PHYSICAL REVIEW VOLUME 122, NUMBER 1 APRIL 1, 1961

Relativistic Particle Systems with Interaction*

LESLIE L. FOLDY
Case Institute of Technology, Cleveland, Ohio
(Received November 17, 1960)

The possibility of covariantly describing a system of a fixed number of particles interacting directly is explored by attempting a direct "integration" of the commutation relations for the inhomogeneous Lorentz group under restrictions appropriate to the term "system of a fixed number of particles." By direct interaction is meant the fact that interaction between the particles is expressed directly in terms of coordinates, momenta, and spins for the particles rather than through the agency of a mediating field. The integration is carried out in considerable generality with the assumption that the infinitesimal generators of the group have expansions in inverse powers of the square of the velocity of light. The result coincides with that obtained earlier by Bakamjian and Thomas, but the method employed yields greater insight into the generality of the result, as well as into how further conditions beyond covariance, such as the property which is here called "separability of the interaction," can be incorporated in the result. The relationship of the result to the complete reducibility of a representation of the inhomogeneous Lorentz group is pointed out. Possible generalizations and applications of the procedures here employed are discussed.

I. INTRODUCTION

THERE appears to exist a common misconception to the effect that it is not possible to construct a relativistically covariant description of a system of interacting particles where the interactions are direct rather than mediated through a field. By a direct interaction we mean one in which the interaction term in the Hamiltonian is expressed explicitly in terms of the dynamical variables of the particles (their positions, momenta, and spin vectors). This misconception has currency in spite of the fact that Bakamjian and Thomas[1] have presented such a description, but may in part be owing to the fact that the approach of Bakamjian and Thomas is a somewhat unfamiliar, if not an unorthodox one; and hence, its relationship to the main stream of development of elementary particle physics is somewhat obscured. Playing an important role in perpetuating this misconception is no doubt the further fact that the term *relativistic covariance* often has different meanings to different workers and in different contexts. It is therefore important that we clarify exactly what is meant in the present context by relativistic covariance. We here take a point of view which has been expounded by Dirac,[2] among others, and particularly emphasized by Wigner.[3] We feel that this is the only viewpoint properly deserving of this title, and we attempt to expound it below.

We wish particularly to emphasize that we are speaking here of the requirement of relativity (combined with what one ordinarily considers a *system of particles*) and not with further extraneous conditions which may nevertheless be necessary to yield a theory which is physically satisfactory, or (if there is any distinction) which describes nature. No doubt, much confusion has also arisen from the fact that the requirements of covariance have often been simultaneously applied with other requirements (often unstated or only implied) in order to set up specific theories. We have no quarrel with such procedures, of course, though it would be better if the tacit assumptions or requirements, beyond covariance, were clearly stated; but it still remains an important question as to what are the requirements of covariance apart from such other considerations. It is to this question that the present paper is primarily addressed.

To begin our discussion of relativistic covariance, we would like first to make clear that we are not in the least concerned with appropriate tensor or spinor equations, or with "manifest covariance" or with any other mathematical apparatus which is intended to exploit the space-time symmetry of relativity, useful as such may be. We are instead concerned with the *group* of inhomogeneous Lorentz transformations as expressing the inter-relationship of physical phenomena as viewed by different equivalent observers in unaccelerated reference frames. That this group has its basis in the symmetry properties of an underlying space-time continuum is interesting, important, but not directly relevant to the considerations we have in mind. We feel that the direct application of the Lorentz transformation equations for space and time coordinates to the coordinates of a *particle* at a particular time instant, while relevant in classical mechanics, cannot be naively carried over into quantum mechanics where the concept of position of a particle at a given time instant is obscured by its probabilistic character, the uncertainty relation, and, perhaps most important, by the internal kinematic structure of relativistic particles exhibited in such phenomena as zitterbewegung.[4] It

* This work was supported in part by the U. S. Atomic Energy Commission.

[1] B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953).

[2] P. A. M. Dirac, Revs. Modern Phys. **21**, 392 (1949).

[3] E. P. Wigner, Nuovo cimento **3**, 517 (1956), and references listed in footnote 6; also the reference to Newton and Wigner in footnote 4.

[4] For the ambiguity in the concept of particle position arising in this last connection, see in particular, L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950). The question of the definition of position (and its relativistic transformation properties) in irreducible representations of the inhomogeneous Lorentz group has been studied in detail by T. D. Newton and E. P.

is not that we wish to reject out-of-hand approaches based on such ideas as a source of valid information, but our viewpoint is simply that if we wish to examine the basic implications or demands of relativistic covariance stripped of all extraneous ideas and free of all preconceptions, we stand on more secure ground in retreating to the position that the Lorentz group expresses the relationship between physical phenomena viewed by different observers; for this consideration must be valid whether or not further conditions supposedly stemming from relativistic covariance can be justified.

We now consider what the above viewpoint implies for quantum mechanics. We deal here with a particular *physical system*, and in particular with the totality of its possible quantum states. We may for convenience label the possible abstract states by symbols ψ, $\phi\cdots$. A particular abstract state will be described by each equivalent observer by a vector in a *private* linear vector space. Thus the abstract state ψ will be described by observer A as a vector ψ_A in his private space $\{A\}$, by the observer B as a vector ψ_B in his private space $\{B\}$, etc. Thus, there is a one-to-one correspondence between the state vectors of the private space of one observer with the state vectors of the private space of every other observer, the correspondence being fixed by the fact that the same abstract state of the physical system is being described by corresponding vectors. Actually, this statement is not quite correct, since states of the physical system are represented by each observer not as a vector in his private vector space, but by a *ray*, since the normalization and phase of the vector is not pertinent to the description of the state. Thus, what one has, in fact, is a correspondence between rays rather than between vectors. That this is of importance has been emphasized by Wigner,[3] but in the interests of simplicity of the present argument, we ignore this fact for the present and discuss briefly its consequences in modifying our argument at a later point.

At this point we must now introduce an assumption, which we prefer to do explicitly, although it is often tacitly assumed. Namely, we assume that the correspondence between vectors described above is in fact an isomorphism; that is, the correspondence is such that the state vector which is a particular linear combination of state vectors for one observer is in correspondence with the state vector which is the same linear combination of the corresponding state vectors for every other observer. The necessity for this assumption arises from the usual physical interpretation given to the mathematical formalism of quantum mechanics. Specifically, if a physical system is in the state ψ and the observer A makes a measurement to determine whether the system is in the state ϕ, then the probability of an affirmative result (for normalized state vectors) is given by $|(\phi_A,\psi_A)|^2$, while for a second observer B, the probability will be given by $|(\phi_B,\psi_B)|^2$. For these probabilities to be the same for all observers, the isomorphism is necessary. Again, this last statement is not quite correct, since there is another possibility, namely that for some pairs of observers one could have a particular linear combination of state vectors for one observer in correspondence with the complex conjugate linear combination (that is, the linear combination with complex conjugate coefficients) of the corresponding state vectors of the other observer. So long as we deal with purely continuous groups such as the *proper* Lorentz group, however, we need not concern ourselves with this possibility if we impose the continuity condition described below. For our present purposes, we therefore omit consideration of such a possibility.

So far we have placed no limitations on the particular type of vector space representation employed by each observer, whether it is, for example, a Heisenberg momentum representation or a Schrödinger coordinate representation, etc. But the relativistic equivalence of the various observers must allow them to employ the same descriptions within their private spaces, if they so choose, and we now assume that this is done but without regard for the particular choice. Then in consequence of the isomorphism of the various private spaces, there may be constructed a single *public* linear vector space (which may be either a new space or the private space of a particular observer) in which all of the abstract physical states of the system can be displayed as vectors. In this public space, however, a particular abstract state of the physical system is associated with a different vector by each different observer, and the correspondence which was previously manifested as an isomorphism between the private spaces of two observers becomes an automorphism of the public space. In other words, the correspondence between a vector in the private space of observer A and a vector in the private space of observer B, arising from the fact that the two vectors represent the same abstract physical state, appears now as a correspondence between two vectors in the public space. Thus the relationship between vectors describing corresponding states associated with any pair of observers becomes a mapping of the public space onto itself. Since the relationship between two observers is itself described by a particular transformation belonging to the Lorentz group, this means that associated with each transformation of the Lorentz group, one has a mapping of the public space onto itself.

The essential requirement which these mappings must possess is the following: If the Lorentz transformation connecting observers A and C is the resultant (product) of the Lorentz transformation connecting observers A and B followed by that connecting observers B and C, then the mapping associated with the pair of observers A and B when followed by the mapping

Wigner, Revs. Modern Phys. **21**, 400 (1949), and adds substantial weight to the argument presented in the text. Further discussion of this point may be found in reference 7.

associated with the pair of observers B and C must be the mapping associated with the pair of observers A and C. This consistency requirement then means that to every transformation of the Lorentz group there is associated a mapping of the public space onto itself such that the mappings have the same composition structure as the abstract group. Combined with the linearity (isomorphism) assumption made earlier, this amounts to the fact that the public space must be a representation space for the Lorentz group, the representation being by linear mappings. Finally, since the public space may be taken to be the private space of any one observer, one can deal with the description of states as practiced in any particular Lorentz frame in establishing the group representation.

