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Solution for the Two-Electron Correlation Function in a Plasma

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The two-electron correlation function, g, responsible for "collisional" corrections to the correlationless (or Vlasov) description of a plasma, is investigated. It is shown that an exact solution of the integral equation for g can be found for a fairly wide class of spatially homogeneous, one-electron distribution functions, f (the ion dynamics being neglected). This is carried out in detail for the simplest member of the class (the resonance shape), and the Landau damping of g to its asymptotic ($t\rightarrow\infty$) form is exhibited explicitly. It is shown that correlations between particles separated by more than the Debye length are damped in a time which exceeds the period of plasma oscillations, ω_p^{-1}, and that these make an appreciable contribution to the "collisional" rate of change of f. It is concluded that for rapidly varying f (as in problems involving plasma oscillations) conventional treatments of the "collision" term should be replaced by a self-consistent solution of the coupled equations for f and g.

I. INTRODUCTION

THE "Vlasov" or "collisionless Boltzmann" equation for a plasma has received considerable attention during the past few years, especially in connection with problems involving plasma oscillations, wave motions, instabilities, etc. As is known from the work of Rosenbluth and Rostoker[1] (RR) and others, this formulation may be considered as the lowest order approximation in a perturbation solution of the exact kinetic equations for a plasma. In the present work we are concerned with the corrections resulting from a consideration of the next order of the perturbation theory. Any of the effects commonly associated with "collisions" such as dissipation, entropy, production, etc., must come from this (or from higher orders) since the correlationless system, (4) and (5), despite the inclusion of long-range, collective interactions, gives a reversible, entropy-conserving picture of the plasma. A description of our program and of its relation to previous work can best be given after a brief resume of the RR perturbation formalism.

The RR procedure may be summarized as follows. (We shall consider only the simplest case: electrons interacting via Coulomb forces in a uniform background of positive charge, with no magnetic field.) Starting with the Liouville equation for the density in phase space, $D(\mathbf{x}_1,\mathbf{v}_1,\mathbf{x}_2,\mathbf{v}_2\cdots\mathbf{x}_N,\mathbf{v}_N,t)$, one integrates over the coordinates and velocity of $N-1$ electrons to obtain an equation for the one-particle distribution function

$$f^{(1)}(\mathbf{x}_1,\mathbf{v}_1,t)=\int d\mathbf{x}_2 d\mathbf{v}_2\cdots d\mathbf{x}_n d\mathbf{v}_n\, D. \tag{1}$$

This involves the two particle function

$$f^{(2)}(\mathbf{x}_1,\mathbf{v}_1,\mathbf{x}_2,\mathbf{v}_2,t)=\int d\mathbf{x}_3 d\mathbf{v}_3\cdots d\mathbf{x}_n d\mathbf{v}_n\, D, \tag{2}$$

which, in turn, obeys an equation obtained by integrating the Liouville equation over the variables of $N-2$ particles, etc. The idea is now to solve the resulting hierarchy or chain of equations for $f^{(1)}$, $f^{(2)}$, $f^{(3)}\cdots$, by a perturbation approach in which the charge, mass, and inverse density (e,m,n^{-1}) are treated as small quantities. (The dimensionless "small parameter" is N_D^{-1}, where N_D is the number of particles in a sphere whose radius is the Debye length.) To lowest order (which we shall henceforth call zero order) an

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[1] M. Rosenbluth and N. Rostoker, Phys. Fluids 3, 1 (1960).

exact solution of the entire chain of equations is given by

$$f^{(s)}(\mathbf{x}_1,\mathbf{v}_1,\cdots\mathbf{x}_s,\mathbf{v}_s,t)=\prod_{i=1}^{s} f(\mathbf{x}_i,\mathbf{v}_i,t), \tag{3}$$

where f satisfies the equation[2]

$$\partial f/\partial t+\mathbf{v}\cdot(\partial f/\partial \mathbf{x})-(e\mathbf{E}/m)\cdot(\partial f/\partial \mathbf{v})=0, \tag{4}$$

with

$$\nabla\cdot\mathbf{E}=4\pi ne\left(1-\int f d\mathbf{v}\right). \tag{5}$$

In view of (3), the term "correlationless" seems more appropriate for (4) than "collisionless."

Continuing the perturbation solution to first order in e, m, and n^{-1}, RR find that again the entire chain is satisfied by taking (with $\xi=\{\mathbf{x},\mathbf{v}\}$)

$$f^{(1)}(\xi,t)=f(\xi,t),$$
$$f^{(2)}(\xi_1,\xi_2,t)=f(\xi_1,t)f(\xi_2,t)+g(\xi_1,\xi_2,t),$$

and expressing all $f^{(s)}$, $s>2$, as symmetrized products of f and g,

$$f^{(3)}(\xi_1,\xi_2,\xi_3,t)=f(\xi_1,t)f(\xi_2,t)f(\xi_3,t)$$
$$+f(\xi_1,t)g(\xi_2,\xi_3,t)+f(\xi_2,t)g(\xi_1,\xi_3,t)+f(\xi_3,t)g(\xi_1,\xi_2,t),$$

etc., provided f and g satisfy the coupled equations

$$\begin{aligned}&\partial f/\partial t+\mathbf{v}\cdot(\partial f/\partial \mathbf{x})-(e\mathbf{E}/m)\cdot(\partial f/\partial \mathbf{v})=\delta f/\delta t,\\ &\delta f/\delta t\equiv(ne/m)\int d\mathbf{x}_2 d\mathbf{v}_2\, \mathbf{F}_{1,2}\cdot[\partial g(\xi_1,\xi_2,t)/\partial \mathbf{v}_1],\end{aligned} \tag{6}$$

and

$$\partial g/\partial t+\sum_1^2 \{\mathbf{v}_i\cdot(\partial/\partial \mathbf{x}_i)-[e\mathbf{E}(\mathbf{x}_i)/m]\cdot[\partial/\partial \mathbf{v}_i]\}g$$
$$=(ne/m)\int d\mathbf{x}_3 d\mathbf{v}_3 g(\xi_2,\xi_3,t)\mathbf{F}_{1,3}\cdot\partial f(\xi_1,t)/\partial \mathbf{v}_1$$
$$+(e/m)f(\xi_2,t)\mathbf{F}_{1,2}\cdot[\partial f(\xi_1,t)/\partial \mathbf{v}_i]+\{1\leftrightarrow 2\}. \tag{7}$$

