Time-Ordered Green's Functions and Electromagnetic Interactions*

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In the present paper we study various aspects of the Ward-Takahashi equations. In perturbation theory the equivalence between this set of equations and the requirement of gauge invariance is shown. It is then shown that these equations are valid for composite particles as well as for elementary particles. Based on our new formulation the definition of composite particles is given, and then we show with the aid of the Ward-Takahashi equations that the photon is an elementary particle.

I. INTRODUCTION

In a previous paper the problem of how to express the requirement of gauge invariance without reference to field equations was discussed.[1] Starting from a gauge-invariant but otherwise quite arbitrary Lagrangian we derived a set of equations which will be referred to as Ward-Takahashi (W-T) equations.[2] These equations are supposed to be equivalent to the requirement of gauge invariance in the absence of a Lagrangian to begin with.

In this paper we shall study some properties of this set of equations.

In Sec. II we shall show that quantum electrodynamics can be reproduced in the lowest few orders in perturbation theory by combining the W-T equations with a few fundamental postulates of the axiomatic field theory. This verifies to some extent the assertion that the complete set of W-T equations is equivalent to the requirement of gauge invariance.

The derivation of the W-T equations casts some doubt as to their validity for composite particles since we have no Lagrangian formulation of composite particles. For this reason we shall show in Sec. III that these equations are valid not only for elementary particles but also for composite particles. The technique of introducing field operators for composite particles which is used in this section will be discussed at length in the Appendix.

In Sec. IV, we shall give a likely definition of bound states. This definition is useful in the problem of distinguishing between elementary and composite particles.

Finally, in Sec. V we shall prove with the aid of the W-T equations that the photon is an elementary particle.

II. QUANTUM ELECTRODYNAMICS IN PERTURBATION THEORY

In a series of papers[3] we have discussed the formulation of field theories in terms of (1) the generalized unitarity condition, and (2) the parametric dispersion relations, and it has been shown that one can reproduce the renormalized perturbation theory based on these two postulates alone. In this section we shall discuss the perturbation theory with the W-T equations as a supplementary condition which selects only the gauge-invariant solution out of many other possible solutions. This problem has already been discussed in a previous paper in terms of retarded functions.[1]

First-Order Vertex Function

The generalized unitarity condition assures us of the vanishing of the absorptive part of the first-order Green's function.[3] Thus in quantum electrodynamics we are led to the equation in the first order,

$$\Box_x D_y \bar{D}_z \langle 0|T[A_\mu(x)\psi(y)\bar{\psi}(z)]|0\rangle - \Box_x D_y \bar{D}_z \langle 0|\tilde{T}[A_\mu(x)\psi(y)\bar{\psi}(z)]|0\rangle = 0, \quad (2.1)$$

where $D_y = \gamma\partial_y + m$, $D_z = \gamma^T\partial_z - m$ and ψ, $\bar{\psi}$, and A_μ refer to the electron and radiation fields.

To the above equation expressing the vanishing of the absorptive part one can apply the subtracted parametric dispersion relation after decomposing the Green's function into a sum of invariants.[4] Then we get a general solution of the following form:

$$\Box_x D_y \bar{D}_z \langle 0|T[A_\mu(x)\psi_\alpha(y)\bar{\psi}_\beta(z)]|0\rangle = (\mathcal{O}_\mu)_{\alpha\beta}\delta(x-y)\delta(x-z), \quad (2.2)$$

where $\mathcal{O}_\mu$ is a vector constructed out of Dirac's γ matrices and the differential operators. In order to determine the operator $\mathcal{O}_\mu$ we use one of the Ward-Takahashi equations, i.e.,

$$\Box_x \frac{\partial}{\partial x_\mu} \langle 0|T[A_\mu(x)\psi_\alpha(y)\bar{\psi}_\beta(z)]|0\rangle = eS_F'(y-z)_{\alpha\beta}[\delta(x-y)-\delta(x-z)], \quad (2.3)$$

where e is the electronic charge and S_F' is defined by

$$S_F'(y-z)_{\alpha\beta} = \langle 0|T[\psi_\alpha(y)\bar{\psi}_\beta(z)]|0\rangle. \quad (2.4)$$

In the first order S_F' may be replaced by the unprimed

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[1] K. Nishijima, Phys. Rev. **119**, 485 (1960).

[2] The references to the generalization of the Ward identity are found in reference 1.

[3] M. Muraskin and K. Nishijima, this issue [Phys. Rev. **122**, 331 (1961)]. Also see reference 1.