Insofar as a particular observer wishes to describe the totality of states of a system, not directly by the state vectors themselves, but by an equation (Schrödinger equation) the totality of whose solutions form the linear vector space, one can translate the essential requirement as follows: One must associate with every Lorentz transformation a mapping of every solution of the equation on to another solution such that the composition properties of the group are preserved. This approach has the advantage that the fact that a solution of the equation is carried into another solution by a mapping can be verified in many cases without the necessity of having explicitly the general solution of the equation. This is the approach that will be followed in the body of the present paper.

It is necessary, for completeness, to add one further restriction which is not obviously connected with any physical consideration. Namely, as is customary in any case in the discussion of group representations for continuous groups, we shall be restricted to representations which are continuous. Roughly put, this means simply that if we have two transformations L and L' of the Lorentz group which are infinitesimally close (that is, the set of continuous parameters which label them differ by infinitesimals), then the mapping associated with $L^{-1}L'$ shall differ infinitesimally from the identity.

It remains only to remark on the effect of the observation that in quantum mechanics one is working in reality with ray rather than vector spaces. This implies that for physical consistency, if L'' is the Lorentz transformation which is the product LL' of the Lorentz transformations L and L', then the mapping associated with L'' need not be precisely the product of the mappings associated with L and L' individually. It need only be a mapping such that its effect on every vector of the space is to yield a vector which differs by a multiplicative constant from that yielded by the product of the two mappings. So long as one deals with unitary mappings (which are all that will be considered below), this multiplier can only be a number of modulus unity and so effects only a change in phase of the resultant vector. The practical consequence of this for what follows is that the Lie-Koenig relations for the commutators of representatives of infinitesimal generators of the group is weakened. The commutator need not be simply a linear combination of the representatives of infinitesimal generators of the group but may contain an additional additive term which is a multiple of the identity. For the Lorentz group itself, Wigner has shown that this is of no practical consequence. However, for the Galilean group associated with the nonrelativistic limit of the Lorentz group, there do exist ray representations which are not the equivalent of vector representations, and in fact, the irreducible vector representations of the Lorentz group go over in the nonrelativistic limit to ray representations of the Galilean group.[5] This poses no grave problems, however, and what minor difficulties do arise are disposed of in heuristic fashion, since it is not felt justified to present at length a more careful treatment in the context of the present problem.

The author apologizes for the somewhat overextended and perhaps unnecessarily elementary discussion presented above of the meaning of Lorentz covariance. It is included in the hope that it may prevent any misunderstanding of the scope and implications of the results which are derived below.

The above considerations apply to any relativistic physical system. The present paper is concerned with their application to a system of a fixed number of *particles*. To define this more precisely, we mean here such systems where all observable quantities are represented by operators which are functions of a basic set of operators; namely, the familiar operators representing the position coordinates, the momentum components, and the spin operators of the n particles constituting the system. We shall define what we mean by *free* and by *interacting* particles more precisely at a later point and remain content at this point with simply remarking that the intent, at least, is that these terms have their simple familiar meaning.

The problem to be solved is now clear; namely, to construct a Schrödinger equation describing what we mean by a system of interacting particles such that the solutions give rise to a representation space for the inhomogeneous Lorentz group. With the restriction to the proper (isochronous) group, this may be carried out by finding appropriate operator representatives for the ten infinitesimal generators of the group (satisfying well-known commutation relations), but subject to certain restrictions, as was pointed out by Dirac.[2] These representations are not irreducible; the irreducible representations have been studied and classified by Wigner and Bargmann[6,7]; they are not appropriate

[5] In this connection see E. Inonu and E. P. Wigner, Nuovo cimento **9**, 705 (1952); V. Bargmann, Ann. Math. **59**, 1 (1954); and M. Hammermesh, Ann. Phys. **9**, 518 (1960).

[6] E. P. Wigner, Ann. Math. **40**, 149 (1939); Z. Physik **124**, 665 (1947). V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1948).

[7] A discussion of certain of these irreducible representations in

to the description of what we mean by a system of particles, but only to a single particle, or to what Wigner calls an *elementary system.* Representations of the type required here were obtained by Bakamjian and Thomas by a somewhat peculiar heuristic procedure. This consisted in starting from the representation which corresponds to a system of free particles (Sec. III, below) and performing a canonical transformation to a new representation. In this representation it is a trivial matter to modify the operator representatives of the infinitesimal generators so as to introduce interaction, without at the same time invalidating the commutation relations which these generators must satisfy. The procedure has the shortcoming that the generality of the result is not at all clear.

The investigation reported in the present paper, which originated independently of the earlier work,[8] follows a different path, namely, the direct "integration" of the commutation relations for the generators of the inhomogeneous Lorentz group, and attempting to do this in complete generality. The procedure which was devised for this purpose consists in making use of an artificial expansion[9] of all quantities in powers of $(1/c^2)$, where c represents the velocity of light. With this method we were able to secure the most general solution for which such an expansion exists, but we have no knowledge at this time whether solutions for which expansibility is not possible may also exist. Our considerations are limited to a system of particles with finite rest mass and finite spin, and we conjecture that our solution is the most general for such a system. Although our solution is obtained by an expansion procedure, the final result is obtained in closed form and coincides with the result of Bakamjian and Thomas. The generality of their solution is thus more clearly defined.

The general result which is obtained shows on examination that without further restrictions it encompasses systems which are quite aphysical in a certain sense in that they fail to possess a property which we call *separability of the interaction.* Although defined more precisely in the text, this term is meant to cover the characteristic that a system of particles when broken up into two spatially remote subsystems should be such that the dynamics of each subsystem are independent of the other. We are able to exploit the methods we have developed to form a basis for limiting the general solution to systems satisfying the separability condition on the interaction, but unfortunately, here we have not been able to put the appropriate requirement in closed form; instead the procedure must be carried out order by order in powers of $(1/c^2)$. Its utility is thereby somewhat impaired insofar as actual applications are concerned.

While this work was primarily motivated by the purely theoretical question as to whether relativistic descriptions of a system of directly interacting particles were possible, and in what generality, its usefulness is not restricted by this consideration. While the theory is certainly not in a form where one can employ it immediately to set up a completely relativistic description, say of even so simple a system as a hydrogen atom, except perhaps approximately, nevertheless it does separate out the essential restrictions of relativity and shows further that these alone do not sufficiently circumscribe reasonable physical systems. Hence, it can yield some insight into the further requirements which systems must possess to be what we call "physically reasonable." Separability of the interaction is here a case in point, but other requirements such as causality and nonpropagation of physical effects with velocities exceeding the velocity of light still require exploration. Furthermore, within the scope of the present result there is established a framework within which any relativistic description of a system of the type considered here, whether exact or approximate, must fall. Thus, effective Hamiltonians for the interaction of particles as derived from a field-theoretical description should be encompassed by the results here obtained. Finally, the results obtained can serve in a most practical fashion to set up equations which are relativistic to some particular order in $(1/c^2)$ starting from a knowledge of the nonrelativistic interaction (as we will show in a future publication).

II. INHOMOGENEOUS LORENTZ AND GALILEAN GROUPS[10]

We take as the ten infinitesimal generators of the proper inhomogeneous Lorentz group the generators of the infinitesimal space translations $(P_1,P_2,P_3)=\mathbf{P}$, the generator of the infinitesimal time translation H, the generators of infinitesimal rotations $(J_1,J_2,J_3)=\mathbf{J}$, and the generators of infinitesimal Lorentz transformations $(K_1,K_2,K_3)=\mathbf{K}$. These satisfy the well-known commutation relations[11]:

a notation and representation corresponding to that employed below will be found in L. L. Foldy, Phys. Rev. **102**, 568 (1956).

[8] Most of the substantive content of the present paper was obtained in 1956 and presented at a seminar at Brookhaven National Laboratory during the summer of that year. Its connection with the work of Bakamjian and Thomas was pointed out to the author at that time by Professor N. M. Kroll. A substantial part of the long delay in submitting this work for publication was occasioned by an unsuccessful search for a more satisfactory method of handling the problem of *separability* of the interaction than that presented later in the paper.

[9] The specific nature of this expansion depends in part on the form in which the fundamental commutation relations are written. The particular choice employed was dictated by considerations discussed in footnote 11.

[10] The discussion of the irreducible representations of the inhomogeneous Lorentz group follows that contained in reference 7.

[11] The specific form of the constant coefficients appearing in these relations depends of course on the "normalization" of the infinitesimal generators, which are undefined to within a constant which may be dimensional. The particular choice here used corresponds to appropriate "physical dimensions" for the various generators such that, for example, $\mathbf{P}$ has the dimensions of a momentum, H an energy, $\mathbf{J}$ an angular momentum, and $\mathbf{K}$ a reciprocal velocity in units in which $\hbar=1$. The occurrence of the

$$\begin{aligned}
&[P_i,P_j]=0,\\
&[P_i,H]=0,\\
&[J_i,H]=0,\\
&[J_i,J_j]=i\epsilon_{ijk}J_k,\\
&[J_i,P_j]=i\epsilon_{ijk}P_k,\\
&[J_i,K_j]=i\epsilon_{ijk}K_k,\\
&[H,K_j]=-iP_j,\\
&[K_i,K_j]=-i\epsilon_{ijk}J_k/c^2,\\
&[P_i,K_j]=-i\delta_{ij}H/c^2.
\end{aligned}\tag{1}$$

Here δ_{ij} is the Kronecker symbol, ϵ_{ijk} the Levi-Civita three-index symbol, c represents the velocity of light, and the summation convention on repeated indices is assumed.