Here

$$\mathbf{F}_{i,j}\equiv e(\partial/\partial \mathbf{x}_i)|\mathbf{x}_i-\mathbf{x}_j|^{-1}, \tag{8}$$

$\mathbf{E}$ is given by (5) and the curly brace in (7) stands for two terms which differ from the first two on the right-hand side only in having $\mathbf{x}_1$, $\mathbf{v}_1$ interchanged with $\mathbf{x}_2$, $\mathbf{v}_2$.

A consistent treatment of the first-order problem would require the simultaneous solution of (5), (6), and (7). To date this has not been achieved. A less ambitious but still interesting problem would be that obtained by linearization of the equations about an equilibrium, time-independent solution, a technique responsible for most of the knowledge we possess of the properties of the correlationless equations, (4) and (5). We formulate this problem below, but have so far been unable to find a complete solution.

The procedure actually adopted heretofore in dealing with (6) and (7) has been along the lines of Bogolyubov's concept of a hierarchy of relaxation times. Applied to the present case, this means that, for given f, the two-particle function g relaxes to its equilibrium value (for that f) in a time short compared to that in which f changes appreciably. *If* this is true, then it is reasonable to solve (7) with f regarded as given, independent of time; to find g; to compute the asymptotic ($t\to\infty$) limit of g; and to use this limiting value, g_∞, in calculating $\delta f/\delta t$. This is essentially the procedure used by Lenard,[3] for the particular case where the given f is homogeneous (independent of $\mathbf{x}$ as well as t). He shows that the resulting $\delta f/\delta t$ agrees very closely with the Boltzmann collision term as computed by Landau[4] for Coulomb forces. The RR "test-particle" technique, although formally different from Lenard's, also employs the $t\to\infty$ form of g in order to obtain essentially the same $\delta f/\delta t$ (plus additional terms associated with the radiation of plasma oscillations by a fast charged particle).

It seems clear, however, that in situations where f is changing rapidly, say in a time compared with a plasma oscillation period, $1/\omega_p$, essential dynamic effects resulting from the interplay of f and g will be entirely lost if one replaces g by its asymptotic limit. Plasma oscillations constitute an important case where just such rapid variation of f will necessarily occur. Of course, the $\delta f/\delta t$ term will be small (as N_D^{-1}) compared to the other terms in (6) so that dramatic new effects are not to be expected so long as we remain within the limits of validity of the RR perturbation theory ($N_D\gg 1$). Nonetheless, even a small term like $\delta f/\delta t$ can produce significant changes from the behavior of the completely correlationless system ($\delta f/\delta t=0$) and if we are to include such a term at all, it should be the correct one.

As a first step in the investigation of these dynamic effects, we present here a problem which is at once an extension and a specialization of Lenard's treatment of the homogeneous case. We generalize to the extent of studying g at finite times, rather than just in the limit $t\to\infty$. Were we able to solve the resulting integral equation for g with f chosen as a Maxwellian distribution, then the linearized problem described above would also be in hand. Failing in this, we have instead specialized to a particular class of f which allows a complete analytical solution of the integral equation for g. It is likely that the principal features of the resulting solution apply, at least qualitatively, to other choices of f as well.

In Sec. II we list, chiefly for reference, the original

[2] A. Simon and E. G. Harris, Phys. Fluids 3, 245 (1960), have shown that inclusion of the radiation field to lowest order gives rise to a $\mathbf{v}\times\mathbf{B}$ term in (4), with $\mathbf{E}$ and $\mathbf{B}$ determined by the full Maxwell equations. For our purposes, however, the interaction via the Coulomb field alone will suffice.

[3] A. Lenard, Ann. Phys. 10, 390 (1960).

[4] L. Landau, Physik Z. Sowjetunion 10, 154 (1936).

system (6) and (7) and the linearized form thereof, Fourier transformed to a wave number representation. The two-particle integral equation for the homogeneous case is analyzed in Sec. III, and a class of one-particle functions is given which make the integral equation soluble. An explicit solution is found for the simplest f of this class and, in Sec. IV, some properties of this solution are deduced.

II. TRANSFORMATION TO WAVE NUMBER SPACE

In studying the correlationless equation, (4), it is usually convenient to deal with the spatial Fourier transform,

$$f(\mathbf{k},\mathbf{v},t)=\int d\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x},\mathbf{v},t). \tag{9}$$

This suggests that we likewise Fourier-transform g and work with $g(\mathbf{k}_1,\mathbf{v}_1,\mathbf{k}_2,\mathbf{v}_2,t)$. It is advantageous to introduce, at the same time, center-of-mass and relative coordinates for the two particles involved in g, i.e., to work with

$$G(\mathbf{k};\,\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)=g(\mathbf{k}/2+\mathbf{p}/2,\ \mathbf{v}_1,\ \mathbf{k}/2-\mathbf{p}/2,\ \mathbf{v}_2,\ t), \tag{10}$$

where

$$\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2 \quad\text{and}\quad \mathbf{p}=\mathbf{k}_1-\mathbf{k}_2 \tag{11}$$

are the transform variables associated with the center-of-mass and relative coordinates, respectively:

$$g(\mathbf{x}_1,\mathbf{v}_1,\mathbf{x}_2,\mathbf{v}_2,t)=(1/8)(2\pi)^{-6}\int d\mathbf{k}\int d\mathbf{p}\, e^{i\mathbf{k}\cdot(\mathbf{x}_1+\mathbf{x}_2)/2} \times e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)/2} G(\mathbf{k};\,\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t). \tag{12}$$

