be noted, is a sufficient degree of convergence to guarantee the existence of the unsubtracted dispersion relation (18).

It was stated above that it is necessary to restrict λ to values such that $D(s)$ has no zeros. This is necessary in order that (21) and (22) be a solution of (18), since T as defined by (18) has no poles except at μ^2. Incidentally, if we had allowed the existence of bound states of the $B\bar{B}$ system, there would be additional poles in Eq. (18) at the square energies of these states; in such an event, of course, we would have to require that D did have zeros at these points. Under no circumstances, though, could we tolerate zeros of D for negative s; their existence would correspond to bound states of imaginary energies, which are commonly called ghosts.

Inspection of Eq. (22) shows that the λ term in D is positive (remember $\lambda<0$) for $s<\mu^2$, while the g^2 term is negative. For sufficiently large negative s, the λ term dominates; however, if g^2 is too large, there could be an intermediate region of s where D becomes negative. A restriction on the size of g^2 relative to λ is therefore necessary to ensure that D will have no zeros. The precise limits on λ and g^2, and the question of zeros of D in general will be discussed in detail in Sec. IV and we shall consequently drop the subject for the present.

We have obtained a solution of the dispersion relation for $B\bar{B}$ scattering, which depends on two coupling constants, λ and g^2, which is valid for certain ranges of values of λ and g^2. Let us now use this information to evaluate further properties of the model.

It will first be convenient to define the phase shift for $B\bar{B}$ scattering—note that only S-wave scattering exists, so only the s-wave phase shift need be discussed. The fact that we have only one nonzero phase shift is a reflection of the lack of crossing symmetry. Define

$$T(s)=-16\pi\left(\frac{s}{s-4M^2}\right)^{\frac{1}{2}}\sin\delta(s)e^{i\delta(s)}. \quad (28)$$

Note that, for $s>4M^2$

$$\mathrm{Im}D(s)=\frac{1}{16\pi}\left(\frac{s-4M^2}{s}\right)^{\frac{1}{2}}(\lambda+g^2/s-\mu^2). \quad (29)$$

[$\mathrm{Im}D(s)=0$, of course, for $s<4M^2$.] Thus, from Eq. (24), we get the result

$$\sin\delta(s)e^{i\delta(s)}=-\frac{\mathrm{Im}D(s)}{D(s)}, \quad (30)$$

as is implied by the notation; therefore, $D(s)$ is the conventional determinental function[12] for S-wave $B\bar{B}$ scattering. This interpretation will be very useful in Sec. IV.

Equation (30) may be rephrased as

$$\mathrm{Im}D(s)=-\tan\delta(s)\,\mathrm{Re}D(s), \quad (31)$$

which, coupled with the analyticity properties of D, yields the integral equation

$$D(s)=1-\frac{s-\mu^2}{16\pi^2}\int_{4M^2}^{\infty}\frac{\mathrm{Re}D(s')\tan\delta(s')}{(s'-\mu^2)(s'-s-i\epsilon)}ds', \quad (32)$$

to which the solution is[13]

$$D(s)=\exp\left\{-\frac{s-\mu^2}{\pi}\int_{4M^2}^{\infty}\frac{\delta(s')}{(s'-\mu^2)(s'-s-i\epsilon)}ds'\right\}. \quad (33)$$

We found in Sec. II that the form factor for the A particle—which is basically the continuation of the amplitude $T_{B\bar{B},A}$ off the mass shell of the A particle—satisfied in our model the equation

$$F(s)=\frac{1}{\pi}\int_{4M^2}^{\infty}\frac{\mathrm{Im}F(s')}{s'-s-i\epsilon}ds', \quad (34)$$

with

$$\mathrm{Im}F(s)=F(s)^*\sin s(s)e^{i\delta(s)}$$
$$=\mathrm{Re}F(s)\tan\delta(s). \quad (35)$$

Comparing this with Eqs. (30) and (31), it is easily shown that[14]

$$F(s)=g/D(s), \quad (36)$$

where we have used the statement that $F(\mu^2)=g$. Note that $F(s)\to0$ like $(\ln s)^{-1}$ when $s\to\infty$, and $\mathrm{Im}F(s)\to0$ like $(\ln s)^{-2}$. The unsubtracted form of the dispersion relation for F, namely Eq. (34), therefore in fact exists. In spite of this, the equation

$$g=\frac{1}{\pi}\int_{4M^2}^{\infty}\frac{\mathrm{Im}F(s')}{s'-\mu^2}ds' \quad (37)$$

[12] M. Baker, Ann. Phys. 4, 271 (1958).
[13] R. Omnes, Nuovo cimento 8, 316 (1958).
[14] This is a special case of a generally true statement in the complete field theory. See for example J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960).

does not put any restrictions on the value of g, but is merely an identity. This is of course obvious since Eqs. (34) and (35) are a homogeneous integral equation for F, and only determine F up to a multiplicative constant.

