PHYSICAL REVIEW VOLUME 122, NUMBER 1 APRIL 1, 1961

Some Considerations Concerning the Nuclear Matrix Element $\sum |B_{ij}|^2$ in Beta Decay*

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(Received November 29, 1960)

Evidence that the nuclear matrix element $\Sigma|B_{ij}|^2$ contributes to first-forbidden transitions may be deduced from the electron-capture–positron-emission ratios observed in these transitions and from an analysis of the energy and Z dependence of the coefficients of all the nuclear matrix elements involved. The analysis of these coefficients for the capture and for the emission processes shows that it is reasonable to expect that, except for $\Sigma|B_{ij}|^2$, these transitions would be characterized by essentially the same capture–positron ratios as those characteristic of allowed transitions. Consequently the deviations of the observed ratios from the allowed values would be due to contributions from $\Sigma|B_{ij}|^2$; and the magnitude of the deviation, in a particular case, together with lifetime information may be used to compute values for this matrix element and for the sum of the others. Values are thus computed from available experimental data for three $2- \rightarrow 2+$ transitions, and these are compared with values derived directly from theory of Rose and Osborne. In each case the value of $\Sigma|B_{ij}|^2$ computed from experiment is nearly as large as the theoretical $\Sigma|B_{ij}|^2$ value calculated for nucleon spin change two and is considerably larger than those calculated for spin change zero or one; furthermore, the "observed" value for the sum of the other matrix elements is considerably smaller than the largest single matrix element calculated for spin change zero or one. These results are consistent with the conclusion that the nuclear states involved are of such a nature that the transitions proceed unhindered only via $\Sigma|B_{ij}|^2\Delta j=2$. One arrives at the same conclusion on the basis of the shell model description of these nuclear states. For $2- \rightarrow 0+$ transitions, which can be effected only by $\Sigma|B_{ij}|^2$, the "observed" and theoretical values of the matrix element are in good agreement; and it is of interest to note that both the theoretical and "observed" values of $\Sigma|B_{ij}|^2$ in each of these $2- \rightarrow 0+$ transitions are about the same or smaller than the values for the $2- \rightarrow 2+$ transition in the same nucleus.

THE determination of the relative magnitudes of the nuclear matrix elements which contribute to first-forbidden beta decay is based upon the analysis of the observed variation of the decay probability with energy, atomic number, and spatial correlation in terms of the energy, Z, and correlation dependent coefficients of the individual matrix elements. With respect to variation with energy, except for the unique ($\Delta J=2$, yes) transitions, the observed shapes deviate so little from the distributions characteristic of allowed transitions that the analysis is, in most cases, impractical. The probability of K capture relative to that of positron emission, however, is more sensitive to the energy-dependent coefficients of the matrix elements than is the spectral shape. In the capture process, neutrinos are emitted of energy greater by two electron masses than those of highest energy associated with positron decay. Deviations of the decay probability from the allowed energy dependence arise from terms in the electron energy and the neutrino energy; and the deviations are, in general, greatest at the high and low ends of the spectrum, where there are the fewest events (i.e., due to the multiplicative terms in p and q, electron and neutrino momenta). Decay by electron capture produces neutrinos of the largest energy possible for a given transition; and the capture process may be looked upon as a negative energy part of the positron emission, where comparatively large deviations from the allowed energy dependence are found.[1,2]

Specifically, the probability of positron emission for the interaction combination $V-A$ in first-forbidden beta decay may be written[2]

$$
\begin{aligned}
&P_+(G_V,G_V',G_A,G_A')\\
&=P_+(\text{permitted})\Big\{(|G_V|^2+|G_V'|^2)\Big[\Big|\int\mathbf{r}\Big|^2\zeta_1{}^A(V)+\Big|\int\boldsymbol{\alpha}\Big|^2\zeta_1{}^B(T)+i\Big(\Big(\int\boldsymbol{\alpha}\Big)\cdot\Big(\int\mathbf{r}\Big)^*-\text{c.c.}\Big)\zeta_1{}^D(T)\Big]\\
&\quad+(|G_A|^2+|G_A'|^2)\Big[\Big|\int\boldsymbol{\sigma}\cdot\mathbf{r}\Big|^2\zeta_1{}^A(A)+\Big|\int\gamma_5\Big|^2\zeta_1{}^B(T)-i\Big(\Big(\int\boldsymbol{\sigma}\cdot\mathbf{r}\Big)\Big(\int\gamma_5\Big)^*-\text{c.c.}\Big)\zeta_1{}^D(T)\\
&\quad+\Big|\int\boldsymbol{\sigma}\times\mathbf{r}\Big|^2\zeta_1{}^D(A)+\sum|B_{ij}|^2\zeta_1{}^E(T)\Big]+(G_VG_A{}^*+G_V'G_A'{}^*)\Big[-i\Big(\Big(\int\boldsymbol{\sigma}\times\mathbf{r}\Big)\cdot\Big(\int r\Big)^*-\text{c.c.}\Big)\zeta_1{}^A(S,T)\\
&\qquad-\Big(\Big(\int\boldsymbol{\sigma}\times\mathbf{r}\Big)\cdot\Big(\int\boldsymbol{\alpha}\Big)^*+\text{c.c.}\Big)\zeta_1{}^B(S,T)\Big]\Big\}. \qquad (1)
\end{aligned}
$$