Upon Fourier-transforming (6) and (7), we obtain

$$(\partial/\partial t+i\mathbf{k}\cdot\mathbf{v})f(\mathbf{k},\mathbf{v},t)-(e/m)(2\pi)^{-3}\int d\mathbf{k}'\mathbf{E}(\mathbf{k}')\cdot\partial f(\mathbf{k}-\mathbf{k}',\ \mathbf{v},\ t)/\partial\mathbf{v}=\delta f/\delta t, \tag{13}$$

$$\delta f(\mathbf{k},\mathbf{v},t)/\delta t=i\omega_p{}^2(2\pi)^{-3}\int d\mathbf{k}'\int d\mathbf{v}'(k')^{-2}\mathbf{k}'\cdot[\partial G(\mathbf{k};\,\mathbf{v},\ \mathbf{v}',\ \mathbf{k}-2\mathbf{k}',\ t)/\partial\mathbf{v}], \tag{14}$$

and

$$\begin{aligned}
&\{\partial/\partial t+(i/2)[\mathbf{p}\cdot(\mathbf{v}_1-\mathbf{v}_2)+\mathbf{k}\cdot(\mathbf{v}_1+\mathbf{v}_2)]\}G(\mathbf{k};\,\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)\\
&\quad-(e/m)(2\pi)^{-3}\int d\mathbf{k}'\,\mathbf{E}(\mathbf{k}')\cdot[\partial G(\mathbf{k}-\mathbf{k}';\,\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{p}-\mathbf{k}',\ t)/\partial\mathbf{v}_1+\partial G(\mathbf{k}-\mathbf{k}';\,\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{p}+\mathbf{k}',\ t)/\partial\mathbf{v}_2]\\
&=i\omega_p{}^2(2\pi)^{-3}\int d\mathbf{v}'\int d\mathbf{k}'\, G(\mathbf{k}/2-\mathbf{p}/2+\mathbf{k}';\,\mathbf{v}_2,\ \mathbf{v}',\ \mathbf{k}/2-\mathbf{p}/2-\mathbf{k}',\ t)(k')^{-2}\mathbf{k}'\cdot[\partial f(\mathbf{k}/2+\mathbf{p}/2-\mathbf{k}',\ \mathbf{v}_1)/\partial\mathbf{v}_1]\\
&\quad+(i\omega_p{}^2/8\pi^3n)\int d\mathbf{k}'(k')^{-2}\mathbf{k}'\cdot[\partial f(\mathbf{k}/2+\mathbf{p}/2-\mathbf{k}',\ \mathbf{v}_1,\ t)/\partial\mathbf{v}_1]f(\mathbf{k}/2-\mathbf{p}/2+\mathbf{k}',\ \mathbf{v}_2,\ t)+\{\mathbf{v}_1\leftrightarrow\mathbf{v}_2,\ \mathbf{p}\leftrightarrow-\mathbf{p}\},
\end{aligned} \tag{15}$$

with

$$\omega_p{}^2=4\pi ne^2/m.$$

Two methods of attacking this rather formidable system suggest themselves. If f (and, therefore, $\mathbf{E}$) is considered as known, (15) is linear in the unknown function G and, hopefully, susceptible to some of the many techniques which have been developed for solving linear problems. However, the more interesting problem would seem to be the fully self-consistent one wherein f is unknown (save for initial conditions) and is co-determined with G. Since the problem is then not only complicated but nonlinear as well, one is tempted to linearize, as is generally done with (4), about some "unperturbed" problem. If we require the latter to be independent of time, one (and probably the only) choice is a space-independent, Maxwellian distribution for f_0,

$$\begin{aligned}
f_0(\mathbf{k},\mathbf{v},t)&=(2\pi)^3\delta(\mathbf{k})f_0(\mathbf{v}),\\
f_0(\mathbf{v})&=(a\pi^{\frac{1}{2}})^{-3}\exp(-v^2/a^2).
\end{aligned} \tag{16}$$

With this f_0, (15) gives for G_0 the Debye-shielded two-particle function

$$\begin{aligned}
G_0(\mathbf{k};\,\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)&=(2\pi)^3\delta(\mathbf{k})f_0(\mathbf{v}_1)f_0(\mathbf{v}_2)Y(\mathbf{p})\\
Y(p)&=-4(k_D{}^2/n)(p^2+4k_D{}^2)^{-1},
\end{aligned} \tag{17}$$

k_D being the Debye wave number

$$k_D=2^{\frac{1}{2}}\omega_p/a.$$

Setting

$$f=f_0+f_1,\quad G=G_0+G_1,$$

we then obtain, in a linear approximation,

$$(\mathbf{k}\cdot\mathbf{v}-i\partial/\partial t)f_1(\mathbf{k},\mathbf{v},t)=\omega_p{}^2\eta k^{-2}\mathbf{k}\cdot df_0/d\mathbf{v}+(2\pi)^{-3}\omega_p{}^2\int d\mathbf{v}'\int d\mathbf{k}'(k')^{-2}\mathbf{k}'\cdot[\partial G_1(\mathbf{k};\mathbf{v},\mathbf{v}',\mathbf{k}-2\mathbf{k}',t)/\partial\mathbf{v}], \quad (18)$$