Next let us turn to the propagator for the A particle. This was given by

$$\Delta(s)=\frac{1}{s-\mu^2}-\frac{1}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}} \times\frac{|F(s')|^2}{(s'-\mu^2)^2}\frac{ds'}{s'-s-i\epsilon}. \quad (38)$$

Here it is not necessary to solve an integral equation to obtain Δ; all we have to do is evaluate an integral over the known function F. The wave function and mass renormalizations of the A particle are obtained from the propagator by the usual relations[10]

$$Z_3^{-1}=1+\frac{1}{16\pi^2}\int_{M4^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\left(\frac{|F(s')|^2}{(s'-\mu^2)^2}\right)ds', \quad (39)$$

$$Z_3^{-1}\mu_0^2=\mu^2+\frac{1}{16\pi^2}\int_{M4^2}^{\infty} \times\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\left(\frac{|F(s')|^2}{(s'-\mu^2)^2}\right)s'ds'. \quad (40)$$

(We use the notation Z_3 for the wave function renormalization of the A particle; Z_2 will be used for that of the B particle.) Equivalently, then, we can say

$$\Delta(s)\rightarrow Z_3^{-1}/(s-\mu_0^2), \quad (41)$$

as $s\rightarrow\infty$.

$\Delta(s)$ may be evaluated as follows: Notice that

$$\mathrm{Im}\frac{1}{D(s)}=-\frac{1}{16\pi^2}\left(\frac{s-4M^2}{s}\right)^{\frac{1}{2}}\left(\lambda+\frac{g^2}{s-\mu^2}\right)\left|\frac{1}{D(s)}\right|^2. \quad (42)$$

Furthermore, $1/D(\mu^2)=1$, and since D has no zeros, $[D(s)]^{-1}$ has no poles. Hence

$$D(s)^{-1}=1-\frac{s-\mu^2}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\left|\frac{1}{D(s')}\right|^2 \times\left(\lambda+\frac{g^2}{s'-\mu^2}\right)\frac{ds'}{(s'-\mu^2)(s'-s-i\epsilon)}. \quad (43)$$

From this a certain amount of algebraic manipulation results in

$$\frac{g^2}{16\pi^2}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\left|\frac{1}{D(s')}\right|^2\frac{ds'}{(s'-\mu^2)^2(s'-s-i\epsilon)} =\frac{1}{s-\mu^2}\frac{1}{1+(s-\mu^2)\lambda/g^2} \times\left[1-\frac{1}{D(s)}-\frac{\lambda}{g^2}(Z_3^{-1}-1)(s-\mu^2)\right]. \quad (44)$$

If this is inserted into Eq. (38), and Eq. (36) is recalled, we obtain an explicit expression for the propagator:

$$\Delta(s)=\frac{1}{s-\mu^2}-\frac{1}{s-\mu^2}\frac{1}{1+(\lambda/g^2)(s-\mu^2)} \times\left[1-\frac{1}{D(s)}-\frac{\lambda}{g^2}(Z_3^{-1}-1)(s-\mu^2)\right]. \quad (45)$$

By allowing s to go to infinity in this equation we can find the self-mass

$$\delta\mu^2\equiv\mu^2-\mu_0^2=g^2/\lambda. \quad (46)$$

This same expression can also be obtained directly from Eq. (40) by performing manipulations analogous to those used in deriving the expression (45) for the propagator. The bare mass is seen to be larger than the physical mass, since λ is negative, as is to be expected.[10]

It is most interesting that the self-mass is finite [Z_3 is obviously finite as well since certainly the integral in (39) converges if that in (40) does]; this results from the fact that $F(s)\rightarrow 0$ as $s\rightarrow\infty$, which in turn comes from the fact that $D(s)\rightarrow\infty$ like $\ln s$. If $\lambda=0$, then $D(s)\rightarrow$ const and $\delta\mu^2$ is logarithmically infinite.

