Statistical Theory of Gamma-Ray Spectra Following Nuclear Reactions*

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A theory predicting γ-ray spectra following intermediate or high-energy nuclear reactions is derived on the basis of the statistical theory. The equations take a particularly simple form if the assumption is made that all radiative transitions are of the electric dipole type. The theory is applied to two specific reactions: inelastic neutron scattering and thermal neutron capture. Numerical calculations of spectra arising from thermal neutron capture by two gadolinium isotopes are shown to compare well with experiments.

I. INTRODUCTION

LET us consider a nuclear reaction leaving a residual nucleus with an excited state of energy E above the ground state. Let $R^0(E)dE$ be the probability that levels between E and $E+dE$ are excited. $R^0(E)$ can be considered as a population density. (See Fig. 1 which illustrates the case of a particular nuclear reaction: inelastic neutron scattering.) An excited state at energy E can be de-excited by emission of a gamma ray of energy E_γ or by some other process like particle emission. In the former case the residual nucleus is left at an energy $E'=E-E_\gamma$. Let $S(E,E')$ be the relative transition probability such that, given an excited state at energy E, $S(E,E')dE'$ is the probability of radiative transition to the excited states between E' and $E'+dE'$.

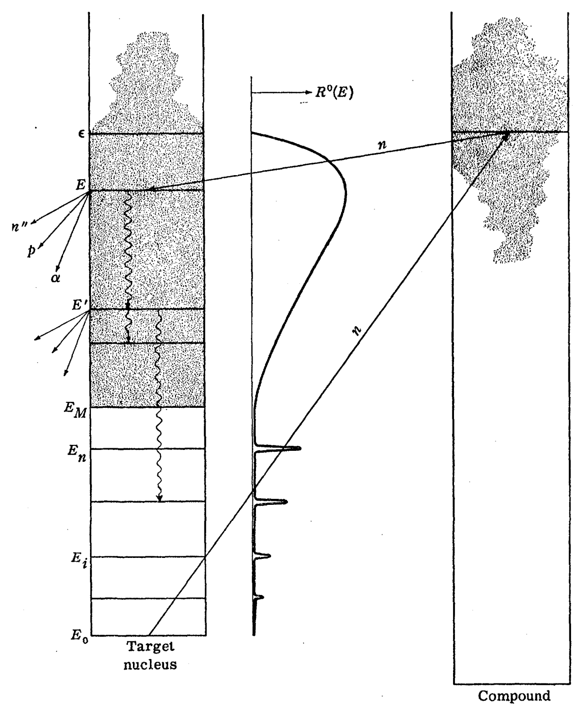

FIG. 1. Schematic energy diagram for inelastic neutron scattering followed by γ-ray emission.

$S(E,E')$ has the normalization $\int_0^E S(E,E')dE'=\Gamma_\gamma(E)/\Gamma(E)=\gamma(E)$, where $\Gamma_\gamma(E)$ and $\Gamma(E)$ are the average radiative and total widths at the energy E. The states at energy E' can be de-excited again by gamma-ray emission or by particle emission, the process ending when the ground state of the nucleus is reached. The complete de-excitation of the nucleus leads therefore to a number of cascade gamma rays. We are interested in the spectrum $P(E_\gamma)$ of all these gamma rays, such that $P(E_\gamma)dE_\gamma$ gives the number of gamma rays of energy between E_γ and $E_\gamma+dE_\gamma$.

Let us define a quantity $R^n(E)$ such that $R^n(E)dE$ gives the expected number of levels between E and $E+dE$ that are excited as the result of the nth gamma ray following the nuclear reaction. $R^n(E)$ satisfies the obvious recursion formula

$$R^n(E)=\int_E^\epsilon R^{n-1}(E')S(E',E)dE', \tag{1}$$

where the upper limit ϵ is such that $R^0(E)=0$ for $E>\epsilon$. Let us define the population density due to any number of transitions:

$$R(E)=\sum_{n=0}^{\infty} R^n(E).$$

Writing Eq. (1) for $n=1, 2 \cdots \infty$ and adding them up, we obtain

$$R(E)=R^0(E)+\int_E^\epsilon R(E')S(E',E)dE', \tag{2}$$

which is an integral equation for $R(E)$.