and

$$\{\mathbf{p}\cdot(\mathbf{v}_1-\mathbf{v}_2)/2+\mathbf{k}\cdot(\mathbf{v}_1+\mathbf{v}_2)/2-i\partial/\partial t\}G_1(k;\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)=-\omega_p{}^2n^{-1}\eta k^{-2}\mathbf{k}\cdot(df_0/d\mathbf{v}_1)4k_D{}^2[(\mathbf{k}-\mathbf{p})^2+4k_D{}^2]^{-1}f_0(\mathbf{v}_2)$$
$$+2\omega_p{}^2\int d\mathbf{v}'\,G_1(\mathbf{k};\mathbf{v}',\mathbf{v}_2,\mathbf{p},t)(\mathbf{k}+\mathbf{p})^{-2}(\mathbf{k}+\mathbf{p})\cdot(df_0/d\mathbf{v}_1)+2\omega_p{}^2n^{-1}(\mathbf{k}+\mathbf{p})^{-2}(\mathbf{k}+\mathbf{p})\cdot(df_0/d\mathbf{v}_1)f_1(\mathbf{k},\mathbf{v}_2)$$
$$+2\omega_p{}^2n^{-1}[(\mathbf{p}-\mathbf{k})^{+2}+4k_D{}^2]^{-1}(\mathbf{p}-\mathbf{k})\cdot[\partial f_1(\mathbf{k},\mathbf{v}_1)/\partial\mathbf{v}_1]f_0(\mathbf{v}_2)+\{\mathbf{v}_1\leftrightarrow\mathbf{v}_2,\ \mathbf{p}\leftrightarrow-\mathbf{p}\}, \quad (19)$$

where

$$\eta(\mathbf{k},t)=\int d\mathbf{v}\, f(\mathbf{k},\mathbf{v},t).$$

III. THE SPATIALLY HOMOGENEOUS SYSTEM

We consider now the case studied by Lenard, where the one-body function is a given function of $\mathbf{v}$, and is independent of $\mathbf{x}$ and t. The electric field $\mathbf{E}$ vanishes, and in wave number space we have only the $\mathbf{k}=0$ component:

$$f(\mathbf{k},\mathbf{v},t)=(2\pi)^3\delta(\mathbf{k})f(\mathbf{v}), \quad (20)$$

$$G(\mathbf{k};\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)=(2\pi)^3\delta(\mathbf{k})G(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t). \quad (21)$$

From (15) we find that this $\mathbf{k}=0$ portion of G must satisfy

$$[-i\partial/\partial t+\mathbf{q}\cdot(\mathbf{v}_1-\mathbf{v}_2)]G(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)$$
$$-(\omega_p{}^2\mathbf{q}/q^2)\cdot\int d\mathbf{v}'[G(\mathbf{v}',\mathbf{v}_2,\mathbf{p},t)df/d\mathbf{v}_1$$
$$-G(\mathbf{v}_1,\mathbf{v}',\mathbf{p},t)df/d\mathbf{v}_2]$$
$$=(\omega_p{}^2\mathbf{q}/nq^2)[f(\mathbf{v}_2)df/d\mathbf{v}_1-f(\mathbf{v}_1)df/d\mathbf{v}_2]$$
$$\equiv H(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p}), \quad (22)$$

where

$$\mathbf{q}=\mathbf{p}/2.$$

We note that the linearized integral equation (19), specialized to $\mathbf{k}=0$, differs from this only with respect to the inhomogeneous term. However, in (22) we have not yet specified our choice of f, whereas in (19) we are necessarily committed to a Maxwellian distribution.

Before solving (22), we digress briefly to explain the connection between it and the equation (derived from Bogolyubov's formalism) which forms the starting point of Lenard's treatment. The Laplace transform of (22) gives a linear integral equation for the Laplace transform of G, $\tilde{G}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},\omega)$, which we may write as

$$\mathcal{L}\tilde{G}=iH/\omega-iG_0, \quad (23)$$

where G_0 denotes the initial ($t=0$) value of G and the factor i/ω arises from the fact that the inhomogeneous term, H, is time independent. The solution, $\tilde{G}$, of this equation will have singularities in the ω plane whose location depends upon the properties of $\mathcal{L}$ and G_0, plus a pole at $\omega=0$. *If* $G_0=0$ and *if* the poles associated with $\mathcal{L}$ are all damped (i.e., lie in the lower half ω-plane), *then* the asymptotic behavior of G is determined entirely by the residue at $\omega=0$, i.e.,

$$G_\infty\equiv\lim_{t\to\infty}G=\lim_{t\to\infty}(2\pi)^{-1}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega t}\tilde{G}(\omega)$$
$$=\lim_{\omega\to 0}(-i\omega\tilde{G}).$$

Since the contour of integration in the ω plane must lie above all singularities of $\tilde{G}$, including the one at $\omega=0$, it is correct (and advisable, as regards possible ambiguities) to set $\omega=i\epsilon$, $\epsilon>0$, and at the end allow ϵ to approach zero. From (23) we obtain an equation for the function $\bar{G}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p})\equiv(-i\omega\tilde{G})_{\omega=i\epsilon}$ which coincides with Lenard's starting point. (Lenard does not completely solve this equation but instead manages, ingeniously, to extract from it just enough information to permit the evaluation of $\delta f/\delta t$ in the $t=\infty$ limit.)

We now return to the solution of the complete equation, (22). The form of this equation prompts us to define

$$\alpha(\mathbf{v}_1,\mathbf{q},t)=\int d\mathbf{v}_2\, G(\mathbf{v}_1,\mathbf{v}_2,2\mathbf{q},t). \quad (24)$$

The other integral which occurs in (22) can be expressed in terms of α by exploiting the fact that the symmetry of g under the interchange $(\mathbf{x}_1,\mathbf{v}_1)\leftrightarrow(\mathbf{x}_2,\mathbf{v}_2)$ implies

$$G(\mathbf{k};\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)=G(\mathbf{k};\mathbf{v}_2,\mathbf{v}_1,-\mathbf{p},t), \quad (25)$$

while the scalar character of G requires

$$G(\mathbf{k};\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)=G(-\mathbf{k};-\mathbf{v}_1,-\mathbf{v}_2,-\mathbf{p},t). \quad (26)$$

It follows that

$$G(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)=G(-\mathbf{v}_2,-\mathbf{v}_1,\mathbf{p},t), \quad (27)$$

and hence that[5]

$$\int d\mathbf{v}_1\, G(\mathbf{v}_1,\mathbf{v}_2,2\mathbf{q},t)=\alpha(-\mathbf{v}_2,\mathbf{q},t). \quad (28)$$

[5] We are indebted to Professor F. Zachariasen for calling our attention to this symmetry.