* Research performed in part under the auspices of the U. S. Atomic Energy Commission.
† Research performed in part as Guest Scientist of the Chemistry Department, Brookhaven National Laboratory.
[1] D. S. Harmer and M. L. Perlman, Phys. Rev. **114**, 1133 (1959); D. S. Harmer and M. L. Perlman, Bull. Am. Phys. Soc. **4**, 229 (1959).
[2] M. L. Perlman, J. P. Welker, and M. Wolfsberg, Phys. Rev. **110**, 381 (1958).

In Eq. (1), $P_+(\text{permitted})=(1/2\pi^3)\int_1^{W_0} pW(W_0-W)^2F_0(Z,W)L_0dW$. The ζ_1 factors, which are derived from the coefficients of the individual matrix elements as given in the C_{1V} and C_{1A} factors of Konopinski and Uhlenbeck,[3] represent the deviation factors of the individual coefficients from the allowed factor, L_0. For example, the coefficient of $|\int \mathbf{r}|^2$, $\zeta_1{}^A(V)$, is, to better than 5% accuracy,

$$\frac{\displaystyle\int_1^{W_0} [\tfrac{1}{3}(W_0-W)^2L_0+2L_1+M_0-\tfrac{2}{3}(W_0-W)N_0]pW(W_0-W)^2F_0(Z,W)dW}{\displaystyle\int_1^{W_0} pW(W_0-W)^2F_0(Z,W)L_0dW}.$$

Similarly, the probability of K capture may be written

$$P_K(G_V,G_V',G_A,G_A')$$
$$=P_K(\text{permitted})$$
$$\times\left\{(|G_V|^2+|G_V'|^2)\left[\left|\int \mathbf{r}\right|^2\zeta_{1K}{}^A(V)+\cdots\right]\right\}, \quad (2)$$

where the ζ_{1K} factors are analogous to those for positron emission, and $P_K(\text{permitted})=(1/4\pi^2)g_K{}^2(W_0+\epsilon_K)^2=(1/4\pi^2)f_{0K}$. Simplified integrated expressions for the ζ_1 factors for both K capture and positron emission have been given in reference 2. The ζ_1 factors may be classified into three groups according to the forms of their energy dependence:

(1) $\zeta_1=a$, for $\left|\int\gamma_5\right|^2$, $\left|\int\boldsymbol{\alpha}\right|^2$, $[a=1]$,

(2) $\zeta_1=a'+f(W_0)$, for $\left|\int\mathbf{r}\right|^2$, $\left|\int\boldsymbol{\sigma}\cdot\mathbf{r}\right|^2$,

$\left|\int\boldsymbol{\sigma}\times\mathbf{r}\right|^2$, $[a'=f(Z)]$,

(3) $\zeta_1=f'(W_0)$, for $\sum|B_{ij}|^2$.

One may then separate the expressions for P_K and P_+ into energy-dependent and energy-independent terms; thus from Eq. (2)

$$P_K=P_K(\text{permitted})$$
$$\times[\sum G_i{}^2|M_i|^2a_i+\sum G_i{}^2|M_i|^2f_{iK}(W_0)$$
$$+G_A{}^2\sum|B_{ij}|^2f_K'(W_0)], \quad (3)$$

or

$$=P_K(\text{permitted})$$
$$\times\sum G_i{}^2|M_i|^2a_i\left[1+\delta_K+\frac{\sum|B_{ij}|^2f_K'(W_0)}{\sum G_i{}^2|M_i|^2a_i}\right], \quad (4)$$

where $G_i{}^2$ is intended to designate $(G_xG_y{}^*)+(G_x'G_y'{}^*)$ and $|M_i|^2$ to designate the appropriate matrix elements and cross-terms, and

$$\delta_K=\sum G_i{}^2|M_i|^2f_{iK}(W_0)/\sum G_i{}^2|M_i|^2a_i.$$

[3] E. J. Konopinski and G. B. Uhlenbeck, Phys. Rev. **60**, 308 (1941); E. Greuling, Phys. Rev. **61**, 568 (1942); A. M. Smith, Phys. Rev. **32**, 955 (1951); D. Pursey, Phil. Mag. **42**, 1193 (1951).