The spectrum $P(E_\gamma)$ which we seek can be obtained from the solutions of (2) through

$$P(E_\gamma)=\int_{E_\gamma}^\epsilon R(E)S(E, E-E_\gamma)dE. \tag{3}$$

The problem is formally solved.

It can be shown similarly that the spectrum $P(E_\gamma)$ can be obtained from the solution of:

$$P(E,E_\gamma)=S(E, E-E_\gamma)+\int_{E_\gamma}^E S(E,E')P(E',E_\gamma)dE', \tag{4}$$

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by evaluating the integral

$$P(E_\gamma)=\int_{E_\gamma}^{\epsilon} R^0(E)P(E,E_\gamma)dE. \qquad (5)$$

The Eqs. (2) and (4) are mutually adjoint.

It will follow from the assumption on the level density $\rho(E)$ that the use of the set of Eqs. (2) and (3) is much more convenient than that of the set (4) and (5).

II. GENERAL ASSUMPTIONS ABOUT THE LEVEL DENSITY

Let us now make some general assumptions on the level density of the nucleus considered. The level spectrum can be considered as composed of a discrete part, with levels at $E=E_i[i=0, 1, 2, \cdots n]$, and of a continuous part with a density $\rho(E)$ such that $\rho(E)dE$ gives the number of levels between E and $E+dE$; $\rho(E)=0$ for $E<E_M$, $E_M>E_n$.

Under these conditions the quantity $R^0(E)$ can be written as

$$R^0(E)=\mathcal{R}^0(E)+\sum_{i=0}^{n} r_i^0\delta(E-E_i), \qquad (6)$$

$\mathcal{R}^0(E)$ being the initial population density in the continuum and r_i^0 being the initial population of the discrete level i. The transition probability $S(E,E')$ can be broken up into three parts:

$$S(E,E')=S_1+S_2+S_3, \qquad (7)$$

where $S_1(E,E')$ is a continuous function for $E>E'>E_M$, and is 0 otherwise;

$$S_2(E,E')=\sum_{i=0}^{n} S^i(E)\delta(E'-E_i),$$

$S^i(E)$ being a continuous function for $E>E_M$, and being 0 otherwise;

$$S_3(E,E')=\sum_{j=0}^{n}\sum_{i=0}^{j-1} S^{ji}\delta_{E,E_j}\delta(E'-E_i),$$

where δ_{E,E_j} is the limit, when $\xi\to 0$, of a function which is equal to 1 when $E_j-\xi<E<E_j+\xi$, and equal to zero otherwise.

Substituting the above expression of $S(E,E')$ in the integral equation (2), we obtain

$$\begin{aligned}R(E)=\mathcal{R}^0(E)&+\sum_i r_i^0\delta(E-E_i)\\ &+\int_E^{\epsilon} R(E')S_1(E',E)dE'\\ &+\sum_{i=0}^{n}\delta(E-E_i)\int_E^{\epsilon} R(E')S^i(E')dE'\\ &+\sum_{j=0}^{n}\sum_{i=0}^{j-1} S^{ji}\delta(E-E_i)\int_E^{\epsilon} R(E')\delta_{E',E_j}dE'. \qquad (8)\end{aligned}$$

For $E>E_M$ most terms of Eq. (8) vanish because of the conditions (7). There remains

$$R(E)=\mathcal{R}^0(E)+\int_E^{\epsilon} R(E')S_1(E',E)dE', \quad (E>E_M), \quad (9)$$

which is an integral equation for $R(E)$, involving only continuous functions.