Upon taking the Laplace transform of G, α, and β,

$$\tilde{G}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},\omega)=\int_0^\infty dt\, e^{i\omega t}G(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t), \quad \text{etc.}, \tag{29}$$

we obtain at once from (22) an integral equation for the Laplace transform, $\tilde{\alpha}$, of α

$$\tilde{\alpha}(\mathbf{v}_1,\mathbf{q},\omega)=\gamma^2\left[(\partial f/\partial u_1)\int d\mathbf{v}_2(u_1+u_2-u)^{-1}\tilde{\alpha}(\mathbf{v}_2,\mathbf{q},\omega)-\tilde{\alpha}(\mathbf{v}_1,\mathbf{q},\omega)\int d\mathbf{v}_2(u_1-u_2-u)^{-1}\partial f/\partial u_2\right]$$
$$+q^{-1}\int d\mathbf{v}_2[i\omega^{-1}H(\mathbf{v}_1,\mathbf{v}_2,2\mathbf{q})-iG_0(\mathbf{v}_1,\mathbf{v}_2,2\mathbf{p})](u_1-u_2-u)^{-1}, \tag{30}$$

where

$$\gamma\equiv\omega_p/q, \qquad u\equiv\omega/q,$$
$$u_1\equiv\mathbf{v}_1\cdot\mathbf{p}/p, \quad u_2\equiv\mathbf{v}_2\cdot\mathbf{p}/p, \tag{31}$$
$$G_0(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p})\equiv G(\mathbf{v}_1,\mathbf{v}_2,\mathbf{p},t)\,|_{t=0}.$$

Since $\mathbf{q}$ defines the only preferred direction in the problem, we can reduce this vector integral equation to a scalar one by integrating out the components perpendicular to $\mathbf{q}$. With

$$A(u_1,\mathbf{q},\omega)\equiv\int d\mathbf{v}_1\,\delta(u_1-\mathbf{v}_1\cdot\mathbf{q}/q)\tilde{\alpha}(\mathbf{v}_1,\mathbf{q},\omega), \tag{32}$$

$$F(u_1)\equiv\int d\mathbf{v}_1\,\delta(u_1-\mathbf{v}_1\cdot\mathbf{q}/q)f(\mathbf{v}_1), \tag{33}$$

$$\mathfrak{F}(u_1)\equiv dF/du_1, \tag{34}$$

and

$$\mathcal{H}(u_1,u_2,q)\equiv(\gamma^2/n)[\mathfrak{F}(u_1)F(u_2)-F(u_1)\mathfrak{F}(u_2)], \tag{35}$$

we obtain at once from (30) (suppressing the $\mathbf{q}$ and ω dependence),

$$A(u_1,u)=\gamma^2\mathfrak{F}(u_1)[1-\gamma^2\bar{\Lambda}(u_1-u)]^{-1}$$
$$\times\int du_2(u_1+u_2-u)^{-1}A(u_2,u)+\tau(u_1,u). \tag{36}$$

Here $\bar{\Lambda}$ is given by

$$\bar{\Lambda}(\zeta)=\int_{-\infty}^{\infty}dx\,\mathfrak{F}(x)(x-\zeta)^{-1} \quad \text{for} \quad \mathrm{Im}\zeta<0, \tag{37}$$

and by the analytic continuation of this if $\mathrm{Im}\zeta\geqslant 0$, this definition being a consequence of the Laplace transform prescription that ω lie in the upper half plane (or, more precisely, above the poles of α). We shall later need the related function defined as

$$\Lambda(\zeta)=\int_{-\infty}^{\infty}dx\,\mathfrak{F}(x)(x-\zeta)^{-1} \quad \text{for} \quad \mathrm{Im}\zeta>0$$
$$=\text{analytic continuation} \quad \text{for} \quad \mathrm{Im}\zeta\leq 0.$$

The inhomogeneous term τ,

$$\tau(u_1,u)\equiv i[1-\gamma^2\bar{\Lambda}(u_1-u)]^{-1}$$
$$\times\int du_2(\mathcal{H}/\omega-G_0/q)(u_1-u_2-u)^{-1}, \tag{38}$$

is (for given f and G_0) a known function of u_1 and u.

At this point, we should like to solve (36) for $A(u_1,u)$, keeping $\mathrm{Im}u>0$; analytically continue into the lower half of the u plane; and locate the poles (or other singularities) of A. This is very difficult to accomplish if f is Maxwellian, since the function $\bar{\Lambda}$ (essentially the error function of complex argument) is already a fairly intractable object. For a rather wide class of f, however, a solution is possible, provided we study not (36) but the equivalent equation resulting from a single iteration of (36):

$$A(u_1,u)=\sigma(u_1,u)+\gamma^4\mathfrak{F}(u_1)[1-\gamma^2\bar{\Lambda}(u_1-u)]^{-1}$$
$$\times\int du_2du_3\,\mathfrak{F}(u_2)A(u_3,u)(u_2+u_3-u)^{-1}$$
$$\times(u_1+u_2-u)^{-1}[1-\gamma^2\bar{\Lambda}(u_2-u)]^{-1}, \tag{39}$$

$$\sigma(u_1,u)=\tau(u_1,u)+\gamma^2\mathfrak{F}(u_1)[1-\gamma^2\bar{\Lambda}(u_1-u)]^{-1}$$
$$\times\int du_2\,\tau(u_2,u)(u_1+u_2-u)^{-1}. \tag{40}$$

The class of f which simplifies the problem consists of those for which $F(u_1)$ is analytic, save for a finite number of poles in the finite plane, and regular at infinity (as distinguished from a Gaussian, which has an essential singularity at infinity). (Of course, its significance as a distribution function requires that $0\leqslant F(u_1)<\infty$ for real u_1.) We shall consider in detail only the simplest member of this class,

$$f(\mathbf{v})=(a/\pi^2)(a^2+v^2)^{-2}, \tag{41}$$

the method of extension to other cases being clear. With this f we have

$$F(u_1)=(a/\pi)(a^2+u_1^2)^{-1}, \tag{42}$$

$$\Lambda(\zeta)=\bar{\Lambda}(-\zeta)=(\zeta+ia)^{-2}. \tag{43}$$

[Of course, this "resonance" shape, (41), has no particular physical justification, but since f approaches a delta function as $a\rightarrow 0$, the results obtained should be approximately valid for any very narrow distribution function.]