The renormalizations associated with the B particle are all trivial; due to the simplicity of our solution of the infinite set of dispersion relations, there is no amplitude of the form $T_{n,B}$ for any n. Therefore the propagator for the B particle is given exactly by the free-particle result:

$$s(s)=1/(s-M^2). \quad (47)$$

The corresponding renormalizations are then

$$\delta M^2=0, \quad (48)$$

and

$$Z_2=1. \quad (49)$$

Finally, we may mention the vertex renormalization (which we call Z_1). There is no rigorously known expression for Z_1 in usual field theory [analogous to Eq. (39) for Z_3, for example] which may be used as a definition of Z_1. It has been conjectured[15] that the relation

$$\lim_{s\rightarrow\infty}F(s)=gZ_1/Z_3 \quad (50)$$

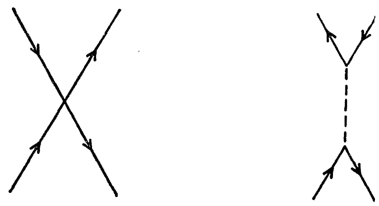
is rigorously true, but no proof exists. If Eq. (50) is accepted as provable in Lagrangian field theory, and therefore used as a definition in the dispersion approach, then we find $Z_1=0$, and this must in fact be the case in any theory with finite mass renormalization.

There are additional processes which may be evaluated exactly; however, these are not of any particular interest in themselves and they are not relevant to any of the more fundamental properties, such as renormalizations. We shall therefore content ourselves with the solutions we have already obtained, and turn our attention to the perturbation theory associated with the model.

[15] K. Symanzik, Nuovo cimento **11**, 269 (1959).

IV

It was observed in Sec. III that the object $D(s)$ defined in Eq. (24), and written explicitly in Eq. (25), is just the determinental function[12] for $B\bar{B}$ scattering. From Eq. (29) it is seen that $\operatorname{Im}D(s)$ is essentially just the two Feynman graphs of Fig. 1, computed by the



(a) (b)

FIG. 1. Feynman graphs for $B\bar{B}$ scattering due (a) to the λ coupling between two B and two $\bar{B}$ particles, and (b) to the g vertex coupling an A, a B, and a $\bar{B}$ particle.

usual Feynman rules. It is therefore to be expected that the entire scattering $T(s)$ is equivalent to the set of Feynman graphs formed by all chains built up out of the basic graphs of Fig. 1. The first few of these are shown in Fig. 2.

That this is in fact true may be verified directly. Let us compute the $B\bar{B}$ scattering produced by the sum of all the chain diagrams of Fig. 2, where the usual Feynman rules are used to compute each graph. From what has been said before, it is not surprising that the determinental method[12] is the most convenient way to evaluate the sum of all these graphs.

In lowest order, the scattering amplitude is produced by the graphs of Fig. 1; the Feynman amplitude for these is

$$F_1=-i\left(\lambda_0+\frac{g_0^2}{s-\mu^2}\right). \tag{51}$$

Here λ_0 and g_0 are the unrenormalized coupling constants associated with the $\bar{B}\bar{B}BB$ and $B\bar{B}A$ vertices. In accord with the prescription of the determinental method, we assume the mass renormalization to have been performed; μ^2 is therefore the physical mass of the A particle.

The phase shift in lowest order is readily computed from Eq. (51), and we find

$$(\sin\delta e^{i\delta})_1=-\frac{1}{16\pi}\frac{q}{\omega}\left(\lambda_0+\frac{g_0^2}{s-\mu^2}\right). \tag{52}$$

Here δ is the center-of-mass system phase shift, ω the

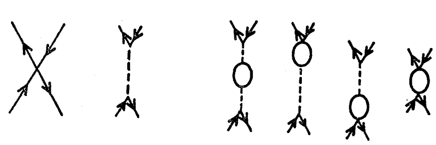


FIG. 2. Chain graphs constructed from the basic graphs of Fig. 1.

center-of-mass energy of one B particle, and q the corresponding momentum. Thus

$$\omega=(s^2/4)^{\frac{1}{2}},\quad q=(s^2/4-M^2)^{\frac{1}{2}}.$$

The determinental approach now instructs us to use as the lowest order $\operatorname{Im}D$ just $(-\sin\delta e^{i\delta})_1$. Thus in lowest order,

$$\operatorname{Im}D_1(s)=\frac{1}{16\pi}\left(\frac{s-4M^2}{s}\right)^{\frac{1}{2}}\left(\lambda_0+\frac{g_0^2}{s-\mu^2}\right). \tag{53}$$

From this, we compute the lowest order D:

$$\begin{aligned}D_1(s)&=1+\frac{s-\mu^2}{\pi}\int\frac{\operatorname{Im}D_1(s')}{(s'-s-i\epsilon)(s'-\mu^2)}ds'\\&=1+\frac{s-\mu^2}{\pi}\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\\&\qquad\times\left(\lambda_0+\frac{g_0^2}{s'-\mu^2}\right)\frac{ds'}{(s'-\mu^2)(s'-s-i\epsilon)}.\end{aligned} \tag{54}$$