For $E<E_M$ the other terms contribute, but let us first transform the last term of (8). Using the definition of the function δ_{E',E_j}, it becomes

$$\sum_{j=0}^{n}\sum_{i=0}^{j-1} S^{ji}\delta(E-E_i)\left[\lim_{\xi\to 0}\int_{E_j-\xi}^{E_j+\xi} R(E')dE'\right].$$

Changing the order of summation, we obtain

$$\sum_{i=0}^{n}\delta(E-E_i)\sum_{j=i+1}^{n} S^{ji}\left[\lim_{\xi\to 0}\int_{E_j-\xi}^{E_j+\xi} R(E')dE'\right]. \qquad (10)$$

Let us write the nonvanishing terms of Eq. (8) for $E<E_M$, using the expression (10) for the last term:

$$\begin{aligned}R(E)=\sum_{i=0}^{n}\delta(E-E_i)\Bigg\{r_i^0&+\int_{E_M}^{\epsilon} R(E')S^i(E')dE'\\ &+\sum_{j=i+1}^{n} S^{ji}\left[\lim_{\xi\to 0}\int_{E_j-\xi}^{E_j+\xi} R(E')dE'\right]\Bigg\}, \quad (E<E_M). \quad (11)\end{aligned}$$

It can be shown rigorously that it follows from (11) that one can write $R(E)$ for $E<E_M$ in the form

$$R(E)=\sum_{i=0}^{n} r_i\delta(E-E_i), \quad (E<E_M),$$

r_i being the population of the discrete level i. Substituting this expression in (11), we obtain

$$r_i=r_i^0+\int_{E_M}^{\epsilon} R(E')S^i(E')dE'+\sum_{j=i+1}^{n} S^{ji}r_j, \qquad (i=0, 1, \cdots n). \quad (12)$$

Writing

$$R(E)=\mathcal{R}(E)+\sum_{i=0}^{n} r_i\delta(E-E_i), \quad (\text{any } E), \quad (13)$$

the solutions (9) and (11) become

$$\mathcal{R}(E)=\mathcal{R}^0(E)+\int_E^{\epsilon}\mathcal{R}(E')S_1(E',E)dE', \qquad (14a)$$

$$r_i=r_i^0+\int_{E_M}^{\epsilon}\mathcal{R}(E')S^i(E')dE'+\sum_{j=i+1}^{n} S^{ji}r_j, \qquad (i=0, 1, \cdots n). \quad (14b)$$

Taking into account (7) and (13), the expression for

the spectrum $P(E_\gamma)$ becomes

$$P(E_\gamma)=\int_{E_\gamma}^{\epsilon}[\mathcal{R}(E)+\sum_k r_k\delta(E-E_k)]$$
$$\times[S_1(E,E-E_\gamma)+\sum_i S^i(E)\delta(E-E_\gamma-E_i)$$
$$+\sum_j\sum_i S^{ji}\delta(E-E_j)\delta(E-E_\gamma-E_i)]dE.$$

Performing the integrations involving δ functions, the expression becomes

$$P(E_\gamma)=\int_{E_\gamma}^{\epsilon}\mathcal{R}(E)S_1(E,E-E_\gamma)dE$$
$$+\sum_{i=1}^{n}\mathcal{R}(E_\gamma+E_i)S^i(E_\gamma+E_i)$$
$$+\sum_{j=0}^{n}\sum_{i=0}^{j-1}r_jS^{ji}\delta[E_\gamma-(E_j-E_i)]. \quad (15)$$

Once the integral equation (14a) is solved, the system of linear equations (14b) can also be solved and the solutions can be used for the calculation of the spectrum by Eq. (15).

It is easy to recognize that the first term of Eq. (15) corresponds to transitions between states in the continuum, the second term to transitions between the continuum and discrete state, and the last term to transitions between discrete states.

III. ASSUMPTION ABOUT THE TRANSITION PROBABILITIES

We are now going to make the simple assumption that all the transitions between levels in the continuum are of the electric dipole type, that is,

$$S_1(E,E')=f(E)(E-E')^3\rho(E'), \quad (16)$$

$f(E)$ assuring a proper normalization of $S(E,E')$.

We have assumed that the transition probability depends only on the energies of the initial and final states and that the average matrix element is constant. The form (16) is to be understood as a relative transition probability, averaged over many initial and many final states. At high excitation energies there is a great number of levels of all spins and parities, and one can indeed assume that the predominant transitions are of the $E1$ type. In that region the assumption (16) is probably reasonable, unless $R^0(E)$ includes only one, or only a few levels of definite spins and parities (as, for instance, in the case of thermal neutron capture). For lower excitation energies, the validity of the above assumption becomes questionable.