By an appropriate transition[12] to the limit $c\to\infty$:

$$\begin{aligned}
&\mathbf{P}\to\mathbf{P}^{(0)},\\
&H\to Mc^2+H^{(0)},\\
&\mathbf{J}\to\mathbf{J}^{(0)},\\
&\mathbf{K}\to\mathbf{K}^{(0)},
\end{aligned}\tag{2}$$

one obtains the infinitesimal generators of the Galilean group which satisfy the same commutation relations (1) with the exception of the last two relations which are replaced by

$$\begin{aligned}
&[K^{(0)}{}_i,K^{(0)}{}_j]=0,\\
&[P^{(0)}{}_i,K^{(0)}{}_j]=-i\delta_{ij}M,
\end{aligned}\tag{3}$$

where M is a constant which we shall call the intrinsic mass. Both the Lorentz and Galilean groups may be extended to include space-inversion and time-inversion transformations, but since this generalization leads to only minor modifications of the discussion which follows, we shall not discuss these in any detail. (See Sec. XI.)

A quantum mechanical system whose state vectors form a representation space for the Lorentz group we shall call a *Lorentz* or *relativistic* system; one whose state vectors (more properly, rays) form a representation space for the Galilean group we shall call a *Galilean* or *nonrelativistic* system.

The irreducible representations of the commutation relations (1) have been studied by Wigner and Bargmann.[6,7] Restricting the present discussion to representations suitable for describing a particle of finite mass and noninfinite spin, each irreducible representation, to within a unitary or antiunitary equivalence, can be designated by two numbers: m, taking any positive value, which we shall call the *mass*, and s, taking positive integral or half-integral values or zero, which we shall call the *spin*. A description of the irreducible representation (m,s) in the language of a Schrödinger coordinate representation can be summarized as follows: The vectors of the unitary representation space as $(2s+1)$-component wave functions $\psi(\mathbf{r},t)$ which are square-integrable (on $\mathbf{r}$) solutions of the equation

$$i\partial\psi(r,t)/\partial t=\omega\psi(\mathbf{r},t),\tag{4}$$

where ω is the integral (nonlocal) operator

$$\omega=(m^2c^4+c^2p^2)^{\frac{1}{2}},\quad \mathbf{p}=-i\nabla,\tag{5}$$

with units chosen so that $\hbar=1$. The scalar product in the representation space is given by

$$(\psi_a,\psi_b)=\int\psi_a{}^*(\mathbf{r},t)\psi_b(\mathbf{r},t)d\mathbf{r}.\tag{6}$$

A representation of the Lorentz group is provided by the following identifications of the infinitesimal generators:

$$\begin{aligned}
&\mathbf{P}=\mathbf{p},\\
&H=\omega,\\
&\mathbf{J}=\mathbf{r}\times\mathbf{p}+\mathbf{s},\\
&\mathbf{K}=(\mathbf{r}\omega+\omega\mathbf{r})/2c^2-[\mathbf{s}\times\mathbf{p}]/(mc^2+\omega)-t\mathbf{p}.
\end{aligned}\tag{7}$$

The symbols $(s_1,s_2,s_3)=\mathbf{s}$ represent three irreducible $(2s+1)$-dimensional matrices satisfying the commutation relations

$$[s_i,s_j]=i\epsilon_{ijk}s_k,\tag{8}$$

velocity of light c in special positions in (1) is then conditioned by this choice. One could equally well have employed $c\mathbf{P}$ as the generator of infinitesimal space translations, and $c\mathbf{K}$ as the generator of infinitesimal Lorentz transformations, in which case no c's would appear. While such a choice has the advantage that the commutation relations are free of dimensional constants, it would have precluded our using the limit $c\to\infty$ as the characteristic nonrelativistic limit leading to the Galilean group. An alternative approach which could have some advantages is to employ $c\mathbf{P}/\sqrt{\alpha}$ as the generator of space translations and $c\mathbf{K}/\sqrt{\alpha}$ as the generator of Lorentz transformations, where α is a positive dimensionless number. The commutation relations would then still have the form (1) except for the replacement of $1/c^2$ by α on the right sides of the last two relations. The nonrelativistic limit would then be given by $\alpha\to0$, but with some care taken about the rest-energy term in H. The expansions which we later use would be expansions in the abstract parameter α rather than in a physical quantity (with dimensions), namely $1/c^2$; this might have some conceptual advantage but in no way really affects the arguments that we employ. We choose, however, to retain the form of (1) as written and to think of the nonrelativistic limit as the limit $c\to\infty$.

[12] The rest energy term in H is the source of some difficulty in going to the nonrelativistic limit. In actuality the correct form of the commutation relations of the Galilean group has zero on the right side of the second equation of (3). In the representations of the Lorentz group we shall later be considering, the limit actually takes the form of (3), and hence the Lie-Koenig relations for the infinitesimal generators of the Galilean group are not obtained, since the right side of the second equation of (3) is not a linear combination of the infinitesimal generators of the group. The reason for this is the fact noted in the introduction, that the nonrelativistic limit of the representations of the Lorentz group which we employ (and which are vector representations) yield *ray* representations of the Galilean group. For such representations, the occurrence of such a term on the right side is perfectly allowable insofar as the commutation relations are applied to operator representatives of the infinitesimal generators of the group. Since this is the only application we make of the commutation relations, we have no cause for concern.

and hence constitute an irreducible representation of the three-dimensional rotation group. One can easily verify that (7) satisfies the fundamental relations (1) and that the transformations induced by these infinitesimal generators carry any solution of (4) into another solution. The above representation may be identified with the description of a free relativistic particle of mass m and spin $\mathbf{s}$; in the conventional interpretation, $\mathbf{r}$ represents its position, $\mathbf{p}$ its momentum, ω its energy, $\mathbf{s}$ its spin angular momentum, and $\mathbf{r}\times\mathbf{p}$ its orbital angular momentum.

Taking the limit $c\rightarrow\infty$ yields a representation of the Galilean group[13] which may be identified with the description of a free nonrelativistic particle. The Schrödinger equation (4) becomes in this limit (after dropping the rest-energy term),

$$i\partial\psi(\mathbf{r},t)/\partial t=(p^2/2m)\psi(\mathbf{r},t), \tag{9}$$

and the second and fourth equations of (7) become

$$\begin{aligned} H^{(0)}&=p^2/2m,\\ \mathbf{K}^{(0)}&=m\mathbf{r}-t\mathbf{p}, \end{aligned} \tag{10}$$

while the other generators are the same as in the Lorentz case.

III. PARTICLE SYSTEMS

The direct product[14] of N irreducible representations of the Lorentz (Galilean) groups leads to a reducible representation which is identified with a relativistic (nonrelativistic) system of N noninteracting particles. Explicitly, we associate with the νth particle a rest mass m_ν, a spin s_ν, a position vector $\mathbf{r}_\nu$, a momentum $\mathbf{p}_\nu=-i\nabla_\nu$, an energy $\omega_\nu=(m_\nu{}^2c^4+c^2p_\nu{}^2)$, and a spin angular momentum $\mathbf{s}_\nu$. The vectors of the representation space are functions with $\prod_{\nu=1}{}^N(2s_\nu+1)$ components, each component being a function of all the $\mathbf{r}_\nu$, but of only one time variable t. These state vectors are the square-integrable (on all $\mathbf{r}_\nu$) solutions of the Schrödinger equation

$$i\partial\psi(\mathbf{r}_1\cdots\mathbf{r}_N;t)/\partial t=H\psi(\mathbf{r}_1\cdots\mathbf{r}_N;t), \tag{11}$$

where the operator H is given in (13) below, and the scalar product is defined by

$$(\psi_a,\psi_b)=\int\cdots\int\psi_a{}^*\psi_b d\mathbf{r}_1\cdots d\mathbf{r}_N. \tag{12}$$

In accordance with the direct product character of the representation, the infinitesimal generators of the Lorentz group are identified as

$$\begin{aligned} \mathbf{P}&=\textstyle\sum_\nu\mathbf{p}_\nu,\\ H&=\textstyle\sum_\nu\omega_\nu,\\ \mathbf{J}&=\textstyle\sum_\nu\mathbf{j}_\nu,\\ \mathbf{K}&=\textstyle\sum_\nu\mathbf{k}_\nu, \end{aligned} \tag{13}$$

where

$$\mathbf{j}_\nu=\mathbf{r}_\nu\times\mathbf{p}_\nu+\mathbf{s}_\nu, \tag{14}$$

$$\mathbf{k}_\nu=(\mathbf{r}_\nu\omega_\nu+\omega_\nu\mathbf{r}_\nu)/2c^2-[\mathbf{s}_\nu\times\mathbf{p}_\nu]/(m_\nu c^2+\omega_\nu)-t\mathbf{p}_\nu. \tag{15}$$

The corresponding representation of a noninteracting Galilean particle system can obviously be obtained in an analogous way.