[4] This means that we should apply the subtracted dispersion relations to the coefficients of the invariants as has been discussed in the case of meson-nucleon interaction. See reference 3.

function S_F and we get

$$\Box_x \frac{\partial}{\partial x_\mu} D_y \tilde{D}_z \langle 0 | T[A_\mu(x)\psi_\alpha(y)\bar{\psi}_\beta(z)] | 0 \rangle = -ie\left(\gamma_\mu \frac{\partial}{\partial x_\mu}\right)_{\alpha\beta} \delta(x-y)\delta(x-z). \quad (2.5)$$

By comparing the solution (2.2) with Eq. (2.5), we find

$$(\Theta_\mu)_{\alpha\beta} \frac{\partial}{\partial x_\mu} \delta(x-y)\delta(x-z) = -ie\left(\gamma_\mu \frac{\partial}{\partial x_\mu}\right)_{\alpha\beta} \delta(x-y)\delta(x-z). \quad (2.6)$$

If we imply that Θ_μ does not involve scalar products of derivatives, this being a reasonable condition for the choice of invariants. We find

$$\Theta_\mu = -ie\gamma_\mu + i\lambda\sigma_{\mu\nu}(\partial/\partial x_\nu). \quad (2.7)$$

This is the most general solution in the first order satisfying Eq. (2.3). As has been discussed previously the first term represents the Dirac-type interaction and the second term gives the Pauli-type interaction, and hence the solution (2.7) certainly is gauge invariant.

Photon Propagator

Our next problem is the calculation of the photon propagator in the second order. The W-T equation for the photon propagator is given by

$$\Box_x \frac{\partial}{\partial x_\mu} \langle 0 | T[A_\mu(x)A_\nu(y)] | 0 \rangle = i \frac{\partial}{\partial x_\nu} \delta(x-y). \quad (2.8)$$

The combination of Eq. (2.8) with Källén-Lehmann representation[5] leads to

$$\langle 0 | T[A_\mu(x)A_\nu(y)] | 0 \rangle = \delta_{\mu\nu} D_F(x-y) + G_{\mu\nu}(x-y),$$

where

$$G_{\mu\nu}(x) = \frac{-i}{(2\pi)^4} \int d\kappa^2\, \sigma(\kappa^2) \int_F d^4k \times e^{ikx}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{1}{k^2+\kappa^2}, \quad (2.9)$$

and C_F denotes the Feynman path for the k_0 integration.

From the unitarity condition one finds in the second order

$$\Box_x \Box_y \langle 0 | T[A_\mu(x)A_\nu(y)] | 0 \rangle + \Box_x \Box_y \langle 0 | \tilde{T}[A_\mu(x)A_\nu(y)] | 0 \rangle = e^2 \operatorname{Tr}[\gamma_\mu S^{(+)}(x-y)\gamma_\nu \tilde{S}^{(+)}(x-y)^T] + (x \rightleftarrows y, \mu \rightleftarrows \nu), \quad (2.10)$$

[5] G. Källén, Helv. Phys. Acta **25**, 417 (1952); H. Lehmann, Nuovo cimento **11**, 342 (1954); M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

where use has been made of the first-order vertex function (2.2) and (2.7). $S^{(+)}$ and $\tilde{S}^{(+)}$ are the contraction functions for the electron and positron defined by

$$S^{(+)}(x) = (\gamma\partial_x - m)\Delta^{(+)}(x), \qquad \tilde{S}^{(+)}(x) = (\gamma^T\partial_x + m)\Delta^{(+)}(x), \quad (2.11)$$

and

$$i\Delta^{(+)}(x) = \frac{1}{(2\pi)^3} \int d^4p\, e^{ipx}\theta(p_0)\delta(p^2+m^2).$$

Making use of the formula

$$\operatorname{Tr}[\gamma_\mu S^{(+)}(x-y)\gamma_\nu \tilde{S}^{(+)}(x-y)^T] = -\frac{1}{(2\pi)^6}\int_{4m^2}^\infty d\kappa^2 \frac{\pi}{3\kappa}(\kappa^2-4m^2)^{\frac{1}{2}}\left(2+\frac{4m^2}{\kappa^2}\right) \times \int d^4k\, e^{ik(x-y)}(\delta_{\mu\nu}k^2 - k_\mu k_\nu)\theta(k_0)\delta(k^2+\kappa^2),$$

we get

$$\Box_x \Box_y \langle 0 | T[A_\mu(x)A_\nu(y)] | 0 \rangle + \Box_x \Box_y \langle 0 | \tilde{T}[A_\mu(x)A_\nu(y)] | 0 \rangle = -\frac{1}{(2\pi)^6}\int_{4m^2}^\infty d\kappa^2 \frac{\pi}{3\kappa}(\kappa^2-4m^2)^{\frac{1}{2}}\left(2+\frac{4m^2}{\kappa^2}\right) \times \int d^4k\, e^{ik(x-y)}(\delta_{\mu\nu}k^2 - k_\mu k_\nu)\delta(k^2+\kappa^2). \quad (2.12)$$