The scattering amplitude, in lowest order determinental method, is finally given by

$$T(s)=16\pi\left(\frac{s}{s-4M^2}\right)^{\frac{1}{2}}\frac{\operatorname{Im}D_1(s)}{D_1(s)}. \tag{55}$$

This result is identical with the solution we obtained before by solving the dispersion relation, except that Eq. (55) involves the unrenormalized coupling constants λ_0 and g_0. However, Eq. (55) is not exact, but represents only the result of the lowest order determinental approximation. The succeeding orders of the determinental method serve only to alter Eq. (55) by replacing λ_0 and g_0 by other constants, which are to be identified with the renormalized coupling constants λ and g.

To see this, let us carry through the second order determinental approximation. First it is necessary to compute the Feynman amplitude to second order. This comes from the graphs of Fig. 2(b), and is given analytically by

$$\begin{aligned}F_2=\int&\frac{d^4p''}{(2\pi)^4}\frac{i}{p''^2-M^2}\frac{i}{(p''-p-\bar{p})^2-M^2}\\&\times\left[(-i\lambda_0)^2+2(-i)^3\lambda_0g_0^2\frac{i}{s-\mu^2}\right.\\&\qquad\left.+(-ig_0)^4\left(\frac{i}{s-\mu^2}\right)^2\right].\end{aligned} \tag{56}$$

From this we may evaluate the second order contribution to the phase shift, applying the usual Feynman parametrization to Eq. (56), and throwing away a self-

mass term, we find

$$(\sin\delta e^{i\delta})_2$$
$$=\frac{1}{16\pi}\frac{q}{\omega}\bigg\{[I(s)-I(\mu^2)]$$
$$\times\left[\left(\frac{\lambda_0}{4\pi}\right)^2+\frac{2}{s-\mu^2}\left(\frac{\lambda_0}{4\pi}\right)\left(\frac{g_0^2}{4\pi}\right)+\left(\frac{g_0^2}{4\pi}\right)^2\left(\frac{1}{s-\mu^2}\right)^2\right]$$
$$-i\alpha\left[\left(\frac{\lambda_0}{4\pi}\right)^2+2\left(\frac{\lambda_0}{4\pi}\right)\left(\frac{g_0^2}{4\pi}\right)\frac{1}{s-\mu^2}\right]\bigg\},\quad (57)$$

where

$$I(s)-I(\mu^2)=\int_0^1 dx\,\ln\left(\frac{x(1-x)s/4-M^2}{x(1-x)\mu^2/4-M^2}\right),$$

so that

$$I(s)=\left(\frac{s-4M^2}{s}\right)^{\frac{1}{2}}[2\cosh^{-1}(s/4M^2)^{\frac{1}{2}}-i\pi],\quad (58)$$

and α is a constant given by

$$\alpha=\int_0^1 dx\int d^4q\left(\frac{1}{q^2+x(1-x)\mu^2/4-M^2}\right)^2.\quad (59)$$

According to the determinental method, now, the second order ImD is found from the relation

$$\mathrm{Im}D_2=-(\sin\delta e^{i\delta})_2-(\sin\delta e^{i\delta})_1D_1.\quad (60)$$

Looking back at the expression for D_1, Eq. (54), and evaluating the integral, gives

$$D_1(s)=1+\left(\frac{\lambda_0}{16\pi^2}+\frac{g_0^2}{16\pi^2}\frac{1}{s-\mu^2}\right)$$
$$\times[I(s)-I(\mu^2)]-\frac{g_0^2}{16\pi^2}\beta,\quad (61)$$

where β is the constant

$$\beta=\int_{4M^2}^{\infty}\left(\frac{s'-4M^2}{s'}\right)^{\frac{1}{2}}\frac{ds'}{(s'-\mu^2)^2}.\quad (62)$$

The function $I(s)$ in Eq. (61), which comes from evaluating the integral in Eq. (54), is identical with the function defined from the Feynman graph by Eq. (58).