If one assumes the form (16) for all energies E, $E_M<E<\epsilon$, the integral equation (14a) can readily be reduced to a fourth-order differential equation.

Making the substitution

$$\psi(E)=[\mathcal{R}(E)-\mathcal{R}^0(E)]/\rho(E),$$

one obtains

$$d^4\psi(E)/dE^4=6f(E)\rho(E)\psi(E)+6f(E)\mathcal{R}^0(E). \quad (17)$$

It is clear that Eq. (14a) can be reduced to a differential equation of order $2(L+1)$ if the more general form

$$S_1(E,E')=\sum_{l=1}^{L}f_l(E)(E-E')^{2l+1}\rho(E') \quad (18)$$

is assumed. We are, however, restricting our calculations to $L=1$.

A few words remain to be said about the transition probabilities S_2 and S_3. In the numerical work reported below, we have assumed

$$S^i(E)=f(E)(E-E_i)^3. \quad (19)$$

This assumption is probably the weakest point of this calculation.

For $E>E_M$, the normalization of $S(E,E')$ yields

$$f(E)=\left[\sum_{i=0}^{n}(E-E_i)^3+\int_{E_M}^{E}(E-E')^3\rho(E')dE'\right]^{-1}\gamma(E). \quad (20)$$

No general assumptions have been made on the transition probabilities S^{ji} between discrete levels. These can usually be derived for each particular case, either from experimental data or, when the spins and parities of the levels are known, from theoretical considerations.

IV. APPLICATION TO INELASTIC NEUTRON SCATTERING

The application of the theory outlined above to specific nuclear reactions requires some specific assumptions on $R^0(E)$. In the case of inelastic scattering, the prediction of the statistical theory[1] gives

$$\mathcal{R}^0(E)=\sigma_c(\epsilon-E)(\epsilon-E)\rho(E)\bigg/\left[\sum_{i=0}^{n}\sigma_c(\epsilon-E_i)(\epsilon-E_i)+\int_{E_M}^{\epsilon}\sigma_c(\epsilon-E')(\epsilon-E')\rho(E')dE'\right], \quad (21)$$

where ϵ has been identified with the incident neutron energy. $\sigma_c(E)$ is the cross section for compound nucleus formation for neutrons of energy E.

V. APPLICATION TO THERMAL NEUTRON CAPTURE

In the case of thermal capture, the form assumed for $R^0(E)$ is

$$R^0(E)=\delta(E-\epsilon), \quad (22)$$

where ϵ is now the neutron binding energy in the residual nucleus.

It can be shown that in this case $R(E)$ can be written in the form

$$R(E)=\delta(E-\epsilon)+R'(E), \quad (23)$$

[1] B. T. Feld *et al.*, Final Report of the Fast Neutron Data Project, NYO-636, 1951 (unpublished).

where $R'(E)$ has no singularity at $E=\epsilon$. It follows from Eq. (2) that $R'(E)$ satisfies the equation:

$$R'(E)=S(\epsilon,E)+\int_{E_M}^{\epsilon} R'(E')S(E',E)dE', \qquad (24)$$

which is an equation similar to that for $R(E)$, with

$$R^{0\prime}(E)=S(\epsilon,E). \qquad (25)$$

It was mentioned in Sec. III that the assumption (16) is usually not valid when $R^0(E)$ does not include a large number of states of all spins and parities. In the case of thermal capture the initial states are of definite parity, and have spins which take only one or two values. If more accurate knowledge is available on the first γ-ray transitions $S^0(\epsilon,\,\epsilon-E_\gamma)$, the results can be improved by taking proper account of it, and by replacing Eq. (24) by