Inserting the representation (2.9) into the left-hand side of Eq. (2.12) one finds the expression for $\sigma(\kappa^2)$ in the second order[6]:

$$\sigma(\kappa^2) = \frac{e^2}{3(2\pi)^2} \frac{\kappa^2+2m^2}{\kappa^5}(\kappa^2-4m^2)^{\frac{1}{2}}\theta(\kappa^2-4m^2). \quad (2.13)$$

This weight function σ is related to Källén's[5] Π by

$$\sigma(a) = \Pi(-a)/a. \quad (2.14)$$

In the above calculation ambiguities related to gauge invariance do not occur at all.

III. THE WARD-TAKAHASHI EQUATIONS FOR COMPOSITE PARTICLES

In reference 1 we introduced the Ward-Takahashi equations which express in an implicit manner the requirement of gauge invariance. This equation is written in a symbolic way as

$$\Box_x \frac{\partial}{\partial x_\mu} T[A_\mu(x)\cdots] = \left[\sum_a e_a\delta(x-x_a) + i\frac{\partial}{\partial x_\mu}\cdot\frac{\delta}{\delta A_\mu(x)}\right] T[\cdots]. \quad (3.1)$$

[6] The weight function σ here differs from the one defined in reference 1 by sign.

Let us write down this equation in a more explicit manner as

$$\Box_x \frac{\partial}{\partial x_\mu} T[A_\mu(x)A_\nu(x')A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]$$
$$=(e_a\delta(x-x_a)+e_b\delta(x-x_b)+\cdots)$$
$$\times T[A_\nu(x')A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]$$
$$+i\frac{\partial}{\partial x_\nu}\delta(x-x')\cdot T[A_\sigma(x'')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]$$
$$+i\frac{\partial}{\partial x_\sigma}\delta(x-x'')\cdot T[A_\nu(x')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]$$
$$+\cdots, \quad (3.2)$$

where e_a, e_b, $\cdots$ are the charges of the quanta of fields φ_a, φ_b, $\cdots$, respectively.

Simple examples of the W-T equations are given by Eqs. (2.3) and (2.8).

Since we derived the W-T equations starting from a gauge invariant but otherwise arbitrary Lagrangian, one might have the impression that these equations are valid only for elementary particles. For this reason we shall show in this section that they are valid for composite particles as well as for elementary particles. This conclusion suggests that one cannot distinguish between elementary and composite particles by means of electromagnetic interactions. This problem will be discussed in the next two sections.

It has been shown in a previous paper[7] that the field operator of a particle c either elementary or composite can be constructed by the following prescription:

$$\lim_{|\xi|\to 0}\lim_{\xi_0\to 0}\frac{:\varphi_a(x+\frac{1}{2}\xi)\varphi_b(x-\frac{1}{2}\xi):}{f(\xi,P)}=\varphi_c(x), \quad (3.3)$$

provided that

$$\langle P,c|T[\varphi_a(\tfrac{1}{2}\xi)\varphi_b(-\tfrac{1}{2}\xi)]|0\rangle=\frac{1}{(2P_0V)^{\frac{1}{2}}}f(\xi,P)\neq 0. \quad (3.4)$$

$|P,c\rangle$ denotes a one-c-particle state with energy-momentum P, and the colons denote the normal product defined by

$$:\varphi_a\varphi_b:=T[\varphi_a\varphi_b]-\langle 0|T[\varphi_a\varphi_b]|0\rangle.$$

In the Appendix it will be shown that the limit $\varphi_c(x)$ does not depend on the direction of the time-like vector P. If we combine Eq. (3.2) with (3.3) one immediately finds

$$\Box_x \frac{\partial}{\partial x_\mu} T[A_\mu(x)A_\nu(x')A_\sigma(x'')\cdots\varphi_c(x_c)\cdots]$$
$$=[e_c\delta(x-x_c)+\cdots]T[A_\nu(x')A_\sigma(x'')\cdots\varphi_c(x_c)\cdots]$$
$$+i\frac{\partial}{\partial x_\nu}\delta(x-x')\cdot T[A_\sigma(x'')\cdots\varphi_c(x_c)\cdots]$$
$$+i\frac{\partial}{\partial x_\sigma}\delta(x-x'')\cdot T[A_\nu(x')\cdots\varphi_c(x_c)\cdots]$$
$$+\cdots, \quad (3.5)$$

where use has been made of the relation

$$\lim_{\xi\to 0}[e_a\delta(x-x_a)+e_b\delta(x-x_b)]=e_c\delta(x-x_c), \quad (3.6)$$

with $e_c=e_a+e_b$, $x_a=x_c+\frac{1}{2}\xi$, $x_b=x_c-\frac{1}{2}\xi$.