The introduction of interaction into the system of particles is to be accomplished by retaining the Schrödinger equation (11) but with H modified by the introduction of an interaction term U, a function of the dynamical variables of the system:

$$H=\textstyle\sum_\nu\omega_\nu+U. \tag{16}$$

Its solutions are still to constitute a representation space for the Lorentz group but with (16) replacing the second equation of (13). The generators $\mathbf{P}$ and $\mathbf{J}$ can be allowed to stand unchanged (by an appropriate choice of representation), but one can no longer retain the identification of $\mathbf{K}$ as given in (13), since in view of (16) such an identification would be in conflict with the last commutation relation of the set (1). Thus it is necessary to assume an interaction term in K as well:

$$\mathbf{K}=\textstyle\sum_\nu\mathbf{k}_\nu+\mathbf{V}, \tag{17}$$

where $\mathbf{V}$ is another function of the dynamical variables of the system. The problem of describing an interacting system of Lorentz particles then consists of determining functions U and $\mathbf{V}$ such that the fundamental commutation relations (1) are still satisfied.[2]

While it would be our hope to obtain a general solution of the commutation relations for U and $\mathbf{V}$ and thus to derive a description of every possible relativistic system of particles with finite mass and noninfinite spin, the methods applied in the present paper are not sufficiently powerful for this purpose. These methods, while lacking in rigor, do yield quite a general solution, which we believe in fact to be *the* most general solution within the context of the present problem, but this last conjecture has not been established. It should also be remarked that while every U and $\mathbf{V}$ such that the commutation relations are satisfied will yield a description of a relativistically invariant system, these systems need not have any resemblance to familiar physical systems unless U and $\mathbf{V}$ satisfy other conditions as well. We shall give some discussion of this point briefly later in the paper.

The problem of establishing a description of interacting Galilean particles can be formulated in close analogy to that for Lorentz systems. Since this problem represents a prelude to the Lorentz problem, we consider it first in the following section.

IV. INTERACTION IN GALILEAN SYSTEMS

In the interest of a simplified appearance for our equations, we shall, *in the present section only*, drop the superscript (0) which we have so far employed in

[13] With intrinsic mass m.
[14] See any book on the theory of groups and group representations.

distinguishing the generators of the Galilean group from those of the Lorentz group. In accordance, then, with the discussion of the last section, we assume for the generators of the Galilean group the forms:

$$\mathbf{P}=\sum_\nu p_\nu, \tag{19}$$

$$H=\sum_\nu p_\nu^2/2m_\nu+U, \tag{20}$$

$$\mathbf{J}=\sum_\nu [\mathbf{r}_\nu\times\mathbf{p}_\nu+\mathbf{s}_\nu], \tag{21}$$

$$\mathbf{K}=\sum_\nu (m_\nu\mathbf{r}_\nu-t\mathbf{p}_\nu)+\mathbf{V}. \tag{22}$$

It is convenient for our present discussion, as well as for the succeeding discussion of the Lorentz group, to introduce conventional center-of-mass and relative coordinates[15]:

$$\begin{aligned} M&=\sum_\nu m_\nu,\\ \mathbf{R}&=\sum_\nu m_\nu\mathbf{r}_\nu/M,\\ \mathbf{P}&=\sum_\nu \mathbf{p}_\nu,\\ \boldsymbol{\varrho}_\nu&=\mathbf{r}_\nu-\mathbf{R},\\ \boldsymbol{\pi}_\nu&=\mathbf{p}_\nu-m_\nu\mathbf{P}/M. \end{aligned} \tag{23}$$

P and **R** are, of course, canonically conjugate, but the $\boldsymbol{\varrho}_\nu$ and $\boldsymbol{\pi}_\nu$ are not independent (and hence not canonically conjugate) since they satisfy the relations:

$$\begin{aligned} \sum_\nu m_\nu\boldsymbol{\varrho}_\nu&=0,\\ \sum_\nu \boldsymbol{\pi}_\nu&=0, \end{aligned} \tag{24}$$

and the commutation relations:

$$\begin{aligned} [(\rho_\nu)_i,(\rho_{\nu'})_j]&=[(\pi_\nu)_i,(\pi_{\nu'})_j]=0,\\ [(\pi_\nu)_i,(\rho_{\nu'})_j]&=-i[\delta_{\nu\nu'}-m_\nu/M]\delta_{ij}. \end{aligned} \tag{25}$$

Actually, we shall require only the fact that **P** and **R** commute with all the $\boldsymbol{\varrho}_\nu$, $\boldsymbol{\pi}_\nu$, and $\mathbf{s}_\nu$. These latter variables we shall call *internal variables*. In terms of these variables, we have

$$\mathbf{J}=\mathbf{R}\times\mathbf{P}+\mathbf{S}, \tag{26}$$

where

$$\mathbf{S}=\sum_\nu [\boldsymbol{\varrho}_\nu\times\boldsymbol{\pi}_\nu+\mathbf{s}_\nu], \tag{27}$$

is the *internal* angular momentum (spin) of the system;

$$H=P^2/2M+T+U \tag{28}$$

where

$$T=\sum_\nu \pi_\nu^2/2m_\nu \tag{29}$$

is the internal kinetic energy of the system; and

$$\mathbf{K}=M\mathbf{R}-t\mathbf{P}+\mathbf{V}. \tag{30}$$

[15] Actually, any transformation of coordinates which yields a coordinate **R**, canonically conjugate to **P**, and internal coordinates commuting with **R** and **P** are satisfactory for the purpose of what follows. The particular choice above is made only because of its familiarity, not because it has some deep-seated significance. Of course, the ordinary center of mass is significant in the nonrelativistic case, but much of its significance is lost when one considers the relativistic situation, and actually we shall continue to use the same variables in this latter situation.

U and **V** may now be regarded as functions of **R**, **P**, and the internal variables. An explicit time dependence of U and **V** is excluded by the requirement that the Schrödinger equation (11) be left invariant under the transformations induced by the infinitesimal generators.

It is seen immediately that (19) satisfies the fundamental commutation relations:

$$[P_i,H]=[J_i,H]=0, \tag{31}$$

provided U is translationally invariant (and hence independent of **R**) and rotationally invariant. Furthermore, (22) satisfies the fundamental relations

$$\begin{aligned} [P_i,K_j]&=-i\delta_{ij}M,\\ [J_i,K_j]&=-i\epsilon_{ijk}K_k, \end{aligned} \tag{32}$$

provided **V** is also translationally invariant (and hence independent of R) and transforms as a vector under rotations. Hence, U must be a scalar and **V** a vector formed from **P** and the internal variables. The remaining two fundamental relations:

$$\begin{aligned} [K_i,K_j]&=0,\\ [H,K_j]&=-iP_j, \end{aligned} \tag{33}$$

then yield the two more complicated conditions:

$$M[R_i,V_j]-M[R_j,V_i]+[V_i,V_j]=0, \tag{34}$$

$$M[U,\mathbf{R}]+[U,\mathbf{V}]+[T,\mathbf{V}]=0. \tag{35}$$

It will be noted that, in contrast to the Lorentz case, these relations can be satisfied with $\mathbf{V}=0$ even if $U\neq 0$. To make this assumption, *ab initio*, however, would shed some doubt on the generality of the solution. We shall instead proceed without this assumption and show that, in fact, $\mathbf{V}=0$ can always be achieved, without loss of generality, by an appropriate choice of representation. Once this has been demonstrated, (35) reduces simply to

$$[U,\mathbf{R}]=0, \tag{36}$$

which states that U is independent of **P** as well as R and hence is an arbitrary rotationally invariant (scalar) function of internal variables only.

Our establishment of the quoted result is conditional on the assumption that **V** may be regarded as a function of a parameter λ, vanishing when $\lambda=0$, which has a regular power series expansion about this point, and that the commutation relations are satisfied for all values of λ within a nonzero radius of convergence of this power series. In this case we may write

$$\mathbf{V}=\lambda^\alpha\sum_{n=0}^{\infty} v^{(n)}\lambda^n, \tag{37}$$

where α is a positive integer. If this expansion is substituted in (34), then the terms of order α yield

$$[R_i,v^{(0)}{}_j]-[R_j,v^{(0)}{}_i]=0. \tag{38}$$

But in a momentum representation, this equation simply states that

$$\mathrm{curl}_{\mathbf{P}}\mathbf{v}^{(0)}=0, \tag{39}$$

and hence that there exists a function Φ such that

$$\mathbf{v}^{(0)}=-\mathrm{grad}_{\mathbf{P}}\Phi=i[\mathbf{R},\Phi]. \tag{40}$$

Furthermore, it is easy to see that Φ can be so chosen that it is Hermitian, translationally invariant, and rotationally invariant, so that it commutes with $\mathbf{P}$ and $\mathbf{J}$. If we now subject our representation to the unitary transformation

$$\mathfrak{U}=\exp(i\lambda^{\alpha}\Phi/M), \tag{41}$$

we have, by virtue of the commutativity of Φ with $\mathbf{P}$ and $\mathbf{J}$, that

$$\begin{aligned}\mathbf{P}'&=\mathfrak{U}\mathbf{P}\mathfrak{U}^{-1}=\mathbf{P},\\ \mathbf{J}'&=\mathfrak{U}\mathbf{J}\mathfrak{U}^{-1}=\mathbf{J}.\end{aligned} \tag{42}$$

Employing an expansion in powers of λ, we have further

$$\begin{aligned}\mathbf{K}'=\mathfrak{U}\mathbf{K}\mathfrak{U}^{-1}&=\mathfrak{U}[M\mathbf{R}-t\mathbf{P}+\mathbf{V}]\mathfrak{U}^{-1}\\ &=M\mathbf{R}-t\mathbf{P}+\mathbf{V}',\end{aligned} \tag{43}$$

where V' is of order $\lambda^{\alpha+1}$ by virtue of (40). Hence formally, at least, we can by repetition of this process eliminate terms of higher and higher order in λ, and thus finally achieve a representation in which $\mathbf{P}$ and $\mathbf{J}$ retain their standard forms and $\mathbf{V}=0$ so that

$$\mathbf{K}=M\mathbf{R}-t\mathbf{P}. \tag{44}$$

Having disposed of $\mathbf{V}$ by a proper choice of representation, we have by our earlier argument that the general solution for U is a rotationally invariant function of the internal variables of the system.