For P_+ one may obtain an expression identical with Eq. (4) except that $f_K'(W_0)$ and δ_K are replaced by $f_+'(W_0)$ and δ_+, respectively. The matrix element $\sum|B_{ij}|^2$ is treated separately because of the strong energy dependence (Group 3, above) and Z independence of its coefficient in contrast to the coefficients of the other matrix elements, which have relatively large energy-independent lead terms. For those matrix elements other than $\sum|B_{ij}|^2$, the energy-independent lead terms are the same for both K capture and positron emission. Thus, the ratio may be expressed as

$$\left(\frac{P_K}{P_{\beta^+}}\right)=\left(\frac{P_K}{P_{\beta^+}}\right)_{\text{allowed}}\left[\frac{1+\delta_K+yf_K'(W_0)}{1+\delta_++yf_+'(W_0)}\right], \quad (5)$$

where $y=G_A{}^2\sum|B_{ij}|^2/\sum G_i{}^2|M_i|^2a_i$, and δ_K and δ_+ are defined above. If a single matrix element in addition to $\sum|B_{ij}|^2$ should be responsible for a transition, then both δ_K and δ_+ would reduce to the ratio of the energy-dependent term, $f(W_0)$, to the energy-independent, a_i, term of the coefficient of that matrix element; i.e.,

$$\delta_{iK}=G_i{}^2f_{iK}(W_0)|M_i|^2/G_i{}^2|M_i|^2a_i=f_{iK}(W_0)/a_i, \quad (6)$$

and the relative contributions, y, of $\sum|B_{ij}|^2$ and the single matrix element could be evaluated from the observed K/β^+ ratio using Eq. (5) and the known energy dependence[2] of the other matrix element. The magnitudes of the two matrix elements could then be determined from the comparative half-life of the transition, using Eq. (4) and the quantities defined above. Thus

$$P_K=(\ln 2)/t_K=P_K(\text{permitted})$$
$$\times G_i{}^2|M_i|^2a_i[1+\delta_{iK}+yf_K'(W_0)], \quad (7)$$

or

$$G_i{}^2|M_i|^2a_i=(4\pi^2\ln 2)$$
$$\times\{f_{0K}t_K[1+f_{iK}(W_0)/a_i+yf_K'(W_0)]\}^{-1}, \quad (8)$$

and

$$G_A{}^2\sum|B_{ij}|^2=yG_i{}^2|M_i|^2a_i. \quad (9)$$

In general, however, several matrix elements contribute to the transition, and a unique solution is not possible. However, one may still set limits to the contributions of certain matrix elements from the observed K/β^+ ratios, since for the matrix elements $|\int\boldsymbol{\alpha}|^2$ and $|\int\gamma_5|^2$, $\delta_K=\delta_+=0$; and for $|\int\mathbf{r}|^2$, $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$,

TABLE I. K-capture–positron ratios, $\zeta_{1K}{}^{i}/\zeta_{1+}{}^{i}$, for the various matrix elements relative to allowed ratios.[a]

Matrix element \ A / Z / W_0	74 / 33 / 1.90	74 / 33 / 2.80[b]	74 / 33 / 5.00	84 / 37 / 2.526[b]	126 / 53 / 1.90[b]	126 / 53 / 5.0
$\sum\|B_{ij}\|^2$	6.44	4.29	3.00	4.72	6.73	2.96
$\|\int\boldsymbol{\alpha}\|^2$	1.00	1.00	1.00	1.00	1.00	1.00
$\|\int\gamma_5\|^2$	1.00	1.00	1.00	1.00	1.00	1.00
$\|\int\mathbf{r}\|^2$	0.937	0.965	1.05	0.960	0.970	1.033
$\|\int\boldsymbol{\sigma}\cdot\mathbf{r}\|^2$	0.907	0.915	0.920	0.923	0.954	0.967
$\|\int\boldsymbol{\sigma}\times\mathbf{r}\|^2$	1.38	1.49	1.76	1.43	1.30	1.57

[a] From reference 2.
[b] These W_0 values correspond to the energies of the $(2-\rightarrow 2+)$ transitions in As^{74}, Rb^{84}, and I^{126}.