With this choice for f, it becomes possible to carry out explicitly the integration over u_2 in (39). The integrand has poles at $u_2=\pm ia$, $u_2=u-u_3$, $u_2=u-u_1$ and $u_2=ia\pm\gamma$. [It is known[6] from the dispersion equation for the Vlasov equation (4) that for any real γ,

$$1-\gamma^2\Lambda(\zeta)=0$$

has roots only in the lower half plane whenever F is a single-humped curve, like (42). A similar argument shows that the roots of

$$1-\gamma^2\bar{\Lambda}(\zeta)=0 \tag{44}$$

all lie in the upper half plane.] The only pole of the integrand in the lower half of the u_2 plane is, therefore, the one at $u_2=-ia$, so that closing the contour below yields

$$\begin{aligned}\int du_2\, \mathfrak{F}(u_2)(u_2+u_3-u)^{-1}(u_2+u_1-u)^{-1}&\\ \times[1-\gamma^2\bar{\Lambda}(u_2-u)]^{-1}&\\ =(d/du)(u_3-ia-u)^{-1}(u_1-ia-u)^{-1}&\\ \times[1-\gamma^2\bar{\Lambda}(-ia-u)]^{-1}.&\end{aligned} \tag{45}$$

This is the kernel for the integral equation (39) and since it is a sum of two separable terms (each being a product of a function of u_1 and a function of u_3), the solution of (39) reduces to an algebraic problem. In fact, substituting (45) into (39) we have

$$\begin{aligned}A(u_1,u)&=\sigma(u_1,u)+\gamma^4\mathfrak{F}(u_1)[1-\gamma^2\bar{\Lambda}(u_1-u)]^{-1}\\ &\quad\times[1-\gamma^2\bar{\Lambda}(-u-ia)]^{-1}\{I_1(u)(u_1-ia-u)^{-2}\\ &\qquad+K(u)(u_1-ia-u)^{-1}\}\\ &=\sigma(u_1,u)+\gamma^4\mathfrak{F}(u_1)[(u_1-u-ia)^2-\gamma^2]^{-1}\zeta(\zeta-1)^{-1}\\ &\qquad\times[I_1+(u_1-ia-u)K],\end{aligned} \tag{46}$$

where

$$\begin{aligned}I_m(u)&\equiv\int_{-\infty}^{\infty}dx\,A(x,u)(x-ia-u)^{-m},\quad m=1,2,\\ K(u)&\equiv I_2-\gamma^2\bar{\Lambda}'(-u-ia)[1-\gamma^2\bar{\Lambda}(-u-ia)]^{-1}I_1.\end{aligned} \tag{47}$$

Substituting (46) into (47) yields two simultaneous equations for I_1 and K, whose solution, together with (46), completes the determination of A. Alternatively, one can simply substitute the expression (46) for A into the original equation, (36), thereby eliminating the possibility that in the course of the algebraic manipulations one might have arrived at a solution of the iterated equation (39) which would not be also a solution of (36). Either of these procedures gives the pair of equations

$$\begin{aligned}(\zeta-2)I_1+\zeta J&=(\zeta-1)\rho_1(u),\\ 4I_1+\zeta(\zeta-2)J&=(\zeta-1)w\rho_2(u),\end{aligned} \tag{48}$$

where

$$\begin{aligned}w=u+2ia,\quad \zeta&=(w/\gamma)^2=(\omega+2iaq)^2\omega_p^{-2},\\ J&=K\gamma^2/w.\end{aligned} \tag{49}$$

The two functions,

$$\rho_m(u)\equiv\int dy\,\tau(y,u)(y-ia-u)^{-m},$$

are, like τ, to be regarded as known. The solution of (48) is

$$\begin{aligned}\zeta I_1(\zeta-1)^{-1}&=[(\zeta-2)\rho_1-w\rho_2](\zeta-4)^{-1},\\ \zeta J(\zeta-1)^{-1}&=[(\zeta-2)\rho_2 w-4\rho_1](\zeta-4)^{-1}\zeta^{-1}.\end{aligned} \tag{50}$$

With (46) and (50) we have the desired solution of the integral equation (36) for $A(u_1,u,q,\omega)$. Retracing the steps which led to (36), we use (30) to find α from A, while (24) and (the Laplace transform of) (22) give G. Some of the properties of this solution and the results which follow therefrom are discussed in the following section. Before taking this up, we remark that had we chosen, in place of (41), a more general example of the class of functions f described above (i.e., one for which $F(u_1)$ has $n>1$ poles in the upper half u_1-plane rather than just one), the explicit evaluation of the kernel of (39) would still be possible. In doing the integration (45) we would still close the contour below, thereby obtaining contributions not only from the n poles of F in the lower half plane but also, possibly, from roots of (44), these being no longer necessarily restricted to the upper half plane. (We are indebted to Dr. Jon Matthews for pointing out this possibility.) The essential point is just that the $2(n+k)$ terms so obtained (k being the number of roots of (44) in the lower half plane) would each be a product of a function of u_1 and a function of u_3, so that again solution of the integral equation would reduce to that of an algebraic system, this time of order $2(n+k)$.