In all that follows, we shall assume that the representation of a Lorentz system is so chosen that in the nonrelativistic limit $\mathbf{V}=0$ and $\mathbf{K}$ assumes the form (44). Representations of the Lorentz group in which (44) holds in the nonrelativistic limit and in which $\mathbf{P}$ and $\mathbf{J}$ have the forms (19) and (21), respectively, will be called *standard* representations. It is of interest to note that by virtue of the zero choice of $\mathbf{V}$ in the Galilean representation, the transformation of $\mathbf{r}_\nu$ and $\mathbf{p}_\nu$ under Galilean transformations is the same in the presence as in the absence of interaction:

$$\begin{aligned}\mathbf{r}_\nu'&=\exp(i\boldsymbol{\xi}\cdot\mathbf{K})\mathbf{r}_\nu\exp(-i\boldsymbol{\xi}\cdot\mathbf{K})=\mathbf{r}_\nu-\boldsymbol{\xi}t,\\ \mathbf{p}_\nu'&=\exp(i\boldsymbol{\xi}\cdot\mathbf{K})\mathbf{p}_\nu\exp(-i\boldsymbol{\xi}\cdot\mathbf{K})=\mathbf{p}_\nu-m_\nu\boldsymbol{\xi},\end{aligned} \tag{45}$$

and in agreement with the classical concept of Galilean kinematics. This may be considered a justification in part for the identification of $\mathbf{r}_\nu$ and $\mathbf{p}_\nu$ with the coordinate and momentum vectors of the νth particle.[15a] No analogous result is valid in the Lorentz case.

[15a] *Note added in proof.* It may be appropriate to remark here that $\mathbf{p}_\nu$ is the momentum canonically conjugate to $\mathbf{r}_\nu$; since U contains the $\mathbf{p}_\nu$, $\mathbf{p}_\nu$ is not necessarily equal to $m_\nu\dot{\mathbf{r}}_\nu$. The same remark is applicable in the relativistic case considered later.

V. INTERACTION IN LORENTZ SYSTEMS (ORDER c^{-2})

We now turn to the problem of integrating the commutation relations for the Lorentz group in the presence of interaction. The method we shall employ will consist in expanding all quantities of interest in powers of the parameter $(1/c^2)$ and then attempting to integrate the commutation relations order by order. To zeroth order, the Lorentz group reduces to the Galilean group so that this problem has been solved in the preceding section. The assumption of expansibility of this type is lacking in elegance and may perhaps pose grave questions of rigor about which we are able to say little. Even on the physical side the meaning of such an assumption is obscure since it is beyond our powers to vary the velocity of light.[16] Our justification for this procedure is thus largely pragmatic in that we are able to secure definite and reasonable results.

In actuality, we shall not quite carry out the full program outlined. We shall instead obtain the results for Lorentz systems valid to order $(1/c^2)$ and use these to infer the result correct to all orders. We are then able to show that the latter is the most general solution in the form of a power series in $(1/c^2)$. Considerable confidence in the generality of our final result is supported by another argument, which we shall give later, drawing on the reducibility of representations of the Lorentz group. The results thus obtained are equivalent to those of Bakamjian and Thomas referred to in the introduction, but our methods suggest a much greater generality than is clear from the work of these authors.

We begin then with the expansibility assumption

$$H=Mc^2+H^{(0)}+H^{(1)}+\cdots, \tag{46}$$

$$\mathbf{K}=\mathbf{K}^{(0)}+\mathbf{K}^{(1)}+\cdots, \tag{47}$$

with superscripts designating the order of the term in powers of $(1/c^2)$. Here $H^{(0)}$ is identical with the H given in (28) of the previous section with U an arbitrary rotationally invariant function of the internal variables, while $\mathbf{K}^{(0)}$ is given by Eq. (44) of the previous section. On substituting (46) and (47) into those of the fundamental commutation relations (1) which involve H and $\mathbf{K}$ and collecting terms of first order, one obtains

$$[P_i,H^{(1)}]=0, \tag{48}$$

$$[J_i,H^{(1)}]=0, \tag{49}$$

$$[J_i,K^{(1)}{}_j]=i\epsilon_{ijk}K^{(1)}{}_k, \tag{50}$$

$$[H^{(0)},K^{(1)}{}_i]+[H^{(1)},K^{(0)}{}_i]=0, \tag{51}$$

$$[K^{(0)}{}_i,K^{(1)}{}_j]-[K^{(0)}{}_j,K^{(1)}{}_i]=-i\epsilon_{ijk}J_k/c^2, \tag{52}$$

$$[P_i,K^{(1)}{}_j]=-i\delta_{ij}H^{(0)}/c^2. \tag{53}$$

The first two of these simply assert that $H^{(1)}$ is translationally and rotationally invariant, while the third asserts that $\mathbf{K}^{(1)}$ transforms as a vector under rotations.

[16] See, however, the remarks made in footnote 11.

The last three relations are more complicated and require detailed consideration.

To obtain a general solution of these, we note that a particular solution of (52) and (53) is given by

$$\mathbf{L}^{(1)}=(\mathbf{R}H^{(0)}+H^{(0)}\mathbf{R})/2c^2-[\mathbf{S}\times\mathbf{P}]/2Mc^2, \tag{54}$$

where use is made of the solutions for $H^{(0)}$ and $K^{(0)}$ obtained in the previous section. It is further clear that the general solution of (53) can be obtained by adding to $\mathbf{L}^{(1)}$ an arbitrary function which commutes with $\mathbf{P}$ and hence is independent of $\mathbf{R}$. On the other hand, Eq. (52) can now be rewritten as

$$[R_i,K^{(1)}{}_j-L^{(1)}{}_j]-[R_j,K^{(1)}{}_i-L^{(1)}{}_i]=0, \tag{55}$$

where use has been made again of the fact that $\mathbf{K}^{(0)}$ is of the form given in Eq. (44) and of our preceding result that the difference between $\mathbf{K}^{(1)}$ and $\mathbf{L}^{(1)}$ commutes with $\mathbf{P}$. Now (55) is again of the same form as Eq. (38) which allows us to conclude that the general solution for $\mathbf{K}^{(1)}$ consists of $\mathbf{L}^{(1)}$ plus the $\mathbf{P}$ gradient of an arbitrary function:

$$\mathbf{K}^{(1)}=\mathbf{L}^{(1)}+i[\mathbf{R},\Phi^{(1)}]. \tag{56}$$

The previously noted conditions allow us to restrict $\Phi^{(1)}$ to an arbitrary rotationally and translationally invariant function.

We turn now to the final equation to be satisfied, namely (51). We note first that

$$[H^{(0)},\mathbf{L}^{(1)}]=[\{H^{(0)}\}^2,\mathbf{R}]/2c^2, \tag{57}$$

where use is made of the fact that $H^{(0)}$ commutes with $\mathbf{P}$ and with $\mathbf{S}$, the latter following in turn from the fact that $\mathbf{J}$ and $\mathbf{R}\times\mathbf{P}$ commute with $H^{(0)}$. We have further, by use of the Jacobi identity,

$$\begin{aligned}[H^{(0)},i[\mathbf{R},\Phi^{(1)}]]=&-i[\Phi^{(1)},[H^{(0)},\mathbf{R}]]\\&-i[\mathbf{R},[\Phi^{(1)},H^{(0)}]]=[i[\Phi^{(1)},H^{(0)}],\mathbf{R}],\end{aligned} \tag{58}$$

where we have employed the fact that the commutator of $H^{(0)}$ with $\mathbf{R}$ is $-i\mathbf{P}/M$ which commutes with $\Phi^{(1)}$. Finally, we note that

$$[H^{(1)},\mathbf{K}^{(0)}]=[MH^{(1)},\mathbf{R}], \tag{59}$$

by using the form (44) for $K^{(0)}$ and the fact that $H^{(1)}$ commutes with $\mathbf{P}$ by (48). Now substituting (56), (57), (58), and (59) into Eq. (51), we find that it can be written

$$[\{H^{(0)}\}^2/2c^2+i[\Phi^{(1)},H^{(0)}]+MH^{(1)},\mathbf{R}]=0, \tag{60}$$

which simply asserts that the first factor of the commutator brackets is independent of $\mathbf{P}$ and hence must be a function, which we call $MW^{(1)}$, of the internal coordinates and of $\mathbf{R}$. Thus the general solution of (51) for $H^{(1)}$ is

$$H^{(1)}=-\{H^{(0)}\}^2/2Mc^2-i[\Phi^{(1)},H^{(0)}]/M+W^{(1)}. \tag{61}$$

The rotational and translational invariance of $H^{(1)}$ as expressed in (48) and (49), then imply that $W^{(1)}$ must be a rotationally invariant function of *internal variables* only.