$|\int\boldsymbol{\sigma}\cdot\mathbf{r}|^2$, $\delta_i=f_i(W_0)/(\alpha Z/2R)^2\sim W_0/(\alpha Z/2R)$. For large values of Z and small values of W_0, δ_K and δ_+ become negligible relative to one, and the K/β^+ ratios for matrix elements other than $\sum|B_{ij}|^2$ correspond to those for allowed transitions. This corresponds to the usual $(\alpha Z/2R)\gg 1$ approximation.[4] For somewhat larger values of W_0 and smaller values of Z, the approximation $(\alpha Z/2R)\gg 1$ breaks down, but δ may still be small compared to one. The energy-independent terms (a_i) in the coefficients of all matrix elements add to the magnitude of the denominator of δ (except for a probably small negative contribution from cross-terms). In the numerator of δ appear the energy-dependent terms, $f_i(W_0)$, which are present in the coefficients of only some of the matrix elements. These terms are generally smaller than the terms in the denominator and furthermore tend to cancel among themselves. It may be pointed out that the "modified $\sum|B_{ij}|^2$ approximation,"[5] according to which only the matrix elements $|\int\boldsymbol{\alpha}|^2$, $|\int\gamma_5|^2$, and $\sum|B_{ij}|^2$ contribute to the transition, would of course make δ identically zero.

In Table I are shown the deviations from allowed values of the K/β^+ ratios which obtain for the various individual matrix elements.[6] It can be seen that for moderate transition energies only the matrix elements $\sum|B_{ij}|^2$ and $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ give rise to positive deviations from allowed ratios, and that $|\int\boldsymbol{\sigma}\cdot\mathbf{r}|^2$ and $|\int\mathbf{r}|^2$ give rise to negative deviations. The matrix elements $|\int\boldsymbol{\alpha}|^2$ and $|\int\gamma_5|^2$ would give rise to allowed K/β^+ ratios. Hence large positive or negative deviations would indicate the predominance of certain matrix elements.

In the case of first-forbidden unique transitions ($\Delta J=2$, yes), only the matrix element $\sum|B_{ij}|^2$ can contribute; and its magnitude therefore is simply determinable from the comparative half-life. The ratio of the coefficients of this matrix element, $(W_0+1)^2/2(W_0^2-1)$, is strongly energy dependent, and the K/β^+ ratios are expected to be much larger than the allowed values. This has been verified experimentally.[2,7] The spectra show a characteristic deviation from the allowed form.[8]

In nonunique transitions ($\Delta J=0$, 1, yes), several matrix elements in addition to $\sum|B_{ij}|^2$ may contribute, and evaluation of their magnitudes is difficult. The spectral shapes of these transitions are observed to be very similar to the allowed shape, and little information can be obtained about the relative importances of the various contributions. On the other hand, the K-capture positron ratio can lead to some additional information due to the larger deviation from the allowed behavior of the K-capture probability for a given transition. Experimental data are available for certain nonunique transitions in which one may expect $\sum|B_{ij}|^2$ to make substantial contributions. The K/β^+ ratios have been measured for the beta decays of the odd-odd nuclides As^{74}, Rb^{84}, and I^{126} to the first-excited states of their respective even-even daughters. These three nuclides are all characterized by spin 2 and odd parity, and the transitions in each case are therefore $2-\rightarrow 0+$ (ground state) and $2-\rightarrow 2+$ (excited state). Shell-model considerations make it appear that in the ground-state transition the decaying nucleon undergoes a spin change of two units. If, in the excited state of the daughter nucleus, the individual nucleons have the same j values as in the ground state, then in the $2-\rightarrow 2+$ transition also, Δj would be 2; and only the matrix element $\sum|B_{ij}|^2$ would be operative. This, of course, would be the extreme single-particle shell-model result.

The observed values of the K/β^+ ratios for these

TABLE II. Observed and calculated allowed K/β^+ ratios for $2-\rightarrow 2+$ transitions.

Nuclide	W_0/mc^2	P_K/P_{β^+} observed	P_K/P_{β^+} calc. allowed[a]	$(P_K/P_{\beta^+})_{\text{obs}}/(P_K/P_{\beta^+})_{\text{allowed}}$
I^{126}	1.90 ± 0.01	145 ± 4[b]	131 ± 6	1.11
Rb^{84}	2.53[c]	5.1 ± 0.4[d]	3.4 ± 0.3	1.51
	2.54 ± 0.03[e]	5.12 ± 0.11[e]	3.3 ± 0.3	1.55
	2.45[f]		4.1	1.25
As^{74}	2.81[g]±0.02	1.5	1.16 ± 0.03	1.29