If c is elementary, Eq. (3.5) is already involved in the complete set of W-T equations, and the present result shows that the W-T equations are compatible with the limiting procedure (3.3). On the contrary, if c is composite we get a new set of equations, but as one can readily notice, the form of the W-T equations is the same for both elementary and composite particles. Thus we are led to draw a conclusion that one cannot distinguish between elementary and composite particles through purely electromagnetic phenomena; for instance, the low-energy limit theorem for Compton scattering holds regardless of whether the target particle is elementary or composite.[8]

IV. DEFINITION OF COMPOSITE PARTICLES

The question of how to distinguish between elementary and composite particles is certainly one of the most fundamental problems in particle physics. In this section we shall give a likely definition of elementary and composite particles. Since we introduced one field operator for each stable particle in the present scheme, our criterion cannot be applied to unstable particles.

In the conventional Lagrangian theory we introduce field operators only for those particles which are supposed to be elementary so that in such a theory we have a clear distinction between elementary and composite particles. Since, however, the Lagrangian is hardly determined from experiments, this is a rather unrealistic definition. In particular, when we formulate field theories in an axiomatic way without assuming the existence of a special Lagrangian, this kind of criterion is really meaningless. As a matter of fact, it has been claimed in the axiomatic approach that there would be no essential difference between elementary and composite particles as far as the fundamental postulates

[7] K. Nishijima, Phys. Rev. **111**, 995 (1958). See also W. Zimmermann, Nuovo cimento **10**, 597 (1958); R. Haag, Phys. Rev. **112**, 669 (1958).

[8] W. Thirring, Phil. Mag. **41**, 1193 (1950); F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).

are concerned and that it is to some extent a matter of convenience to call a particle elementary or composite.[7] If we proceed one step further from the fundamental set of postulates, however, a possibility arises of making a distinction between them. The fundamental set of postulates is something like a set of equations and allows in general many solutions, and in order to choose a special solution one has to give boundary conditions. Then it might be possible to distinguish between them by giving different kinds of boundary conditions to elementary and composite particles, and this is exactly what we are going to do in this section.

We have the generalized unitarity condition and parametric dispersion relations as the fundamental set of equations. A set of boundary conditions will be given by specifying the subtraction constants in the subtracted dispersion relations. Thus the problem is reduced to the question of how one can introduce two different kinds of subtractions. A clue to this question is already found in the Lagrangian theory. Suppose that the nucleon and the pion are elementary and that the deuteron is composite, then the Lagrangian theory implies that all parameters related to the deuteron, such as the rest mass, magnetic moment, must be determined as functions of more fundamental parameters such as the nucleon mass, the pion mass, and the pion-nucleon coupling constant. Therefore no arbitrary parameters should appear for composite particles. Thus we are led to the following definition:

If no arbitrary parameters are introduced through the parametric dispersion relations for all the $\mathcal{G}$ functions involving a special field operator φ_c for a stable particle c, we call c a composite particle. A stable particle which is not composite is called an elementary particle.

This definition does not necessarily mean that a $\mathcal{G}$ function involving φ_c always satisfies a nonsubtracted dispersion relation, i.e., take the two-point $\mathcal{G}^{(2)}$ function for the c field, then we need two subtractions in the parametric dispersion relation for $\mathcal{G}^{(2)}$. In this case, however, the subtraction constants are not arbitrary, but they are determined subject to the renormalization condition[9]:

$$\lim_{p^2+m_c^2\to 0}\frac{\mathcal{G}^{(2)}(p^2)}{p^2+m_c^2}=-1, \tag{4.1}$$

where we assumed that c is a scalar particle for the sake of simplicity. Another example is the electromagnetic interaction which will be discussed in the next section.

Except for special cases as discussed above, however, this definition implies nonsubtracted parametric dispersion relations for $\mathcal{G}$ functions involving composite-particle variables.