Thus our problem has been solved to order $1/c^2$. It is now a straightforward matter to obtain the interaction terms U and $\mathbf{V}$ to first order using (16) and (17). The calculation is a little lengthy, and we simply quote the results:

$$\begin{aligned}U^{(1)}=&-\frac{P^2U^{(0)}}{2M^2c^2}-\frac{(T+U^{(0)})^2}{2Mc^2}+\sum_\nu\frac{1}{8m_\nu}\\&\times\left\{\frac{\pi_\nu{}^2}{m_\nu c}+\frac{2(\pi_\nu\cdot\mathbf{P})}{Mc}\right\}^2-\frac{i}{M}[\Phi^{(1)},H^{(0)}]+W^{(1)},\end{aligned} \tag{62}$$

$$\begin{aligned}\mathbf{V}^{(1)}=&\frac{\mathbf{R}U^{(0)}}{c^2}-\frac{1}{c^2}\sum_\nu\bigg[\frac{1}{4M}\{\rho_\nu(\boldsymbol{\pi}_\nu\cdot\mathbf{P})+(\boldsymbol{\pi}_\nu\cdot\mathbf{P})\boldsymbol{\varrho}_\nu\\&+\boldsymbol{\pi}_\nu(\boldsymbol{\varrho}_\nu\cdot\mathbf{P})+(\boldsymbol{\varrho}_\nu\cdot\mathbf{P})\boldsymbol{\pi}_\nu\}+\frac{1}{2m_\nu}\{\boldsymbol{\varrho}_\nu\pi_\nu{}^2+\pi_\nu{}^2\boldsymbol{\varrho}_\nu\\&-[\mathbf{S}_\nu\times\boldsymbol{\pi}_\nu]\}\bigg]+i[\mathbf{R},\Phi^{(1)}].\end{aligned} \tag{63}$$

Some simplifications in the expressions (62) and (63) can be achieved by noting that the terms $-(T+U^{(0)})^2/2Mc^2+\sum_\nu \pi_\nu{}^4/8m_\nu{}^2c$ in $U^{(1)}$ are functions of internal variables only and can therefore be incorporated into $W^{(1)}$:

$$\begin{aligned}U^{(1)}=&-P^2U^{(0)}/2M^2c^2\\&+\tfrac{1}{2}\textstyle\sum_\nu[(\boldsymbol{\pi}_\nu\cdot P)^2/m_\nu M^2c^2+\pi_\nu{}^2(\boldsymbol{\pi}_\nu\cdot P)/m_\nu{}^2Mc^2]\\&-i[\Phi^{(1)},U^{(0)}]/M+W^{(1)}.\end{aligned} \tag{64}$$

Furthermore $V^{(1)}$ can be written as

$$V^{(1)}=RU^{(0)}/c^2+i[R,\Phi^{(1)}+\Phi^{(1)\prime}], \tag{65}$$

where

$$\begin{aligned}\Phi^{(1)\prime}\equiv&\frac{1}{2c^2}\sum_\nu\bigg[\frac{1}{2M}\{(\boldsymbol{\varrho}_\nu\cdot\mathbf{P})(\boldsymbol{\pi}_\nu\cdot\mathbf{P})+(\boldsymbol{\pi}_\nu\cdot\mathbf{P})(\boldsymbol{\varrho}_\nu\cdot P)\}\\&+\frac{1}{2m_\nu}\{(\boldsymbol{\varrho}_\nu\cdot\mathbf{P})\pi_\nu{}^2+\pi_\nu{}^2(\boldsymbol{\varrho}_\nu\cdot\mathbf{P})-\mathbf{S}_\nu\times\boldsymbol{\pi}_\nu\cdot\mathbf{P}\}\bigg].\end{aligned} \tag{66}$$

It is of some practical interest that the results of this section allow one to generalize any nonrelativistic Hamiltonian to one which is relativistically invariant to order $1/c^2$, but that in view of the arbitrary functions $\Phi^{(1)}$ and $W^{(1)}$, the generalization is by no means unique. Conversely, a Hamiltonian which is purported to correspond to a theory which is relativistically invariant to terms of order $1/c^2$ should be subsumed in our result if it is, in fact, as general as we believe. In particular, one can verify that the reduction of the Breit Hamiltonian[17] for two interacting charged particles in positive

[17] G. Breit, Phys. Rev. 51, 248 (1937).

energy states is comprehended by the expressions obtained here, and presumably the same is true (though we have not attempted a check) of similar reductions of two-particle Dirac Hamiltonians obtained by Chraplevy *et al.*[18]

VI. A UNITARY TRANSFORMATION

Clearly the next step in our program would consist in attempting to repeat the procedure just employed in order to obtain the terms of order $1/c^4$. Eventually, one would have to justify by induction the successive steps and show that the commutation relations at each order can indeed be integrated. In view of the complications encountered already in order $1/c^2$, such a procedure is bound to be extremely involved, though we have no doubts it could be successful. In the present paper, however, we turn to a slightly modified procedure.

The representation obtained to order $1/c^2$ is a *standard* representation in the sense this term was defined. It is not thereby unique, however, as can be seen from the following argument. One can obtain a new representation through the agency of a unitary transformation. If this unitary transformation commutes with $\mathbf{P}$ and with $\mathbf{J}$, it does not disturb the standard form of these generators. If, furthermore, the unitary transformation differs from the identity by terms of order $1/c^2$ or higher, it will not disturb the standard form of the nonrelativistic limit of $\mathbf{K}$. Thus, since the infinitesimal generator of a unitary transformation can be selected in an infinite number of ways and still be translationally and rotationally invariant and of order $1/c^2$, there are an infinite number of unitary transformations which preserve the standard form but modify the generators H and $\mathbf{K}$ (even to order $1/c^2$) profoundly. In view of the complicated forms already obtained for H and K in second order, there would clearly be advantages in exploiting the freedom thus offered of simplifying these expressions. We will show presently that indeed we may eliminate the term in $\Phi^{(1)}$ from Eq. (56) by a unitary transformation with a corresponding simplification of the expression for $H^{(1)}$. Before doing this, however, we remark that such a procedure also has certain disadvantages. These stem from the fact that any such unitary transformation at the same time modifies the *operator representatives* of *physical observables*. Thus the price of the simplification achieved lies in an obscuration of the direct physical interpretation of the operators which appear in the expressions for the generators of the Lorentz group. We shall discuss this point at greater length in a later section.

We consider now the elimination of the arbitrary function $\Phi^{(1)}$ from the expressions of the previous section through the unitary transformation

$$\mathfrak{U}=\exp(i\Phi^{(1)}/M). \tag{67}$$

[18] F. N. Glover and Z. V. Chraplevy, Phys. Rev. **103**, 821 (1956), and further references contained therein.

Since $\Phi^{(1)}$ commutes with $\mathbf{P}$ and $\mathbf{J}$, these generators are left unchanged. On the other hand, we have

$$\mathfrak{U}\mathbf{K}^{(0)}\mathfrak{U}^{-1}=\mathbf{K}^{(0)}-i[\mathbf{R},\Phi^{(1)}]+(\text{terms of order } c^{-4}), \tag{68}$$

$$\mathfrak{U}K^{(1)}\mathfrak{U}^{-1}=\mathbf{K}^{(1)}+(\text{terms of order } c^{-4}), \tag{69}$$

$$\mathfrak{U}H^{(0)}\mathfrak{U}^{-1}=H^{(0)}+i[\Phi^{(1)},H^{(0)}]/M+(\text{terms of order } c^{-4}), \tag{70}$$

$$\mathfrak{U}H^{(1)}\mathfrak{U}^{-1}=H^{(1)}+(\text{terms of order } c^{-4}). \tag{71}$$

We thus obtain

$$\mathfrak{U}\mathbf{K}\mathfrak{U}^{-1}=\mathbf{K}^{(0)}+\mathbf{L}^{(1)}+(\text{terms of order } c^{-4}), \tag{72}$$

$$\mathfrak{U}H\mathfrak{U}^{-1}=Mc^2+H^{(0)}-\{H^{(0)}\}^2/2Mc^2+W^{(1)}+(\text{terms of order } c^{-4}), \tag{73}$$

with $W^{(1)}$, an arbitrary rotationally invariant function of internal variables only, now remaining as the only arbitrary term of second order.