[a] M. L. Perlman and M. Wolfsberg, Brookhaven National Laboratory Report, BNL-485, T-110 (unpublished).
[b] See reference 1.
[c] C. W. Wu and N. Benczer-Koller (private communication); W. O. Doggett, Ph.D. thesis, University of California (unpublished).
[d] J. P. Welker and M. L. Perlman, Phys. Rev. **100**, 74 (1955).
[e] J. Konijn, B. van Nooijen, H. L. Hagedoorn, and A. H. Wapstra, Nuclear Phys. **9**, 296 (1959).
[f] C. M. Huddleston and A. C. G. Mitchell, Phys. Rev. **88**, 1350 (1952).
[g] S. Johansson, Y. Cauchois, and K. Siegbahn, Phys. Rev. **82**, 275 (1951).

[4] It may be noted that in this frequently used approximation, in which the ζ factors are expanded in terms of $\kappa=\alpha Z/2R$ and all terms except the first are neglected (i.e., $\kappa\gg W_0$), all matrix elements other than $\sum|B_{ij}|^2$ would produce K/β^+ ratios *exactly* equal to those characteristic of allowed transitions.

[5] R. W. King and D. C. Peaslee, Phys. Rev. **94**, 1284 (1954); T. Kotani, Phys. Rev. **114**, 795 (1959); M. Morita and R. S. Morita, Phys. Rev. **109**, 2048 (1959).

[6] These values were taken from Table VII of reference 2 and include corrections for screening.

[7] J. Konijn, H. L. Hagedoorn, H. van Krugten, and J. Slobben, Physica **24**, 931 (1958).

[8] The large K/β^+ ratio characteristic of the unique matrix element, $\sum|B_{ij}|^2$, may be used to identify a $\Delta J=2$, yes transition in cases where identification from spectral shape is somewhat ambiguous due to the presence of higher energy positron transitions.

TABLE III. Limits of the magnitudes[a] of $\Sigma|B_{ij}|^2$ and of[b] $G|M_i|^2\zeta_i$ from experimental and calculated[c] K/β^+ ratios for $2-\to 2+$ transitions.

$\lvert M_i\rvert^2$	I^{126}, $W_0=1.90$ $\Sigma\lvert B_{ij}\rvert^2$	$G\lvert M_i\rvert^2\zeta_i$	As^{74}, $W_0=2.80$ $\Sigma\lvert B_{ij}\rvert^2$	$G\lvert M_i\rvert^2\zeta_i$	Rb^{84}, $W_0=2.53$ $\Sigma\lvert B_{ij}\rvert^2$	$G\lvert M_i\rvert^2\zeta_i$	Rb^{84}, $W_0=2.45$ $\Sigma\lvert B_{ij}\rvert^2$	$G\lvert M_i\rvert^2\zeta_i$
$\lvert\int\boldsymbol{\alpha}\rvert^2$ or $\lvert\int\gamma_5\rvert^2$	0.28	1.4	1.6	4.5	1.6	2.2	1.0	2.9
$\lvert\int\mathbf{r}\rvert^2$	0.35	1.4	1.8	4.3	1.8	2.1	1.2	2.7
$\lvert\int\boldsymbol{\sigma}\cdot\mathbf{r}\rvert^2$	0.39	1.3	1.9	4.2	1.8	2.0	1.3	2.6
$\lvert\int\boldsymbol{\sigma}\times\mathbf{r}\rvert^2$	0	d	0	d	0.27	3.6	0	d

[a] All magnitudes in this table have been multiplied by 10^4. The units $\hbar=m=c=1$ have been used.
[b] $G|M_i|^2\zeta_i=(G_i^2/G_A^2)|M_i|^2\zeta_i$.
[c] See references 1 and 2.
[d] If $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ makes a predominant contribution, widely varying contributions from the other matrix elements are necessary to obtain the observed K/β^+ ratios.

$2-\to 2+$ transitions are shown in Table II, together with the ratios which would obtain if these transitions were allowed. It should be noted that these ratios are all larger than allowed, indicating contributions to the transitions from $\sum|B_{ij}|^2$ and/or $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$. Using Eqs. (5) and (8), the maximum and minimum values of $\sum|B_{ij}|^2$ which can obtain with contributions from various other matrix elements may be calculated. The values so obtained are shown in Table III for the $2-\to 2+$ transitions listed in Table II.