It is now a matter of straightforward algebra to substitute (57) and (61) into (60), and obtain

$$\mathrm{Im}D_2(s)=\frac{1}{16\pi}\left(\frac{s-4M^2}{s}\right)^{\frac{1}{2}}\left(\lambda+\frac{g^2}{s-\mu^2}\right),\quad (63)$$

where we define

$$\lambda=\lambda_0+i\alpha\left(\frac{\lambda_0}{4\pi}\right)^2,$$
$$g^2=g_0^2+2i\alpha\left(\frac{\lambda_0}{4\pi}\right)\left(\frac{g_0^2}{4\pi}\right)-\beta\left(\frac{g_0^2}{4\pi}\right)^2\quad (64)$$
$$-\beta\left(\frac{\lambda_0}{4\pi}\right)\left(\frac{g_0^2}{4\pi}\right).$$

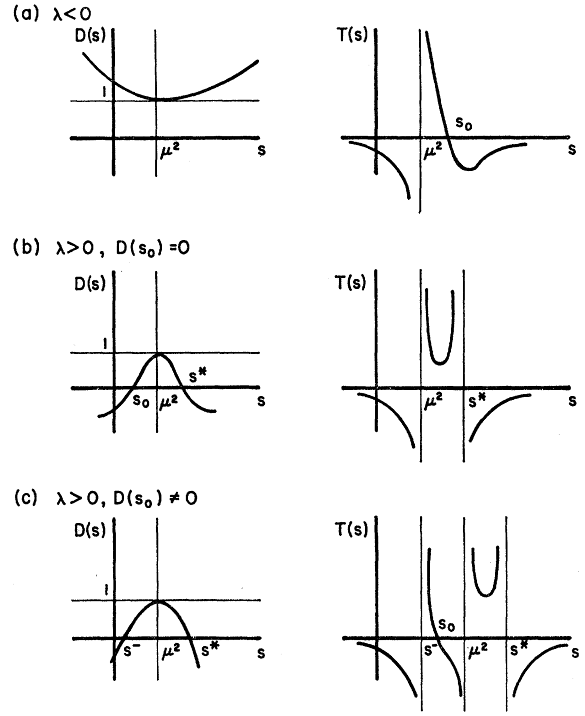


FIG. 3. Schematic plots of $D(s)$ and $T(s)$ for various choices of λ and g^2. In case (a), $\lambda<0$ and g^2 small enough so that $D(s)$ has no zeros. In (b), $\lambda>0$ and $g^2=16\pi^2/\beta$, so $D(s_0)=0$. In (c), $\lambda>0$ and $D(s_0)\neq 0$.

Thus ImD_2 has precisely the same form as ImD_1, in terms of the new constants λ and g^2 defined by Eq. (64). The only effect of the next order of the determinental approximation, then, has been to change the constants in Eq. (54). It is not difficult to convince oneself that this will be the only effect of *any* number of orders of the approximation.

Finally, we may write the result of summing all the Feynman graphs of Fig. 2:

$$T(s)=\frac{\lambda+(g^2/s)-\mu^2}{1+\dfrac{s-\mu^2}{16\pi^2}\displaystyle\int_{4M^2}^{\infty}\left(\lambda+\frac{g^2}{s'-\mu^2}\right)^{\frac{1}{2}}\frac{ds'}{(s'-\mu^2)(s'-s-i\epsilon)}},\quad (65)$$

which is identical with the dispersion theoretic result of Sec. III.

This method, however, gives Eq. (65) as the $B\bar{B}$ scattering amplitude with no restriction on the values of λ and g^2, while the dispersion relation required that $\lambda<0$, among other things, and we must ask why this difference appears. The answer to this was essentially given already in Sec. III: The form of the dispersion relation used there assumed that there were no $B\bar{B}$ bound states. But if, for example, $\lambda>0$ in Eq. (65), $T(s)$ *must* have a pole below and may have a pole above μ^2. The upper pole exists if the denominator vanishes for

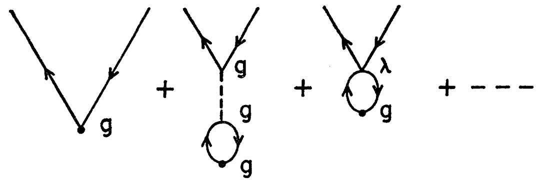


FIG. 4. Feynman graphs for the $AB\bar{B}$ form factor.

$\mu^2<s<4M^2$. Above $4M^2$ only the real part of the denominator can vanish; if it does $T(s)$ has a resonance at that point.

The lower pole, on the other hand, must represent a bound state if it occurs for $s>0$, and this violates the assumed dispersion relation. If the lower pole occurs below $s=0$, it represents an unphysical state, and the model is not self consistent.

The various possibilities are perhaps more conveniently described by the schematic graphs of $D(s)$ and $T(s)$ given in Fig. 3. We know $D(s)\to-(\lambda/16\pi^2)\ln|s|$ as $s\to\infty$, so $T(s)\to\sim(\ln|s|)^{-1}$. That is, $T(s)\to 0$ from below at both + and − infinity independent of the sign of λ. T has a pole at $s=\mu^2$, with a positive residue, so $T\to-\infty$ below the pole and $+\infty$ above. If $\lambda<0$ and g^2 is small enough, D has no zeros, so μ^2 is the only pole in T. Hence we have case (a) of Fig. 3. If $\lambda>0$, ReD must have zeros above and below μ^2. If one of these (it must be the lower since $s_0<\mu^2$ for $\lambda>0$) coincides with s_0, we have Fig. 3(b). If not, we have Fig. 3(c).