IV. PROPERTIES OF THE SOLUTION

In the analysis of the linearized Vlasov equation, one is particularly interested in the dispersion equation, for its roots give information regarding the asymptotic behavior at large time. The analogous equation in the present analysis is $\Delta=0$ where

$$\Delta=\zeta^2(\zeta-4) \tag{51}$$

is the determinant of the system of Eqs. (48). It has a single root at $\zeta=4$, i.e.,

$$\omega=\pm2\omega_p-iap, \tag{52}$$

and a double root at $\zeta=0$ or

$$\omega=-iap. \tag{53}$$

[6] J. D. Jackson, J. Nuclear Energy 1, 171 (1960).

In the Vlasov initial-value problem[6] the Laplace transform of the density is a quotient of two functions of ω. The numerator, N, depends upon the initial values of the distribution function, while the zeros of the denominator, D, are just the roots, ω_n, of the dispersion equation. Without specification of the initial conditions, we cannot really predict the time dependence of the density (even at large times!) For instance, the initial values can be so chosen that $N(\omega_n)=0$. (The use of just such initial conditions enabled Van Kampen[7] to obtain solutions without Landau damping.) Nonetheless, the ω_n are of interest since for most initial conditions, i.e., those for which $N(\omega_n)\neq 0$, the time dependence will be a sum of terms $e^{-i\omega_n t}$ plus contributions associated with the singularities of the N. About the latter, nothing general can be said, whereas the former have, in a sense, a "universal" character.

In our present problem, similar conclusions can be drawn. Barring special choices of the initial condition, G_0, we can say that $A(t)$ will contain terms of the form

$$e^{\pm 2i\omega_p t - apt};\quad e^{-apt}. \tag{54}$$

In addition, there will be terms associated with the u_1-dependent poles of (46), just as in the Vlasov problem the distribution function has, besides the poles of the density, also a pole at $\omega=\mathbf{k}\mathbf{v}$.

However, there exists in the present problem one particularly simple initial condition—namely, $G_0=0$. In contrast to the Vlasov case, a zero initial condition permits a nontrivial solution, for the inhomogeneous term of (30) or (36) comes only partly from G_0. The term H, bilinear in the single-particle function, gives rise to a nonvanishing G for $t>0$ even if $G=0$ at $t=0$. It is therefore of interest to examine this case. For all f of the class defined in Sec. III, the auxiliary functions involved in our solution—τ, σ, ρ_1, ρ_2, I_1, J—can then be evaluated in closed form, permitting an explicit expression for A and α. With the particular choice (41) we find

$$\begin{aligned} A(u_1,\mathbf{q},\omega)=(i\gamma^2/\omega n)[(u_1-u-ia)^2-\gamma^2]^{-1} \\ \times\{F(u_1)+\mathfrak{F}(u_1)[(u_1-ia-u)(\zeta-2)-w] \\ \times(\zeta-4)^{-1}\}, \end{aligned} \tag{55}$$

while $\tilde{\alpha}(\mathbf{v}_1,\mathbf{q},\omega)$ is obtained from this by the substitutions

$$F(u_1)\rightarrow f(\mathbf{v}_1),\quad \mathfrak{F}(u_1)\rightarrow \partial f(\mathbf{v}_1)/\partial u_1. \tag{56}$$

(The expressions for the auxiliary functions are given, for reference, in the Appendix.)

Comparing (55) with (46) and (50), we see that the singularities are now somewhat different than in the more general ($G_0\neq 0$) case. The pole at $\zeta=0$ is no longer present, and we find that, aside from the velocity-dependent poles noted above, A, $\tilde{\alpha}$, and $\tilde{G}$ [as obtained from $\tilde{\alpha}$ and the Laplace transform of (22)] have poles at

$$\omega=0\quad\text{and}\quad \omega=\pm 2\omega_p - iap. \tag{57}$$

[7] N. Van Kampen, Physica **21**, 949 (1955).

The one at $\omega=0$ just corresponds to Lenard's asymptotic solution, while the other two, like the "universal" poles (52) and (53), are Landau damped with decay time $(ap)^{-1}$. For values of p small compared to ω_p/a, the damping will be negligible during the course of one plasma oscillation.

One can, of course, ask whether these weakly damped (i.e., small p) components of the two-particle function play any significant role in the physics. Insofar as the one-particle function, f, is concerned, the effects of G enter solely through the term $\delta f/\delta t$, defined by (14). For the homogeneous problem discussed here, where only $\mathbf{k}=0$ is involved, we have

$$(\delta f/\delta t)(\mathbf{v}_1,t)=(\omega_p{}^2/8\pi^3 i)\int d\mathbf{q}\, q^{-1}(\partial/\partial\mathbf{u}_1)\alpha(\mathbf{v}_1,\mathbf{q},t), \tag{58}$$

and the question is whether the small q region of the integration makes a significant contribution to $\delta f/\delta t$. As usual, no general statement, applicable to all initial conditions, can be made, but we can look a little further into the special case $G_0=0$. We first invert the Laplace transform to obtain $\alpha(\mathbf{v}_1,\mathbf{q},t)$ from $\tilde{\alpha}(\mathbf{v}_1,\mathbf{q},\omega)$. The singularities of (55) consist of simple poles at

$$\omega=0;\quad \omega=qu_1\pm\omega_p-iaq;\quad \omega=\pm 2\omega_p-2iaq \tag{59}$$

(where we have now included, as we must, the velocity-dependent poles). Evaluating the residues, we find

$$\alpha(\mathbf{v}_1,\mathbf{q},t)=\sum_{j=0}^{4}\alpha_j(\mathbf{v}_1,\mathbf{q},t), \tag{60}$$

where α_0 is independent of time (being just Lenard's asymptotic solution for our special case),

$$\begin{aligned}\alpha_0=(2\pi\omega_p{}^2/n)[q^2(u_1-ia)^2-\omega_p{}^2]^{-1} \\ \times\{f(\mathbf{v}_1)+(\partial f/\partial u_1)(a^2q^2+\omega_p{}^2)^{-1} \\ \times[(u_1-ia)a^2q^2+\tfrac{1}{2}\omega_p{}^2u_1]\},\end{aligned} \tag{61}$$

and the damped poles contribute

$$\begin{aligned}\alpha_{1,2}=\pm\pi\omega_p n^{-1}[q(u_1-ia)\pm\omega_p]^{-1}\{f(\mathbf{v}_1) \\ \mp(\partial f/\partial u_1)(u_1+ia)\omega_p[q(u_1+ia)\mp\omega_p]^{-1}\} \\ \times\exp[t(-qa-iqu_1\mp i\omega_p)],\end{aligned} \tag{62}$$

and

$$\begin{aligned}\alpha_{3,4}=\pm(\pi\omega_p{}^3/2nq)(iaq\pm\omega_p)^{-1}[q(ia+u_1)\pm\omega_p]^{-1} \\ \times(\partial f/\partial u_1)\exp[(-2aq\pm 2i\omega_p)t].\end{aligned} \tag{63}$$