Since very little is known about $\mathcal{G}$ functions, it will be instructive to recapitulate some properties of the vertex functions which are very closely related to the $\mathcal{G}$ functions. Take, for simplicity, the neutral scalar theory; then the ρ functions and ρ' functions are defined by

$$\begin{aligned}\langle 0|T[\varphi(x_1)\cdots\varphi(x_n)]|0\rangle_{\text{conn}}\\ &=\int d^4y_1\cdots d^4y_n\,\Delta_F(x_1-y_1)\cdots\\ &\qquad\times\Delta_F(x_n-y_n)\rho(y_1\cdots y_n)\\ &=\int d^4y_1\cdots d^4y_n\,\Delta_F'(x_1-y_1)\cdots\\ &\qquad\times\Delta_F'(x_n-y_n)\rho'(y_1\cdots y_n),\end{aligned} \tag{4.2}$$

where Δ_F' denotes the Feynman propagation function with radiative corrections. The Fourier transform of ρ is the $\mathcal{G}$ function, and the Fourier transform of ρ' defines the vertex function Γ. Both $\mathcal{G}$ and Γ are functions of scalar products of four-momenta, and in particular they are equal on the mass shell, i.e.,

$$\mathcal{G}(p_\alpha p_\beta)=\Gamma(p_\alpha p_\beta),\text{ for } p_1^2+m^2=\cdots=p_n^2+m^2=0. \tag{4.3}$$

We know some interesting properties of the Γ's from which we can conjecture on the properties of $\mathcal{G}$'s.

Let us first consider a three-point vertex function $\Gamma(p_1^2,p_2^2,p_3^2)$ and put two of the p^2's on the mass shell; then we have a function of a single invariant variable. What we know about is the vertex function of this type.

Determination of the Binding Energy

Let us consider the vertex operator corresponding to

$$n+p\to d, \tag{4.4}$$

where n, p, and d denote neutron, proton, and deuteron, respectively. Putting n and d on the mass shell, we get a vertex function which depends only on

$$x=-(p_d-p_n)^2, \tag{4.5}$$

where p_d and p_n are the four-momenta of the deuteron and the neutron on the mass shell. Assuming a nonsubtracted dispersion relation for $\Gamma(x)$, Blankenbecler and Cook[10] have shown that the binding energy of the deuteron can be determined in terms of other fundamental parameters of the theory such as the rest masses of the nucleon and the pion, and the pion-nucleon coupling constant.

This result suggests the following possibility: If a composite-particle field is specified by nonsubtracted dispersion relations, we will get only the free-field solution—but not necessarily so for other elementary particle fields specified by subtracted dispersion relations—unless certain relationships among rest masses and subtraction constants are satisfied. In other words, we have a possibility of determining the rest mass of the composite particle in terms of other parameters in the theory.

[9] See Eq. (4.12) in reference 3.

[10] R. Blankenbecler and L. F. Cook, Jr., Phys. Rev. **119**, 1745 (1960).

Problem of Higher Spins

It has long been conjectured that the spins of elementary particles should be either 0 or $\frac{1}{2}$ and that consequently particles with higher spins would be composite. We shall discuss here the connection between this conjecture and our definition of composite particles.

Let us first refer to the work of Lehmann, Symanzik, and Zimmermann (LSZ) on the vertex function.[11]

As an example we consider the interaction between the nucleon and neutral pseudoscalar meson fields. The vertex function ρ_5' in position space is defined by

$$\langle 0|T[\varphi(x)\psi(y)\bar{\psi}(z)]|0\rangle=\int d^4\xi d^4\eta d^4\zeta \times S_F'(y-\eta)\rho_5'(\eta\zeta:\xi)S_F'(\zeta-z)\cdot\Delta_F'(\xi-x), \quad (4.6)$$

where S_F' and Δ_F' are the Feynman propagation functions for the nucleon and the meson, respectively. The vertex function Γ_5 in momentum space is defined by

$$\rho_5'(yz:x)=\frac{-i}{(2\pi)^8}\int d^4p_1 d^4p_2 \times e^{ip^1(y-x)+ip^2(x-z)}\Gamma_5(p_1,p_2), \quad (4.7)$$

Γ_5 is a function of p_1^2, p_2^2, and $(p_1-p_2)^2$ multiplied by certain invariants. If we put $ip_1\gamma+M=ip_2\gamma+M=0$, Γ_5 is a function of $\kappa^2=-(p_1-p_2)^2$ alone, and we can put

$$\Gamma_5=i\gamma_5 f(\kappa^2). \quad (4.8)$$

It has been concluded by LSZ that this function f must satisfy

$$\frac{1}{8\pi^2}\int_{4M^2}^{\infty}\frac{\kappa(\kappa^2-4M^2)^{\frac{1}{2}}}{(\kappa^2-m^2)^2}|f(\kappa^2)|^2 d\kappa^2<1. \quad (4.9)$$