As drawn in Fig. 3, case (a) is perfectly consistent. However, it is possible with a sufficiently large value of g^2 to make the $D(s)$ curve dip below the axis, so that D has two zeros, and it is even possible to adjust the various parameters so that one of the zeros occurs on the negative axis.

The case 3(b) is also one which could be unphysical. If the lower zero of D is negative, there again exists a bound state of negative s, and therefore imaginary energy, which certainly reflects an unphysical situation and an inconsistency. We may ask for what values of λ and g^2 these difficulties occur. From Eq. (61), we have

$$D(s)=1+\frac{1}{16\pi^2}\left(\lambda+\frac{g^2}{s-\mu^2}\right)[I(s)-I(\mu^2)]-\frac{g^2}{16\pi^2}\beta. \quad (66)$$

The function $I(s)-I(\mu^2)$ can be calculated explicitly, and is seen to have the following properties: it $\to-\infty$ as $s\to\pm\infty$. It increases monotonically below μ^2, and becomes positive there. It remains positive up to a value $s^*>4M^2$, beyond which it decreases monotonically again. It has a single maximum, reached between μ^2 and $4M^2$.

Using these properties we can draw several conclusions. First, if $g^2=16\pi^2/\beta$, the only possible zeros are s_0 and s^*. s_0 is uninteresting since a zero there does not produce a pole in T. s^* is above $4M^2$, so only the real part of D vanishes, and this represents a resonance in T. Second, if $g^2<16\pi^2/\beta$, a zero of D must be such that the product

$$[\lambda+(g^2/s)-\mu^2][I(s)-I(\mu^2)]$$

is positive at the zero. Now $\lambda+(g^2/s)-\mu^2<0$ if $s<s_0$, $\lambda+(g^2/s)-\mu^2>0$ if $s>s_0$, and $I(s)-I(\mu^2)$ is positive only between μ^2 and s^*. Hence, if $\lambda<0$, zeros of D must be confined to the region $\mu^2<s<s_0$, while if $\lambda>0$, they must be in the region $s_0<s<\mu^2$; otherwise, they can only exist for $s>s^*>4M^2$, in which case they correspond to resonances. The only time an unphysical situation can therefore arise is if $\lambda>0$, and $s_0<0$, i.e., if $0<\lambda<g^2/\mu^2$. Finally, if $g^2>16\pi^2/\beta$, zeros must be confined to the complementary regions.

It should be remarked, by the way, that the conditions obtained here which allow zeros of D do not require that D really have zeros. The actual existence of zeros depends in a complicated way on the values of g^2, λ, μ^2, and M^2; since $D(s)$ is an explicitly known function, whether or not it has zeros is an answerable question, but the answer is still very complicated, and will not be given here. The important point is that the above analyses show there are values of the various parameters for which perfectly good physical solutions exist.

We may remark, incidentally, that a theory defined in terms of a set of dispersion relations, as ours was, can of course not have any ghosts. The analyticity properties of the amplitudes are explicitly assumed by the selection of the dispersion relations, and poles on the negative axis are thereby excluded once the amplitude $T(s)$ is defined as the solution of Eq. (18). The theory defined by Eq. (18) exists only for certain values of λ and g^2—those, in fact, such that $\lambda<0$, and for which D has no zeros—however, once we extrapolate from the dispersion relations to a well-defined set of Feynman graphs, the graphs may be used to define a theory which is now valid for all values of λ and g^2. It is in this extended theory that the ghost question takes on meaning. For certain ranges of λ and g^2, the extended theory agrees with the original one defined by the dispersion relations. For other values of λ and g^2, the extended theory agrees with a modified dispersion type theory which assumes a certain number of $B\bar{B}$ bound states; this is so for values of λ and g^2 such that D has zeros above $s=0$. For the

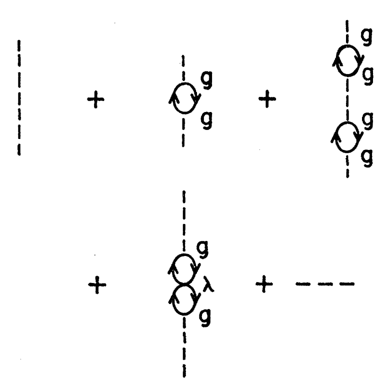


FIG. 5. Feynman graph for the A-particle propagator.