$$R'(E)=S^0(\epsilon,E)+\int_{E_M}^{\epsilon} R'(E')S(E',E)dE'. \qquad (26)$$

We would also like to mention that, in the case of capture, one can show that

$$\int_0^{\epsilon} E_\gamma P(E_\gamma)dE_\gamma=\epsilon \qquad (27)$$

for any $S(E,E')$, provided that

$$\int_0^{E} S(E,E')dE'=1.$$

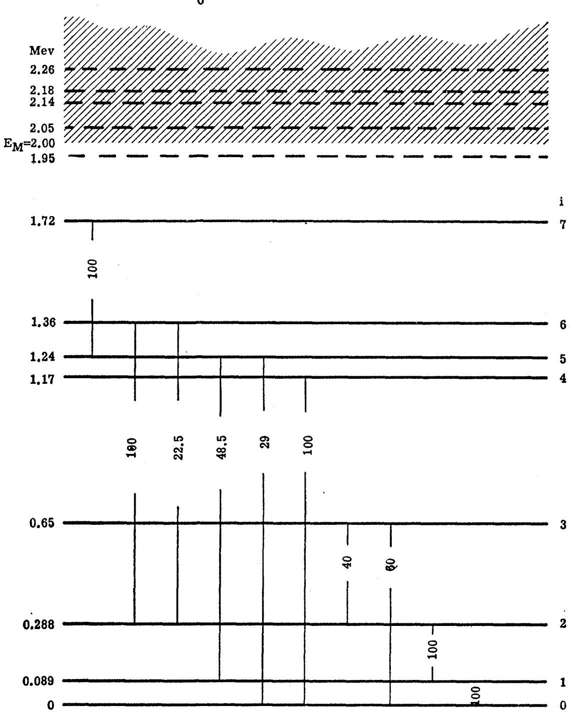

FIG. 2. The scheme of low-lying levels of Gd^{156} (reference 2). Energies are given in Mev. The assumed branching ratios are given in percent.

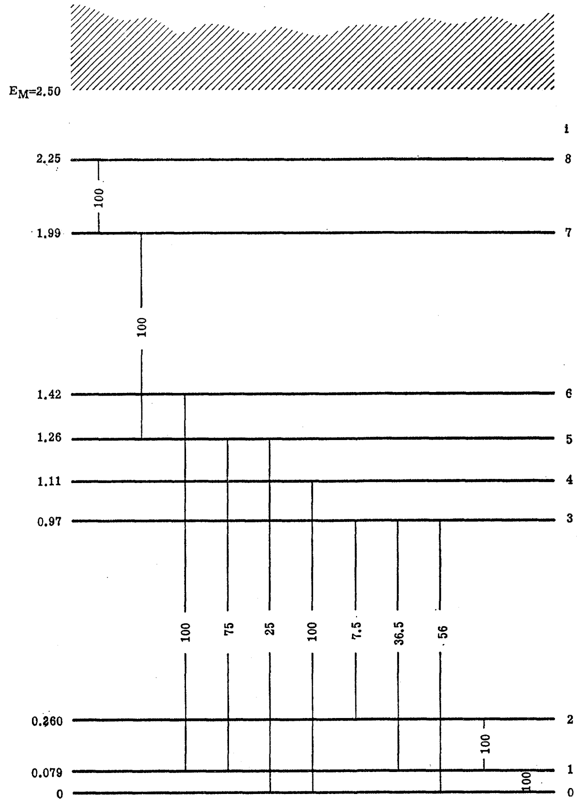

FIG. 3. The scheme of low-lying levels of Gd^{158} (reference 2). Energies are given in Mev. The assumed branching ratios are given in percent.

VI. NUMERICAL CALCULATIONS, RESULTS, AND CONCLUSIONS

It turns out that in both particular cases considered above, the differential equation (17) can be transformed to a homogeneous one [we assume $\sigma_c(E)=\text{const}$ in the case of inelastic scattering, and are solving Eq. (24) in the case of thermal capture].