We replace the nucleon field by a charged scalar field and define the vertex function ρ' by

$$\langle 0|T[\varphi(x)\Phi(y)\Phi^*(z)]|0\rangle=\int d^4\xi d^4\eta d^4\zeta \times\Delta_F'(y-\eta)\rho'(\eta\zeta:\xi)\Delta_F'(\zeta-z)\cdot D_F'(\zeta-z), \quad (4.10)$$

where Δ_F' and D_F' refer to the charged scalar nucleon and the neutral meson, respectively. We then define the vertex function Γ from ρ' by Eq. (4.7). Then put $p_1^2+M^2=p_2^2+M^2=0$, where M is the rest mass of the charged scalar nucleon, and Γ turns out to be a function of $\kappa^2=-(p_1-p_2)^2$, $f(\kappa^2)$.

Corresponding to Eq. (4.9), we can derive

$$\frac{1}{16\pi^2}\int_{4M^2}^{\infty}\frac{(\kappa^2-4M^2)^{\frac{1}{2}}}{\kappa(\kappa^2-m^2)^2}|f(\kappa^2)|^2 d\kappa^2<1. \quad (4.11)$$

What we can see from these results is the following: We can always derive inequalities of the form

$$\int c(\kappa^2)|f(\kappa^2)|^2 d\kappa^2<1, \quad (4.12)$$

and the behavior of the coefficient $c(\kappa^2)$ for large values of κ^2 is completely determined by the spins of the particles on the mass shell. The function $c(\kappa^2)$ in Eq. (4.11) is simply given by

$$c(\kappa^2)=\sigma^{(2)}(\kappa^2)/g^2, \quad (4.13)$$

where $\sigma^{(2)}(\kappa^2)$ is the second-order weight function in the meson propagator given by Eq. (5.8) of reference 3.

For large values of κ^2, $c(\kappa^2)$ is asymptotically given, apart from trivial numerical factors, by

$$\begin{aligned} &c(\kappa^2)\sim\kappa^{-2} \quad \text{for a spinor nucleon,}\\ &c(\kappa^2)\sim\kappa^{-4} \quad \text{for a scalar nucleon.}\end{aligned} \quad (4.14)$$

The function $c(\kappa^2)$ is determined completely kinematically as shown by Eq. (4.14) and we can make a general statement that the power of κ in $c(\kappa^2)$ increases as the spins of particles on the mass shell increase. This implies in turn that the vertex operator $f(\kappa^2)$ must fall off more rapidly as the spins of particles increase. Thus for vertex operators involving particles of sufficiently high spins we shall not need any subtractions in the dispersion relations for an arbitrary choice of particles on the mass shell. Thus if a high spin particle is involved in a $\mathcal{G}$ function, we shall not need any subtraction in the parametric dispersion relation in view of the close connection between the vertex functions and the $\mathcal{G}$ functions. From our definition of composite particles it follows that particles with higher spins would necessarily be composite. In this way we understood a possible connection between the conjecture on higher spins and our definition of composite particles, although nothing could be proved rigorously.

Furthermore, since composite particles are distinguished from elementary particles by the absence of subtractions in the dispersion relations, we might guess that the cross sections of reactions involving composite particles would behave at high energies in quite a different way from those involving elementary particles alone so that there would be an experimental means to make a distinction between elementary and composite particles by observing their high-energy behaviors.

V. ELEMENTARITY OF THE PHOTON

Based on the definition of composite particles given in the previous section, we shall study the elementarity of various particles. Since so little is known about strong interactions, however, we shall confine ourselves to the discussion of the electromagnetic interactions. Study of the electromagnetic interactions alone is

[11] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento 2, 425 (1955).

already very useful and bears a very rich content for the present purpose. The electromagnetic interactions are characterized among other things by the W-T equations on which we base our arguments.

First let us write down one of the W-T equations for a charged scalar particle either elementary or composite:

$$\Box_z \frac{\partial}{\partial x_\mu}\langle 0|T[A_\mu(x)\varphi(y)\varphi^*(z)]|0\rangle = e\Delta_F'(y-z)[\delta(x-y)-\delta(x-z)], \quad (5.1)$$

where e is the charge of the quantum belonging to the field, and Δ_F' is the Feynman propagator of this charged scalar field. The function corresponding to this electromagnetic vertex is defined by

$$(-i)^3\Box_x K_y K_z\langle 0|T[A_\mu(x)\varphi(y)\varphi^*(z)]|0\rangle = \rho_\mu(x;yz). \quad (5.2)$$

Then this ρ function satisfies the equation

$$\frac{\partial}{\partial x_\mu}\rho_\mu(x;yz) = ie[K_z\Delta_F'(x-z)\cdot K_y\delta(x-y) - K_y\Delta_F'(x-y)\cdot K_z\delta(x-z)]. \quad (5.3)$$