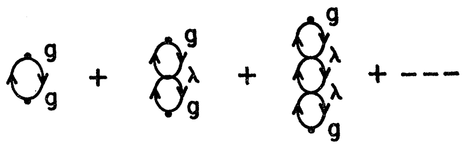


FIG. 6. Feynman graphs contributing to the A-particle self-mass.

remaining values of λ and g^2, the D of the extended theory has zeros for $s<0$ so the theory has ghosts, and is therefore unphysical.

To summarize the situation thus far, we have found the perturbation series equivalent to the model as it was defined in terms of its set of dispersion relations, and the perturbation expansion of the scattering amplitude exists. We can say in addition that there is a wide range of values of the coupling constants and masses for which there are no unphysical singularities of the scattering amplitude, i.e., for which there are no "ghost" states.

It is next of interest to see if we can obtain the expressions of Sec. III for the form factor and propagator as well as the scattering from the Feynman graphs.

The form factor should be clearly given by the sum of all graphs of the type shown in Fig. 4. But the scattering was the sum of all graphs of the type shown in Fig. 2. Evidently if we multiply the graphs of Fig. 4 by $1/\{g[\lambda+(g^2/s-\mu^2)]\}$ we reproduce the graphs of Fig. 2. We should therefore expect

$$F(s)=\frac{gT(s)}{\lambda+(g^2/s-\mu^2)}=\frac{g}{D(s)}, \tag{67}$$

and this is in fact in agreement with Eq. (36).

Similarly, the propagator is given by the graphs of Fig. 5. The scattering graphs of Fig. 2 may be produced from these by the following procedure.

First subtract the free propagator $(1/s-\mu^2)$. Next remove the external legs. Then multiply each end by $(1/g)[\lambda+(g^2/s-\mu^2)]$. Then add the two lower order scattering graphs, $\lambda+(g^2/s-\mu^2)$. Thus we expect

$$T(s)=\lambda+\frac{g^2}{s-\mu^2}+\frac{(s-\mu^2)^2}{g^2}\left(\lambda+\frac{g^2}{s-\mu^2}\right)^2 \times[\Delta(s)-(1/s)-\mu^2]; \tag{68}$$

except not quite. Due to the renormalization prescription, we should replace T and Δ in (68) by Z_3T and $Z_3\Delta$, where T and Δ are now the renormalized scattering and propagator. With this alteration and some algebra, Eq. (68) is easily shown to coincide with Eq. (45) of Sec. III.

Thus all the results of the dispersion approach can be obtained from the Feynman graphs. Even the self-mass can be found, in agreement with Eq. (46), by a pseudo-argument based on the Feynman graphs. If we write the unrenormalized propagator in the form

$$\Delta_0(s)=1/[s-\mu_0^2-\Sigma(s)], \tag{69}$$

where $\Sigma(s)$ is then given by the sum of all *proper* graphs,

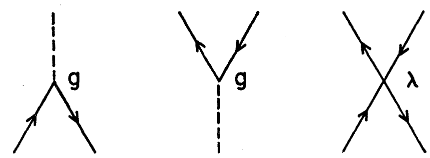


FIG. 7. Feynman graphs contributing to Z_1.

then the self-mass is conventionally defined

$$\delta\mu^2=\Sigma(s=\mu^2), \tag{70}$$

where *renormalized* coupling constants, etc., are to be used in Σ. Now for our model Σ in the sum of all graphs shown in Fig. 6. Note that each term in this series is logarithmically divergent; that is to say, perturbation theory gives an infinite mass. If B represents the basic bubble of Fig. 6, it is clear that

$$\begin{aligned}\delta\mu^2=\Sigma(\mu^2)&=g^2B-g^2\lambda B^2+g^2\lambda^2B^3+\cdots\\&=g^2B/(1+\lambda B).\end{aligned} \tag{71}$$

However, B is infinite, so this reduces to the same result we had earlier.

An argument like this is, of course, not reliable, and the result is believable only because it was obtained in Sec. III by unambiguous mathematics.

The same kind of argument can be used to evaluate Z_1. Z_1 is defined, in perturbation theory, as the value of the sum of all proper vertex graphs when all three particles are on the mass shell. The proper vertex graphs are shown in Fig. 7. Thus we find

$$Z_1=1/(1+\lambda B), \tag{72}$$

which gives $Z_1=0$ since B is logarithmically infinite.

It is unclear if this statement that $Z_1=0$ means anything; however, it does agree with the conjectured relation (50).