VII. THE "GENERAL" SOLUTION

The results expressed by Eqs. (72) and (73) allow one to make a reasonable inference of what we would have obtained had we continued our procedure to all orders in $1/c^2$. To see this we need only rewrite these equations in another form, equivalent to the original to order $1/c^2$. This form is

$$H=[h^2+c^2P^2]^{\frac{1}{2}}, \tag{74}$$

$$\mathbf{K}=(\mathbf{R}H+H\mathbf{R})/2c^2-[\mathbf{S}\times\mathbf{P}]/(h+H)-t\mathbf{P}, \tag{75}$$

where h is a rotationally invariant function of internal variables only and explicitly is

$$h=Mc^2+h^{(0)}+h^{(1)}+\cdots, \tag{76}$$

with

$$h^{(0)}=\sum\nolimits_\nu \pi_\nu^2/2m_\nu+U^{(0)}, \tag{77}$$

$$h^{(1)}=-\{h^{(0)}\}^2/2Mc^2+W^{(1)}. \tag{78}$$

The addition of the row of dots in (76) is purely gratuitous at this stage but is meant to suggest that had the problem been carried out to still higher order in $1/c^2$ the result would still be of the form of (74) and (75) except for the addition of higher order terms to h. But the remarkable feature of Eqs. (74) and (75) is their close analogy in structure to the irreducible representation of the Lorentz group given in Eq. (7) if we identify h with m, $\mathbf{P}$ with $\mathbf{p}$, and $\mathbf{S}$ with $\mathbf{s}$.

With the strong suggestion that (74) and (75) are the general solution to our problem, it is now a simple matter to substitute these directly into the fundamental commutation relations given in Eq. (1). One finds indeed that these are satisfied provided only that h commutes with $\mathbf{P}$, $\mathbf{R}$, and $\mathbf{J}$ which simply asserts that h is a rotationally invariant function of internal variables only. The only question which then remains is whether we have lost any generality in the leap from our general second-order result to our final result

expressed in (74) and (75). We now give an argument which shows that in fact no generality is lost provided $\mathbf{K}$ and H have power series expansions in $1/c^2$ as prescribed earlier. We need only assume that a more general solution is of the form,

$$\mathbf{K}'=\mathbf{K}+\Delta K, \tag{79}$$

$$H'=H+\Delta H, \tag{80}$$

with H and $\mathbf{K}$ given by (74) and (75). On substituting these into the last equation of (1), we have

$$[P_i,\Delta K_j]=-i\delta_{ij}\Delta H/c^2, \tag{81}$$

showing that whatever the order of $\Delta\mathbf{K}$ in powers of $1/c^2$, ΔH is of one lower order. We now substitute (79) and (80) into the second-last equation of (1) and obtain

$$[K_i,\Delta K_j]-[K_j,\Delta K_i]=-[\Delta K_i,\Delta K_j]. \tag{82}$$

Since the deviation ΔK of $\mathbf{K}'$ from $\mathbf{K}$ must be of some definite order (not zero) in $1/c^2$, say of nth order, we have for the terms of nth order in Eq. (82) simply

$$M[R_i,\Delta K_j]-M[R_j,\Delta K_i]=0, \tag{83}$$

which by an argument now familiar implies that $\Delta\mathbf{K}$ has the form $i[\mathbf{R},\Phi^{(n)}]$ and hence that the term of nth order in $\Delta\mathbf{K}$ can be eliminated by a unitary transformation $\mathfrak{U}^{(n)}=\exp[i\Phi^{(n)}/M]$. It then follows from (81) with $i=j$ that ΔH is zero in the order $n-1$. By repetition of the argument we may show that by a series of unitary transformations of this type $\Delta\mathbf{K}$ and ΔH may be made zero to any order. While these unitary transformations do not leave H and $\mathbf{K}$ invariant, these are modified at most in order n and $n+1$, respectively, so that the resultant changes can be incorporated into ΔH and $\Delta\mathbf{K}$ of these orders, respectively. Thus we have established that (74) and (75) constitute indeed the most general solution which has a power series expansion in $1/c^2$. If there exist solutions to our original problem which are not so expansible, we can say nothing about them.

VIII. DISCUSSION OF THE GENERAL SOLUTION

At first sight it may seem remarkable that we were able to obtain so general a solution to our problem. We shall now show that in a way the result is trivial and might have been anticipated from other considerations. We remark first that any representation of a relativistic *system* consisting of more than one particle is reducible, whether or not there is interaction. Since any unitary reducible representation of the inhomogeneous Lorentz group is completely reducible, in an appropriate sense, it must be equivalent to the direct sum of irreducible representations. But this is exactly the nature of the result which we have in Eqs. (74) and (75). To see this, we note that since h commutes with $\mathbf{P}$, $\mathbf{R}$, and $\mathbf{J}$, it also commutes with $\mathbf{S}$ and hence with S^2. Furthermore, S^2 commutes with all the generators of the inhomogeneous Lorentz group. The complete representation space can then be decomposed into subspaces each associated with a definite eigenvalue h' of h and a definite eigenvalue $S'(S'+1)$ of S^2, with S' a positive integer, half-integer, or zero. Each such subspace is left invariant under all the transformations of the inhomogeneous Lorentz group and is therefore a representation space itself. The representation in each of these subspaces is then equivalent to a direct sum of irreducible representations belonging to the mass h' and the spin S'. The further decomposition (relative to any remaining degeneracy) can be effected by finding further observables constructed from the internal variables of the system alone, which commute with h, S^2, and the generators of the Lorentz group until one has a complete set which decomposes the subspace into invariant and irreducible subspaces.

The eigenvalues h' of h are nothing more than the internal energies of the system, and the numbers S' associated with the eigenvalue $S'(S'+1)$ of S^2 are the associated internal angular momentum (spin) of the system in these internal energy states. *Neither the spectrum of h, nor of S^2, nor the nature of the other quantum numbers required to label a particular irreducible representation occurring in the general representation has anything fundamentally to do with the Lorentz covariance of the system as a whole.* The lack of any restrictions on h other than those already noted is thus made clear. It should be obvious now that had we approached our problem originally from this point of view, we could well have written down the answer immediately. One may then well ask whether the lengthy calculation presented earlier is anything more than an overly circuitous route to a trivial result that could have been anticipated from the beginning.

We believe that this is not the case and that our detailed derivation has real intrinsic value in understanding the physical content of the final result expressed in Eqs. (74) and (75). The usefulness of our procedure, however, lies in an area which transcends the problem of relativistic covariance alone (this being fully comprehended in the solution) but is more directly concerned with the important question of the identification of operator representatives of physical observables and with further restrictions on the character of inter-particle interactions. We now consider these questions.

IX. SEPARABILITY OF THE INTERACTION

To illuminate the final remark made in the preceding section, let us assume that we had indeed followed the procedure outlined and had argued from the general reducibility of the representation of a Lorentz system that the general solution was indeed that found above and, in particular, that H was of the form

$$H=[h^2+c^2P^2]^{\frac{1}{2}}, \tag{84}$$

with h an arbitrary function of the internal variables $\boldsymbol{\varrho}_\nu$, $\boldsymbol{\pi}_\nu$, and $\mathbf{s}_\nu$. Recognizing in fact that h is the internal energy of the system, we might then assume for h a seemingly reasonable form, namely,

$$h=\sum_\nu [m_\nu^2c^4+c^2\pi_\nu^2]^{\frac{1}{2}}+u, \tag{85}$$

where u is a function of the internal variables only and represents the "interaction in the center-of-mass coordinate frame." It would then seem reasonable to choose u so that interaction between a pair of particles vanished as their separation approached infinity. We then pose the question whether the resultant representation of the inhomogeneous Lorentz group would describe a physically reasonable system. [It should be remembered that what we are implicitly assuming here is that the internal variables $\boldsymbol{\varrho}_\nu$, $\boldsymbol{\pi}_\nu$, and $\mathbf{s}_\nu$, or their equivalents in terms of $\mathbf{r}_\nu$, $\mathbf{p}_\nu$, and $\mathbf{s}_\nu$, as given through the relations (23) are indeed the operator representatives of the physical position coordinates, physical momenta, and physical spin angular momenta in the usual sense of these terms as physical observables.] Our answer to this query would then be no, because the system lacks an important physical property which we shall call *separability* of the interaction.

We define a system to have a *separable interaction* if it has the following property: in every frame of reference, and for every division of the system into two subsystems I and II, and for all configurations of the particles such that every particle belonging to subsystem I is infinitely separated from every particle belonging to subsystem II, the interaction potential U assumes the separated form $U_{\text{I}}+U_{\text{II}}$, where U_{I} involves dynamical variables referring to particles belonging to subsystem I only, and U_{II} involves dynamical variables referring to particles belonging to subsystem II only.

Obviously, the property of separability is essential in order that the idealization of isolating a system from its physical surroundings should have any meaning. (We believe that this is a necessary property of a physical system in every area of physics except in the case of the general theory of relativity applied in integral fashion to the universe as a whole so as to incorporate the ideas of Mach.)

Now the mere fact that the interaction u of Eq. (85) in the center-of-mass frame is separable does not at all guarantee that the interaction is separable in another frame. In a general frame, the interaction is, according to our definition (16) with H given by (84),

$$U=[h^2+c^2P^2]^{\frac{1}{2}}-\sum_\nu [m_\nu^2c^4+c^2p_\nu^2]^{\frac{1}{2}}, \tag{86}$$

and this is not separable. To see this, we simply remark that if we effect the separation of the two subsystems I and II, then in the center-of-mass frame, this requires that $u=u_{\text{I}}+u_{\text{II}}$. If, however, we substitute this via (85) in (86), we obtain a result which is certainly not the sum of two terms U_{I} and U_{II} of the required character. This is the case even if the interaction u is identically zero!