Letting

$$X(E)=\mathcal{R}(E)/\rho(E),$$

we obtain

$$d^4X(E)/dE^4=6f(E)\rho(E)X(E). \qquad (28)$$

The boundary conditions are

$$\begin{aligned} &X(\epsilon)=0,\\ &X'(\epsilon)=-1\Big/\left[\sum_{i=0}^{n}(\epsilon-E_i)+\int_{E_M}^{\epsilon}(\epsilon-E')\rho(E')dE'\right], \qquad (29)\\ &X''(\epsilon)=0,\\ &X'''(\epsilon)=0,\end{aligned}$$

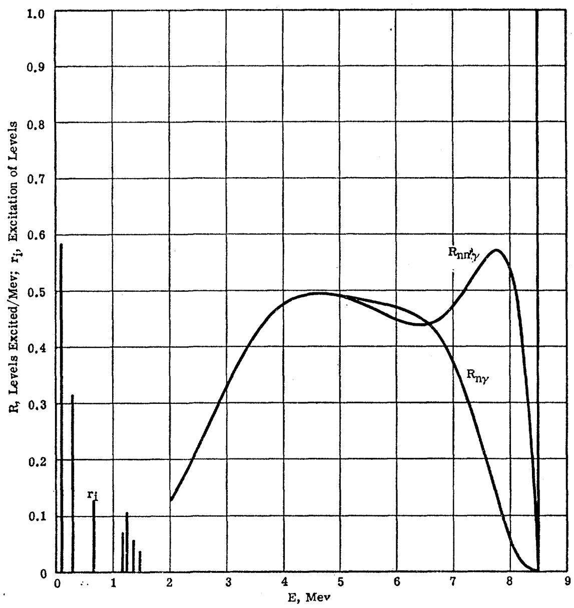

FIG. 4. The population of levels in Gd^{156}: $\mathcal{R}_{nn'\gamma}$—population density in the continuum, following inelastic neutron scattering at 8.5 Mev; $\mathcal{R}_{n\gamma}$—population density in the continuum, following thermal neutron capture by Gd^{155}; r_i—population of the discrete level at energy E, following either 8.5-Mev inelastic scattering on Gd^{156}, or thermal capture on Gd^{155} (the two sets of r_i coincide).

for inelastic scattering, and:

$$X(\epsilon)=0,$$

$$X'(\epsilon)=0,$$

$$X''(\epsilon)=0,$$

$$X'''(\epsilon)=-6\Big/\left[\sum_{i=0}^{n}(\epsilon-E_i)^3+\int_{E_M}^{\epsilon}(\epsilon-E')^3\rho(E')dE'\right],$$

for capture.

A code has been prepared for the NDA Datatron for the solution of Eq. (28) and the integrations involved in (14b) and (15). The level density law assumed is

$$\rho(E)=B\exp[2(aE)^{\frac{1}{2}}]\quad\text{for}\quad E>E_M. \tag{30}$$

Calculations have been performed on two gadolinium isotopes. The low-energy levels,[2] values of E_M, and assumed branching ratios are shown in Figs. 2 and 3. The parameter a was taken following Igo[3] ($a=0.1$ A) as 16 for both isotopes. The parameter B was obtained by normalizing the level density (30) to the observed[2] one in Gd^{156} around 2 Mev: $B\approx6\times10^{-5}$ for both isotopes. It should be noted that the level densities observed just above the neutron binding energy could not be used directly, as only 1^- and 2^- levels are detected by $l=0$ neutrons on $J=\frac{3}{2}^-$ targets.

The variation of $R(E)$ in Gd^{156} is shown on Fig. 4 for two different cases: inelastic neutron scattering at $\epsilon=8.5$ Mev and thermal neutron capture in Gd^{155} ($\epsilon=8.5$ Mev). It is seen that at high energies $R(E)\approx R^0(E)$. At low energies ($E<5$ Mev) both solutions coincide and are therefore independent (except for normalization) on the boundary conditions at ϵ. It can be shown in general that Eq. (28) has four independent solutions: two oscillatory solutions, a solution $X(E)=C/f(E)$, all three being regular as $E\rightarrow0$, and a solution which is irregular as $E\rightarrow0$. This is this last solution which governs the behavior of $R(E)$ at low E.