It is reasonable to suppose that reduction effects of configuration mixing on the magnitudes of the nuclear matrix elements would be more or less the same on those matrix elements which have the same selection rules [i.e., $|\int\gamma_5|^2$ and $|\int\boldsymbol{\alpha}\cdot\mathbf{r}|^2$, ($\Delta J=0$, yes); $|\int\mathbf{r}|^2$, $|\int\boldsymbol{\alpha}|^2$, and $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ ($\Delta J=0,\pm1$, yes); $\sum|B_{ij}|^2$ ($\Delta J=0,\pm1,\pm2$, yes)]. In the absence of a cancellation effect[5] among the lead terms of the matrix elements, roughly equal contributions of the matrix elements $|\int\mathbf{r}|^2$, $|\int\boldsymbol{\alpha}|^2$, and $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ (plus any amounts of $|\int\gamma_5|^2$ and $|\int\boldsymbol{\sigma}\cdot\mathbf{r}|^2$) would give rise to nearly allowed K/β^+ ratios. Only if $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ were considerably larger than the other matrix elements could the observed K/β^+ ratios be explained without a contribution from $\sum|B_{ij}|^2$.[9] One can argue that the marked uniformity of the ft values of β^+ and β^-, ($2-\to 2+$), transitions and the uniformity of the increase of these ft values over those for $\Delta J=0$ transitions in odd-A nuclei indicate approximately the same reduction of the matrix elements other than $\sum|B_{ij}|^2$ in these transitions. It is indicated also that analogous nuclear structures are involved.[5] The accompanying ground state ($2-\to 0+$) transitions, which show a similar uniformity in f_1t, also appear to have analogous nuclear structures as predicted by the shell model. It can be seen from Table III that the value of $\sum|B_{ij}|^2$ is roughly independent of choice of other contributing matrix elements unless $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ makes the major contribution.

Proceeding on the assumption that the sum of contributions of the matrix elements other than $\sum|B_{ij}|^2$ would produce allowed values of the K/β^+ ratios (i.e., δ_K and $\delta_+\ll1$), one may compute the contribution to the transition rate made by $\sum|B_{ij}|^2$ relative to the contributions made by all other matrix elements from Eq. (5), which reduces[1,2] to

$$(P_K/P_{\beta^+})_{2-\to 2+}=(P_K/P_{\beta^+})_{\text{allowed}}\frac{[1+(1/12)(W_0+1)^2y]}{[1+(1/24)(W_0^2-1)y]}. \qquad (10)$$

The values of y for the As^{74}, Rb^{84}, and I^{126} cases, as obtained from Eq. (10) with values of P_K/P_{β^+} and W_0 taken from Table II, are 0.34, 0.72,[10] and 0.18 respectively.[11] It should be noted that these values have attached to them uncertainties of approximately 30% associated with uncertainties in W_0 and in K/β^+ values.

From the y values and the observed half-lives and decay schemes one may calculate the values of $\sum|B_{ij}|^2$ and of $\sum(G_i^2/G_A^2)|M_i|^2\zeta_i$ for these transitions. In Table IV are given the observed values of $\sum|B_{ij}|^2$ and of $\sum(G_i^2/G_A^2)|M_i|^2\zeta_i$ thus computed, together with theoretical values for matrix elements calculated from single-particle wave functions by the method described by Rose and Osborne[12,13] for $\Delta J=0$ and $\Delta j=0$, 1, and 2. For calculation of values from theory, the nucleon configurations were assumed to be those given in the Appendix. In Table V the observed and theoretical

[9] J. Konijn, B. van Nooijen, and A. H. Wapstra, Nuclear Phys. **16**, 683 (1960). Examination of the assumption made by these authors that only $\Delta j=1$ matrix elements contribute to the $2-\to 2+$ transitions shows that, in the range of parameter values chosen, only $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ contributes, due to nearly complete cancellation among the other matrix elements. It may be noted that even for $\Delta j=0$ these parameter choices lead to a similar result. If indeed $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ is responsible for nearly the entire transition rate, the deviation from the allowed spectrum shape would be rather marked at the low-energy end; it may be possible to detect such a deviation experimentally.

[10] Value calculated for $W_0=2.53$.

[11] Measurements of the higher energy $2-\to 2+$ transition in the decay of Tl^{200} made by B. van Nooijen, H. van Krugten, and A. H. Wapstra, Nuclear Phys. (to be published) (quoted in reference 9), would indicate that $y=0.9$ in this case. Here the characteristic $\Sigma|B_{ij}|^2$ spectrum shape may be observable in measurements made in coincidence with the 368-kev gamma ray because of the large fraction (50%) of the transitions that may be effected by this matrix element. This measurement could determine whether $\Sigma|B_{ij}|^2$ or $|\int\boldsymbol{\sigma}\times\mathbf{r}|^2$ is responsible for the enhanced K/β^+ ratio in this $2-\to 2+$ transition. Beta-gamma correlation measurements should show these effects even more clearly.