When we substitute these into (58) to compute $\delta f/\delta t$, all of the integrals will be convergent save for that involving α_0, which is logarithmically divergent at large q, as noted by Lenard, who used a cutoff $q_{\max}\approx(na^2/\omega_p{}^2)$. The integrations can only be carried out numerically (save for the case of α_0, where Lenard was able to find a good approximation), but the precise values are not of great interest. More significant is the fact that α_3 and α_4, the residues at the velocity independent pole, vary as q^{-1} for small q. Since α is multi-

plied by q before the integration on q, one would *a priori* expect the small q (weakly damped) contributions to be suppressed. However, this $1/q$ behavior of α_3 and α_4 just cancels the phase space factor and results in their contributing to $\delta f/\delta t$ a term comparable, in general, with that resulting from α_0. It follows, then, that so far as the effect of G upon f is concerned, use of the asymptotic ($t=\infty$) form of G is justified only for situations where the rate of change of $\ln f$ is small compared to ω_p.

V. CONCLUSIONS

The equation, (7), for the two-particle function g is, in a general way, sufficiently similar to the correlationless one-particle equation (4) that one would, *a priori*, expect solutions of the former to exhibit a Landau damping similar to that familiar for the latter. The exact solution of (7) presented here confirms this expectation, at least for the particular case of spatially homogeneous one-particle functions F having a finite number of poles in the complex velocity plane. That similar conclusions apply to more general forms of the velocity dependence or to nonhomogeneous problems appears probable, although an explicit demonstration would be desirable.

Since the decay rate is of order qa (q= wave number, a= velocity spread in f), deviations between the true g and its asymptotic ($t=\infty$) form will be least damped, and therefore of most importance, at small q. Their effects will be greatest in the case of rapidly varying f, e.g., for plasma oscillations. The significant $\mathbf{q}$ values are then the ones below ω_p/a, corresponding to correlations between particles separated by a distance exceeding the Debye length. Although we have not succeeded in solving the equation for g when f is Gaussian, analogy with the analysis of the correlationless problem (4), as given for instance by Jackson,[6] suggests that the Landau damping for small q would there be much smaller than qa, approaching zero as $\exp[-(\omega_p/qa)^2]$ rather than algebraically. The difference between $g(t)$ and $g(t=\infty)$ for $qa\ll\omega_p$ would then persist longer than is indicated by the present analysis.

Of course, in a certain sense our analysis is inconsistent, for while we have assumed f to be time-independent in calculating g and $\delta f/\delta t$, we find that the departures from the asymptotic values would only be significant on a time scale of ω_p^{-1}, i.e., for a rapidly varying f. This simply emphasizes the necessity of solving the self-consistent problem, in which f and g are codetermined, at least in a linear approximation. Only in this way can one really answer the question of how correlation corrections to the Vlasov equation affect such phenomena as plasma oscillations. The importance of this question is underlined by the numerous recent suggestions that growing waves (due to two-stream phenomena or anisotropic distributions) could result in a quasi-dissipative behavior (as in electrical resistivity or shock formation) greatly in excess of that predicted by a simple picture of two-particle collisions. The failure, to date, to deduce such phenomena directly from the correlationless equation alone may indicate the necessity of providing, so to speak, a dissipative seed in the form of a small but nonzero $\delta f/\delta t$ term.

One must then decide what to use for that term; while the conventional Boltzmann or Fokker-Planck term is no doubt adequate for low-frequency phenomena, it is incorrect for treating plasma oscillations or high-frequency growing waves. There, a self-consistent treatment of (6) and (7) or of (18) and (19) is required. So far as the linear problem is concerned, it is (19) which presents the principal difficulty. The problem solved in this paper is of course a much simpler one, involving only the $\mathbf{k}=0$ form of the linear operator which appears in (19). Nonetheless, the properties, deduced here, of this restricted operator may be of help in the solution of the $\mathbf{k}\neq 0$ case.

APPENDIX

For reference purposes, we list here the functions τ, σ, ρ_1, ρ_2, I_1 and J corresponding to the choice (41) for the one-body function and the initial condition $G=0$ at $t=0$.

$$\tau(u_1,u)=(i\gamma^2/\omega n)[1-\gamma^2\bar{\Lambda}(u_1-u)]^{-1}\times[\mathfrak{F}(u_1)(u_1-ia-u)^{-1}+F(u_1)(u_1-ia-u)^{-2}],$$

$$\sigma(u_1,u)=(i\gamma^2/\omega n)[(u_1-u-ia)^2-\gamma^2]^{-1}\times[(u_1-ia-u)(\zeta^2+1)(\zeta-1)^{-2}\mathfrak{F}(u_1)-w\mathfrak{F}(u_1)(\zeta-1)^{-1}+F(u_1)],$$

$$\rho_1(u)=-(i/\omega nw)(3\zeta-1)(\zeta-1)^{-2},$$

$$\rho_2(u)=(2i/\omega nw^2)(2\zeta-1)(\zeta-1)^{-2},$$

$$I_1(u)=-3i/\omega nw(\zeta-4),$$

$$J(u)=2i(2\zeta+1)/\omega nw\zeta(\zeta-1)(\zeta-4).$$