To conclude this section, it may be remarked that the sets of Feynman graphs corresponding to any other transition amplitudes of the model are easily identified. Namely, the only allowed graphs are those built up from the basic diagrams of Fig. 8. As an example, think of AB scattering. The Feynman graphs for this process appear in Fig. 9; Fig. 9(a) shows explicitly the structure in terms of the basic graphs of Fig. 8, and Fig. 9(b) shows the same graphs "stretched out" to emphasize that they are Feynman graphs and that no time ordering of the vertices is implied. It is clear that this set of graphs corresponds to the set of dispersion relations for AB scattering and its related processes as given in Sec. II.

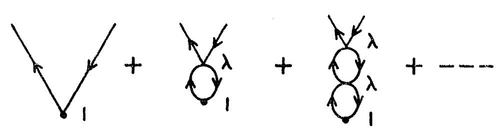


FIG. 8. Basic Feynman graphs for the model.

The same prescription evidently will give the Feynman graphs for any process of the model.

V

To summarize, we have constructed a model field theory which does not suffer from any ghost difficulty, at least for a large range of the coupling constants, and in which the mass and wave function renormalizations are finite. This finiteness is achieved at a very small price, namely the existence of a four-particle vertex, perhaps a reflection of the $\lambda\varphi^4$ interaction in pion physics.

It is appropriate to remark that the drawback—if it is a drawback—of this model in not having a Lagrangian or Hamiltonian can be removed if one gives up the relativistic invariance. The interaction Hamiltonian,

$$H_{\text{int}}=g_0 \sum_{p,\bar{p}'} \frac{1}{(8E_pE_{\bar{p}'}\omega_{p+\bar{p}'})^{\frac{1}{2}}}(a_{p+\bar{p}'}{}^{\dagger}b_p\bar{b}_{\bar{p}'}+a_{p+\bar{p}'}b_p{}^{\dagger}\bar{b}_{\bar{p}'}{}^{\dagger})$$
$$+\lambda_0 \sum_{p,\bar{p}',p''} \frac{1}{(16E_pE_{\bar{p}'}E_{p''}E_{p+\bar{p}'-p''})^{\frac{1}{2}}} \times b_p{}^{\dagger}\bar{b}_{\bar{p}'}{}^{\dagger}b_{p''}\bar{b}_{p+\bar{p}'-p''}$$

defines a soluble theory which has no ghost difficulties (again for a large choice λ and g^2) and in which we still have

$$\delta\mu^2=g^2/\lambda.$$

The graphs which occur are the same as those illustrated previously for the relativistic case, except that they are now to be interpreted as time ordered rather than as Feynman graphs. This model is very similar to the Lee model[1]; it differs only in the presence of recoil and of the λ term. Thus, a fairly slight modification of the Lee model results in removing the ghost difficulty.

One final comment. The model presented here has been exhibited as an exactly soluble but distorted field theory. It may equally well be viewed as the first approximation to an undistorted field theory. In Sec. IV

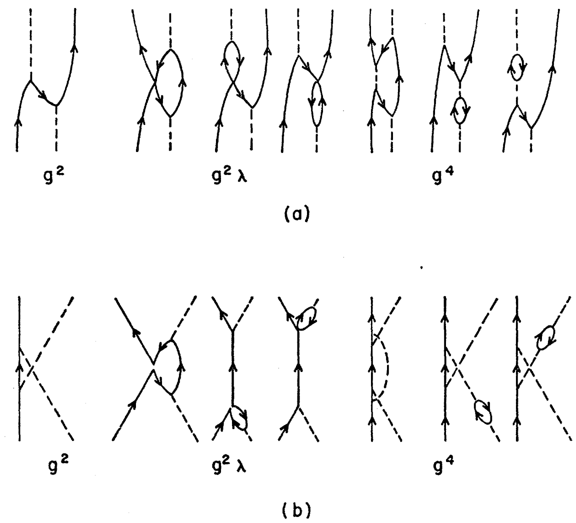


FIG. 9. Feynman graphs for AB scattering.

it was shown that for this model the lowest order determinental approximation was in fact exact; hence, for a true field theory with an interaction Lagrangian

$$\mathfrak{L}_{\text{int}}=g_0\varphi_A(x)\varphi_B{}^2(x)+\lambda_0\varphi_B{}^4(x),$$

our model is precisely the lowest order determinental approximation. One may thus take the view that we have here a full-scale field theory (and therefore, of course, an insoluble one) to which the first of a well-defined sequence of approximations gives finite mass shifts, etc. This might tend to support the view that the *exact* mass shifts in the true field theory are also finite.

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