We rewrite this equation in momentum space and get

$$i(q-p)_\mu \mathcal{G}_\mu(p,q) = ie\left[\left(1+(q^2+m^2)\int\frac{\sigma(\kappa^2)d\kappa^2}{q^2+\kappa^2-i\epsilon}\right)(p^2+m^2) - \left(1+(p^2+m^2)\int\frac{\sigma(\kappa^2)d\kappa^2}{p^2+\kappa^2-i\epsilon}\right)(q^2+m^2)\right], \quad (5.4)$$

where σ is the Källén-Lehmann weight function for Δ_F' and $\mathcal{G}_\mu$ is defined by

$$\rho_\mu(x;yz) = \frac{-i}{(2\pi)^8}\int d^4p d^4q\, e^{ip(y-x)+iq(x-z)}\mathcal{G}_\mu(p,q). \quad (5.5)$$

Next we decompose $\mathcal{G}_\mu$ into a sum of invariants. There are only two linearly independent vectors p_μ and q_μ, but it is more convenient to take $(p+q)_\mu$ and $(p-q)_\mu$ and the basic invariants. Furthermore, from the C or CP invariance of the theory one can easily see that $\mathcal{G}_\mu$ must be symmetric in p and q. Therefore, $(p-q)_\mu$ must appear always multiplied by a factor (p^2-q^2). Then it is not difficult to verify that the general form of $\mathcal{G}_\mu$ is given by

$$\mathcal{G}_\mu(p,q) = (p+q)_\mu\mathcal{G}_a + [(p+q)_\mu(p-q)^2 - (p-q)_\mu(p^2-q^2)]\mathcal{G}_b, \quad (5.6)$$

where $\mathcal{G}_a$ and $\mathcal{G}_b$ are the functions of scalar products p^2, q^2 and $(p-q)^2$ alone, and symmetric in p and q. It must be remarked here that the second invariant is so chosen that the product of the second invariant with $(q-p)_\mu$ vanishes.

Inserting the representation (5.6) into Eq. (5.4), one finds

$$\mathcal{G}_a = -e\left[1+(p^2+m^2)(q^2+m^2)\times\int\frac{\sigma(\kappa^2)d\kappa^2}{(p^2+\kappa^2-i\epsilon)(q^2+\kappa^2-i\epsilon)}\right]. \quad (5.7)$$

The function $\mathcal{G}_a(p^2,q^2)$ clearly cannot satisfy a nonsubtracted parametric dispersion relation, but it satisfies a parametric dispersion relation with one subtraction. The boundary condition to determine the subtraction constant is given by

$$\mathcal{G}_a(p^2=-m^2,\ q^2=-m^2) = -e. \quad (5.8)$$

A similar argument can be presented in the case of a charged spinor field.

From the above discussions we can draw a conclusion that the photon is an elementary particle as a consequence of the definition of composite particles given in the previous section.[12] As for the charged scalar particle discussed above, however, one cannot immediately draw the same conclusion as we shall see below.

The problem that we have to discuss is concerned with the arbitrariness of the subtraction constant e. This subtraction constant is nothing but the charge and we have to be very careful about the arbitrariness of this quantity. In Sec. III we have shown that a linear relation

$$e_c = e_a + e_b \quad (5.9)$$

holds if $\langle 0|T[\varphi_a\varphi_b]|c\rangle \neq 0$. This relation shows that at least one charge is not arbitrary but determined by the charges of other particles. Therefore, what we have to do is to exhaust linear relations of this kind and find the number of linearly independent charges. Take, for instance, the charges of the proton, neutron and neutral K meson as the basic charges, then the charge of a strongly interacting particle is given by[13]

$$Q = e_1 I_3 + e_2 N/2 + e_3 S/2, \quad (5.10)$$

where

$$e_1 = e(p) - e(n),\quad e_2 = e(p)+e(n), \quad (5.11)$$

and

$$e_3 = e_1 + e(K^0).$$

This means that within the approximation of retaining only the strong interactions there must be at least three independent subtraction constants. This will further mean that there must be at least three elementary particles among the strongly interacting particles although we cannot tell which three would be elementary.[14]

[12] Rigorously speaking this proves just the elementarity of the scalar and longitudinal photons. We made an implicit assumption here that the elementarity of the scalar and longitudinal photons implies the elementarity of the transverse photons.

[13] If we assume $e_1 = e_2 = e_3 = e$ as implied by experiments we get the familiar strangeness formula $Q = e(I_3 + N/2 + S/2)$.