This apparent paradox arises, of course, from our unwarranted assumption that the operators for internal variables occurring in h can be identified directly with the usual physical observables. The virtue of our original derivation lies in the fact that it explicitly exhibits the change in the identification of the operator representatives of physical observables consequent on the unitary transformations we had to perform in order to bring the representation into the form given in Eqs. (74) and (75). Since the unitary transformations involved $\mathbf{P}$ and internal variables, the connections between the original operator representatives and the final operator representatives of any physical observable (in particular the internal variables) involve $\mathbf{P}$. The only operator representatives which are, in general, left unchanged by the transformations are those for the total momentum and total angular momentum. Thus, the usefulness of (84) is very much in question unless one can reconstruct the unitary transformation which carries one from a representation in which $\mathbf{r}_\nu$, $\mathbf{p}_\nu$, and $\mathbf{s}_\nu$ are the operator representatives of the position, momentum, and spin angular momentum of the particles of the system (which we shall call the *physical representation*) to the representation in which H and $\mathbf{K}$ have the forms given by Eqs. (74) and (75), which we shall call a *reduced representation*.[19] In a physical representation (16) must be valid with U a separable interaction in order that the representation which we have shall be physically acceptable.

Unfortunately, we have not been able to find, nor is it likely that there exists a *simple* condition on the function h in Eq. (84) which corresponds to the interaction in the physical representation being *separable*. The best that we can do is outline a procedure by which one can construct those functions h which have this property. Unfortunately, the procedure is one requiring an infinite number of steps, in general, and hence its practical utility is somewhat limited. We outline this procedure in the following section.

X. CONSTRUCTION OF SEPARABLE INTERACTIONS

In this section we shall employ H' to designate the generator of infinitesimal time translations (the Hamiltonian) in the reduced representation, and H to designate the corresponding generator in the physical representation. A corresponding notation with primes can be used for the operator representatives of other physical observables in the reduced representation, though we shall not in general require these symbols. We do, however, remark that $\mathbf{P}=\mathbf{P}'$, that is, the operator representative of the total momentum (generator

[19] Reduced representations are not unique since any unitary transformation which involves only internal variables and is rotationally invariant will leave the form of Eqs. (74) and (75) unchanged, and will only replace h by a new function of the internal variables.

of an infinitesimal space translation) is in fact the same in both representations, since the unitary transformation which carries us from one representation to the other is translationally invariant. If we write this transformation as

$$\mathfrak{U}=\exp(i\Phi/M), \tag{87}$$

we then have

$$H=e^{-i\Phi/M}H'e^{i\Phi/M}, \tag{88}$$

with

$$H'=[h^2+c^2P^2]^{\frac{1}{2}}, \tag{89}$$

and

$$H=\sum_\nu \omega_\nu+U. \tag{90}$$

We now assume an expansion for h in powers of $(1/c^2)$ of the form given in Eq. (76), from which we obtain a corresponding expansion of H':

$$H'=Mc^2+H^{(0)\prime}+H^{(1)\prime}+\cdots, \tag{91}$$

with

$$H^{(0)\prime}=\frac{P^2}{2M}+h^{(0)}, \tag{92}$$

$$H^{(1)\prime}=-\frac{P^4}{8M^3c^2}-\frac{P^2h^{(0)}}{2M^3c^2}+h^{(1)}, \tag{93}$$

etc. We also expand Φ in such a power series:

$$\Phi=\Phi^{(1)}+\Phi^{(2)}+\cdots. \tag{94}$$

From (88), we then obtain H in a corresponding power series:

$$H=Mc^2+H^{(0)}+H^{(1)}+\cdots, \tag{95}$$

with

$$H^{(0)}=\frac{P^2}{2M}+h^{(0)}, \tag{96}$$

$$H^{(1)}=-\frac{P^4}{8M^3c^2}-\frac{P^2h^{(0)}}{2M^3c^2}+h^{(1)}-\frac{i}{M}[\Phi^{(1)},h^{(0)}], \tag{97}$$

etc. By expanding H as given by (90) and comparing with (95), we may obtain explicit expressions for $U^{(0)}$, $U^{(1)}$, etc., in terms of $h^{(0)}$, $h^{(1)}$, $\cdots$ and $\Phi^{(1)}$, $\Phi^{(2)}$, $\cdots$. We may now consider the last two sets of quantities as unknowns which are to be determined in such a manner that $U^{(0)}$, $U^{(1)}$, $\cdots$ are separable. This requires in particular that they be independent of $\mathbf{P}$, and this can be achieved by appropriate conditions on these unknown quantities remembering always that h is a function of internal variables only and Φ is a function of $\mathbf{P}$ as well. The condition that one obtains on $U^{(0)}$ is that it be an arbitrary rotationally invariant function of internal variables only, which is itself a separable interaction. The most general choice of $h^{(1)}$ and $\Phi^{(1)}$ such that $U^{(1)}$ is separable is a considerably more complicated problem.

The procedure above outlined is an arduous one, even in a relatively low order. We do not discuss it further here, but defer to a later paper an example where the most general separable interaction valid to terms of order $(1/c^2)$ between two particles will be derived in detail. This result will then represent a substantial generalization of the well-known work of Wigner and Eisenbud on the most general interaction between two particles correct to first-order terms in the momenta of two particles. The usefulness of the considerations presented in this paper to some problems of practical interest will thus be established.

XI. CONCLUDING REMARKS

We conclude this paper with some general and some specific remarks concerning its contents.

(1) We may first note that the problem of taking into account invariance under space and time inversion can be easily incorporated into our treatment by further conditions on the function h in a well-known way. No new problems of principle appear to arise here.

(2) While we believe that Lorentz covariance and the separability condition are essential to any theory of interacting particles, it is not at all certain that these sufficiently delineate physically acceptable theories. There is no place in our treatment where the question of causality, whether in a local or an extended sense, makes it appearance. We thus have no guarantee that all of the theories subsumed under the above conditions have a property corresponding to the fact that physical effects are not propagated with a velocity greater than the velocity of light. It is not even a simple matter to define clearly what this means mathematically in our framework. Obviously a means of incorporating a condition of this character into the formalism developed would greatly enhance its value as a basis for discussing the interactions in a relativistic system independently of the means by which the interaction is propagated, the latter being the additional element contained in a field-theoretical description. Even a more succinct mathematical formulation of the condition of separability is most desirable.

(3) It may be argued that the condition that the number of particles in the system remains fixed already imposes severe limits on the theory which are contradicted by experience. This is certainly true, but there are no obvious barriers in the way of making an extension of the theory (complicated though it may be) to incorporate the creation and annihilation of particles, by passing to a Fock representation through second quantization. This requires, of course, the introduction first of Bose-Einstein or Fermi-Dirac statistics for identical particles, but it is clear that there is no apparent bar to incorporating these conditions into the theory for a fixed number of particles. The limitations which Lorentz covariance (in the sense in which it is applied in this paper) impose on the creation and destruction of particles would appear to be a most intriguing problem.

We would like to emphasize, however, that in spite

of the validity of the objection raised here, the utility of what has been presented is not thereby impaired in many problems of practical interest. Just as nonrelativistic quantum theory has an appropriate domain of validity, the domain of validity of the theory here presented encompasses those situations in which relativistic corrections to nonrelativistic quantum theory are of importance, but where the creation or annihilation of *real* particles is not important. The fact that virtual particles may be created and destroyed is presumably already accounted for in the direct interaction between real particles. Thus, for example, the interpretation of nucleon-nucleon scattering in terms of an effective interaction, even though energy dependent or nonlocal, should be encompassed within the framework of the theory given here. Only where real particle production gives rise to a substantial non-Hermitian part to the interaction in consequence of its reactive effects or where real particle production occurs, should the type of description envisaged here fail severely. This would be true at energies sufficiently above threshold for particle production such that the inelastic cross section represented a substantial part of the total cross section in a collision.

(4) A question which has not been more than superficially discussed in the text is that dealing with the identification of the particular operator representatives of individual particle observables. We have essentially satisfied ourselves with postulating a particular representation, what we call the physical representation, where this identification has been made once and for all. In the type of approach which is employed in this paper, it would be desirable to lay down abstract criteria for recognizing the physical representation (from other representations related to it by unitary transformations) and thus for making a firm and unambiguous identification of basic observables, much along the lines of that employed by Newton and Wigner.[4] We have not been successful in attempts at this nor do we know whether it is possible to go beyond what we have taken above as acceptable. Of course, the same deficiency runs through most of quantum mechanics where one ordinarily starts from a Hamiltonian in which the identification of physical observables is assumed known from the start, even though they may be somewhat mystical, such as "bare particle" operators. But the fact that Newton and Wigner were able to go further than this in one context whets ones appetite for extending this type of approach to clarify ones understanding of what is necessary and why in quantum physics, in contrast to looking only at "what works."

From the point of view of application, the question of identification of at least position of a particle is very pertinent to the problem of interaction with external fields. We are not in a position, from what has been done so far, to extend our considerations to a relativistic system of particles interacting also with an external field such as the electromagnetic field. This is a problem also worthy of study.[20]

(5) Lastly, it may be remarked that the considerations of this paper can be taken over largely unchanged into classical theory (for whatever interest there may be in this) by regarding the infinitesimal generators as the generators of infinitesimal contact transformations with the usual analog of Poisson brackets and commutators.

[20] In this connection, see the last section of the paper mentioned in footnote 7.