[12] M. E. Rose and R. K. Osborne, Phys. Rev. **93**, 1326 (1954).

[13] The radial integral $\mathfrak{F}_1^2$ is taken to be equal to 0.5 $(R/\lambda_c)^2$, where λ_c is Compton wavelength, 3.86×10^{-11} cm, and R is $1.2\times10^{-13}A^{\frac{1}{3}}$ cm.

TABLE IV. Observed and theoretical values of nuclear matrix elements for $2- \rightarrow 2+$ electron-capture–positron transitions.[a,b]

1	2	3	4	5	6	7	8	9	10	11
Nuclide	t_K (sec) $2- \rightarrow 2+$	$\Sigma\|B_{ij}\|^2$ Observed[e]	$\Sigma\|B_{ij}\|^2$ Theoretical[f] $\Delta j=2$	$\Sigma\|B_{ij}\|^2$ Theoretical[f] $\Delta j=1$	$\Sigma\|B_{ij}\|^2$ Theoretical[f] $\Delta j=0$	$G\Sigma\|M_i\|^2\zeta_i$[d] Observed[e]	$\|\int\boldsymbol{\alpha}\|^2$, $\kappa^2\|\int\mathbf{r}\|^2$,[c] $\kappa^2\|\int\boldsymbol{\sigma}\times\mathbf{r}\|^2$ Theoretical[f] $\Delta j=1$	$\|\int\gamma_5\|^2$, $\kappa^2\|\int\boldsymbol{\sigma}\cdot\mathbf{r}\|^2$ Theoretical[f] $\Delta j=0$	$\|\int\boldsymbol{\alpha}\|^2$, $\kappa^2\|\int\mathbf{r}\|^2$ Theoretical[f] $\Delta j=0$	$\kappa^2\|\int\boldsymbol{\sigma}\times\mathbf{r}\|^2$ Theoretical[f] $\Delta j=0$
I^{126}	4.5×10^6	2.8×10^{-5}	1.9×10^{-4}	1.1×10^{-7}	1.7×10^{-5}	1.4×10^{-4}	3.4×10^{-3}	1.3×10^{-3}	7.9×10^{-6}	7.9×10^{-4}
Rb^{84}	5.2×10^6	1.6×10^{-4}	1.6×10^{-4}	1.4×10^{-7}	1.0×10^{-7}	2.2×10^{-4}	1.6×10^{-3}	6.3×10^{-3}	9.5×10^{-6}	6.1×10^{-4}
As^{74}	4.0×10^6	1.6×10^{-4}	1.4×10^{-4}	1.3×10^{-7}	0.9×10^{-5}	4.5×10^{-4}	1.3×10^{-3}	4.9×10^{-3}	7.4×10^{-6}	4.8×10^{-4}

[a] It should be noted that the uncertainties in K/β^+ ratios and transition energies are reflected in corresponding uncertainties in observed values of the matrix elements ($\pm30\%$ in $\Sigma|B_{ij}|^2$).
[b] In units $\hbar=m=c=1$.
[c] $\kappa=(\alpha Z/2R)$.
[d] $G\Sigma|M_i|^2\zeta_i=\Sigma(G_i^2/G_A^2)|M_i|^2\zeta_i$.
[e] Calculated from the observed ratios and f_0t_K values assuming that the matrix elements other than $\Sigma|B_{ij}|^2$ collectively would give rise to an allowed K/β^+ ratio. The values $G_V=2.97\times10^{-12}$ and $G_A=-1.25G_V$ were used in the calculation.
[f] Calculated by the method of Rose and Osborne, reference 12.

values of $\sum|B_{ij}|^2$ for the corresponding $2-\rightarrow0+$ ground-state transitions are given.