The spectra $P(E_\gamma)$ obtained are shown in Figs. 5 and 6. A capture spectrum calculated for Gd^{158} ($\epsilon=7.9$ Mev) is given in Fig. 7. In the case of inelastic scattering (Fig. 5), the different terms of Eq. (15) have been plotted separately: P_c corresponds to transitions between the levels in the continuum and P_d corresponds to transitions between levels in the continuum and discrete levels. The discontinuous form assumed for $\rho(E)$ (a continuum starting abruptly at $E=E_M$) gives rise to a discontinuous shape of the P_d spectrum. The sharp oscillations in the low-energy end of the composite spectrum P should not therefore be taken seriously.

The discrete lines obtained [last term in Eq. (15)] are given in Tables I and II.

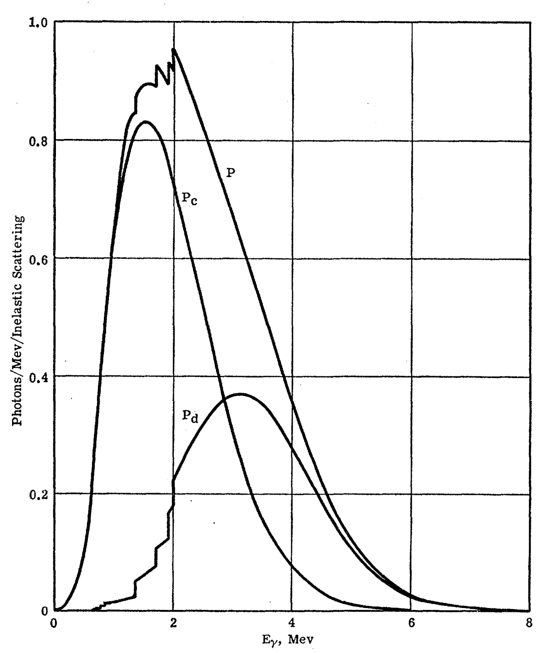

FIG. 5. The spectrum of γ rays following inelastic scattering of 8.5-Mev neutrons on Gd^{156}. P_c—part of the spectrum due to transitions between levels in the continuum; P_d—part of the spectrum due to transitions from levels in the continuum to discrete levels. $P=P_c+P_d$. Additional low-energy discrete lines are given in Table I.

[2] L. V. Groshev, A. M. Demidov, V. N. Lutsenko, and V. I. Pelekhov, *Atlas of Thermal Neutron Capture γ-Ray Spectra* (Atomizdat, Moscow, 1958), translated by J. B. Sykes, (Pergamon Press, London, 1959); and Atomnaya Energ. **4**, 5 (1958) [translation: Soviet J. Atomic Energy 4, 1 (1958)].

[3] G. Igo and H. E. Wegner, Phys. Rev. **102**, 1364 (1958).

TABLE I. Calculated and observed low-energy discrete lines following thermal neutron capture in Gd^{155}. The calculated intensities of low-energy γ ray following inelastic scattering of 8.5-Mev neutrons on Gd^{156} coincide with those given in Column 2.

	I_γ, photons/100 captures		
E_γ, Mev	Calc	Obs	References
0.089	58.3	50	4
0.199	31.4	34	4
0.36	5.1	...	
0.48	3.6	...	
0.65	7.7	8[a]	2
0.96	2.3	7	2
1.06	5.7	6	2
1.17	12.2	15	2
1.24	3.0	9	2

[a] A 0.64-Mev line was observed for natural Gd; $I_\gamma = 2$.

In the case of capture, comparison with experiment is possible. The data[2,4] are given together with the calculated spectra in Figs. 6 and 7 and in Tables I and II. It should be understood that the comparison is absolute: there is no normalization involved.

The difference in the spectra of the two isotopes (up to $E_\gamma \sim 4$ Mev) is mainly accounted for by the difference in the scheme of the low-lying levels. A correct inclusion of these levels seems therefore imperative for a good prediction of the spectrum. At higher energies,

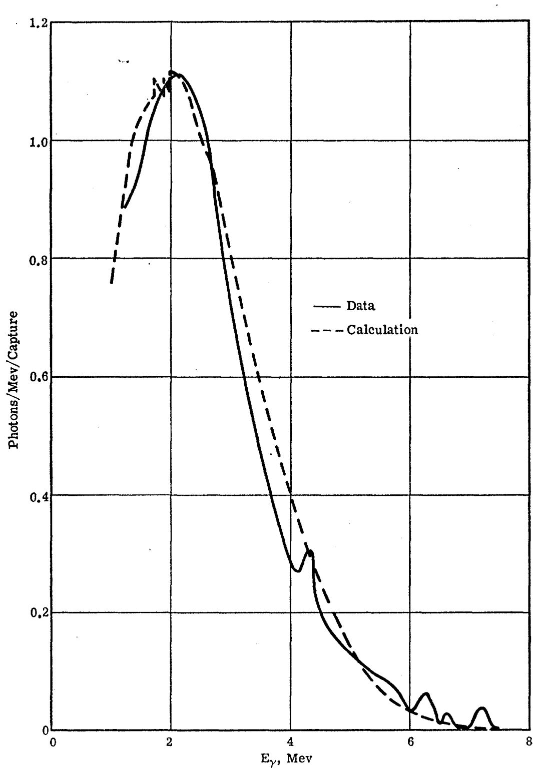

FIG. 6. The spectrum of γ rays following thermal neutron capture in Gd^{155}. The data are from reference 2. Additional low-energy discrete lines are given in Table I.

[4] V. V. Sklyarevskiy, E. P. Stepanov, and B. A. Obinyakov, Atomnaya Energ. 4, 22 (1958) [translation: Soviet J. Atomic Energy 4, 19 (1958)].

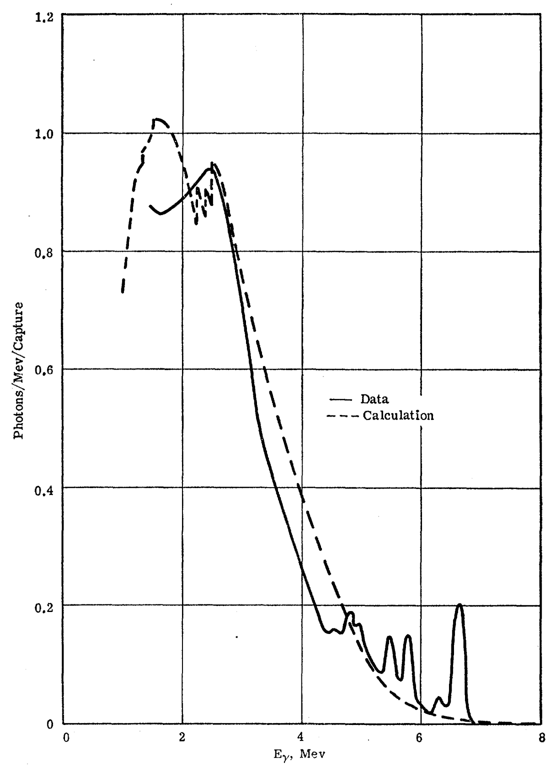

FIG. 7. The spectrum of γ rays following thermal neutron capture in Gd^{157}. The data are from reference 2. Additional low-energy discrete lines are given in Table II.

TABLE II. Calculated and observed low-energy discrete lines following thermal neutron capture in Gd^{157}.

	I_γ, photons/100 captures		
E_γ, Mev	Calc	Obs	References
0.079	59.7	53	4
0.182	19.7	29	4
0.26	1.8	...	
0.69	0.7	1.7[a]	2
0.73	4.5	...	
0.90	3.5	8	2
0.96	5.4	13	2
1.11	8.2	8	2
1.185	8.6	9	2
1.26	2.9	3	2
1.33	5.7	2	2

[a] A 0.69-Mev line was observed for natural Gd; $I_\gamma = 1$.

the difference between the binding energies is also of some importance. Some high-energy transitions are observed experimentally. We could have, but did not attempt to account for them by using a proper $S^0(\epsilon,E)$ in Eq. (26). Equation (24) was used as is.

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