For the ground-state ($2-\rightarrow0+$) transitions the agreement between observed and theoretical values of $\sum|B_{ij}|^2$ is remarkably good; and calculations made for sixteen well-characterized $2-\rightarrow0+$ negative beta transitions demonstrate a similar agreement. For the $2-\rightarrow2+$ beta transitions, only in the case of $\Delta j=2$ (column 4, Table IV) are the theoretical values of $\sum|B_{ij}|^2$ sufficiently large to account for the "observed" $\sum|B_{ij}|^2$ values (column 3); the agreement between the "observed" and theoretical values for $\Delta j=2$ is similar to the agreement found for $2-\rightarrow0+$ transitions. It may be noted that in each case the theoretical value of $\sum|B_{ij}|^2$, $\Delta j=2$, in the $2-\rightarrow2+$ transition tends to be larger than the value for this matrix element in the $2-\rightarrow0+$ transition. The "observed" values show the same trend. If, indeed, the $2-\rightarrow2+$ transitions are characterized to a large extent by $\Delta j=2$, then the "observed" values for the contributing matrix elements other than $\sum|B_{ij}|^2$ (column 7, Table IV) should be smaller than the largest matrix elements which may contribute via $\Delta j=0$ or 1 (columns 8, 9, 10, 11). This is seen to be the case. The argument, in summary, is that in each case the "observed" value for $\sum|B_{ij}|^2$ is nearly as large as the theoretical one calculated for $\Delta j=2$ and is considerably larger than that calculated for $\Delta j=0$ or 1; furthermore, the observed value for the sum of the other matrix elements is considerably smaller than the largest single matrix element calculated for $\Delta j=0$ or 1. Thus, comparison between "observed" and theoretical values for matrix elements in these $2-\rightarrow2+$ transitions is consistent with the conclusion that the states are of such a nature that only the transition via $\sum|B_{ij}|^2\Delta j=2$ proceeds relatively unhindered. It has been pointed out that in the $3-\rightarrow2+$ transitions occurring in Sb^{124} and Eu^{152} beta-gamma correlation measurements have demonstrated that $\sum|B_{ij}|^2$ is responsible for a considerable fraction of the transition rate.[14,15] Furthermore, beta-spectrum measurements made by Langer and Smith[16] in these and other $3-\rightarrow2+$ hindered transitions show nonstatistical shapes, which are interpretable in terms of substantial contributions from $\sum|B_{ij}|^2$.

TABLE V. Observed and theoretical values of $\Sigma|B_{ij}|^2$ for the $2-\rightarrow0+$ electron-capture–positron transitions.

Nuclide	t_K (sec) $2-\rightarrow0+$	$\Sigma\|B_{ij}\|^2$ Observed	$\Sigma\|B_{ij}\|^2$ Theoretical[a]
I^{126}	5.8×10^6	0.41×10^{-4}	0.78×10^{-4}
Rb^{84}	2.6×10^7	0.16×10^{-4}	0.56×10^{-4}
As^{74}	6.9×10^7	0.11×10^{-4}	0.52×10^{-4}

[a] Calculated by the method of Rose and Osborne, reference 12.

If the parent nuclei discussed here are in zero-phonon ground states and if the first-excited states of the even-even daughter nuclei were pure phonon states, the beta transitions to these excited states should be forbidden relative to the transitions to the ground states. The fact that this is not observed to be the case may be evidence that the first-excited states of these even-even nuclei are not pure phonon states. It may be that these excited states are mixed collective–single-particle states such that the beta-decay may proceed, relatively unhindered, mainly through the single-particle components, and the gamma-ray de-excitation may proceed mainly through the collective components because of the enhanced speed of these collective $E2$ transitions.

ACKNOWLEDGMENTS

The authors wish to acknowledge the kind help and valuable suggestions and criticisms of Dr. Joseph Weneser of Brookhaven National Laboratory. One of the authors (D.S.H.) wishes to express his appreciation to the Chemistry Department, Brookhaven National Laboratory, for the guest appointment under which part of the work reported here was performed.

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APPENDIX

The nucleon configurations chosen for the calculation of the nuclear matrix elements for beta decay are shown in Table VI. In the case of $\Delta j=2$ the configurations represent very reasonable choices which are consistent with all information about these nuclei. For $\Delta j=0$ and 1 the assignments are intended merely for demonstration and are such as to provide the possibility of the spin and parity changes needed for the argument.

TABLE VI. Nucleon configurations assumed for calculation of beta-decay matrix elements according to method of Rose and Osborne.[a]

Nuclide		$p, n \to 2n$	
	$\Delta j=2$	$\Delta j=1$	$\Delta j=0$
As^{74}	$f_{5/2}, g_{9/2} \to (g_{9/2})^2$	$f_{7/2}, g_{9/2} \to (g_{9/2})^2$	$f_{7/2}, g_{7/2} \to (g_{7/2})^2$
Rb^{84}	$f_{5/2}, g_{9/2} \to (g_{9/2})^2$	$f_{7/2}, g_{9/2} \to (g_{9/2})^2$	$f_{7/2}, g_{7/2} \to (g_{7/2})^2$
I^{126}	$g_{7/2}, h_{11/2} \to (h_{11/2})^2$	$g_{9/2}, h_{11/2} \to (h_{11/2})^2$	$g_{9/2}, h_{9/2} \to (h_{9/2})^2$

[a] See reference 12.