[14] One possibility is to take n, p; and Λ as the elementary particles. S. Sakata, Progr. Theoret. Phys. (Kyoto) **16**, 686 (1956).

If we introduce weak interactions we immediately find that $e(K^0)=0$, or $e_3=e_1$, from $K^0\to 2\pi^0$ or $\pi^++\pi^-$. Thus the number of independent charges is reduced to 2 if we do not take account of leptons.

Thus we have to modify the previous result into the following: There must be at least two elementary particles among the strongly interacting particles.[15] The reduction of the minimum number of elementary particles corresponds to the very unlikely possibility that one out of three fundamental particles in the strong interactions might be a composite particle formed by means of weak interactions. In view of the very weakness of weak interactions and the different transformation properties of strong and weak interactions in both charge and position spaces, it will not be unreasonable to conclude that the minimum possible number of elementary particles among strongly interacting particles is three.

If we further introduce leptons, we get one or two additional neutrino charges depending on whether the neutrino has two components or four components, e.g., (1) two-component theory:

$$e(e^-)=e(\mu^-)=e(\nu)-e_1,$$

(2) four-component theory:

$$e(e^-)=e(\nu_1)-e_1,\quad e(\mu^-)=e(\nu_2)-e_1,$$

where ν_1 and ν_2 denote two different two-component neutrinos. We assumed the lepton conservation in each case.[16]

In the former case there must be at least one elementary lepton, and in the latter case at least two. In view of the weakness of the leptonic interactions, however, it is very likely that all leptons are elementary.

In order to reduce the number of linearly independent charges to one, as demanded by experiments, we perhaps have to introduce very weak interactions that violate all the conservation laws except for the conservation of charge,[17] such as those interactions proposed by Yamaguchi.[18]

To conclude, we have seen how useful the W-T equations are for the discussion of the elementarity of various particles.

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[15] The number 2 stems from the two conservation laws, i.e., conservation of charge and baryon number.

[16] K. Nishijima, Phys. Rev. **108**, 907 (1957).

[17] A similar consideration has been made by G. Feinberg (unpublished).

[18] Y. Yamaguchi, Progr. Theoret. Phys. (Kyoto) **22**, 373 (1959).

APPENDIX. CONSTRUCTION OF THE COMPOSITE-PARTICLE FIELD OPERATORS

In the text a method of constructing a field operator for a given stable particle has been given by Eqs. (3.3) and (3.4). The field operators constructed in this way satisfy (1) the Lorentz invariance, (2) the microcausality, and (3) the asymptotic condition and hence the unitarity. The last two properties are clear from the construction of the field operator φ_c as has been discussed in reference 7, but the first one may not be very clear at a glance. It is the purpose of this Appendix to show the Lorentz invariance of the composite-particle field operators. Assuming for simplicity that particles a, b, and c are all spinless, we shall show that the operator $\varphi_c(x)$ in Eq. (3.3) does not depend on the direction of the vector P, the energy-momentum vector of the composite particle c.

First we study how to carry out the limiting process in Eq. (3.3). When both the denominator and numerator have finite limits separately the problem is rather simple, so let us assume that both are singular at the origin $|\xi|\to 0$. In order to study the nature of the singularity at the origin we shall appeal to the integral representation of the Feynman amplitudes.[19]

The denominator $f(\xi,P)$ will be expressed by

$$f(\xi,P)=\frac{1}{(2\pi)^4}\int d^4p\, e^{ip\xi}g(p,P), \qquad \text{(A.1)}$$

and

$$g(p,P)=\int_{-1}^{1}d\zeta\int_0^\infty ds\frac{\sigma(\zeta,s)}{\left[\left(p-\zeta\frac{P}{2}\right)^2+s-i\epsilon\right]^N}, \qquad \text{(A.2)}$$

where N is a certain positive integer. By integration by parts with respect to s one can reduce the power N to unity, but it is not possible in general to increase the power N beyond a certain maximum value. From now on we understand that N always denotes this maximum value. With the help of the integral representation (A.2) one can study the singularity of the function $\lim_{\xi_0\to 0}f(\xi,P)$ at the origin $|\xi|=0$. They are given, respectively, by

$$\begin{aligned}&(1)\ N=1\quad f(\xi,P)=\lim_{\xi_0\to 0}f(\xi,P)\sim c_1/\xi^2,\\ &(2)\ N=2\quad f(\xi,P)\sim c_2\ln|\xi|,\\ &(3)\ N\geq 3\quad f(\xi,P)\sim\text{finite}.\end{aligned} \qquad \text{(A.3)}$$

In the first case the limiting value of the ratio will be given by the ratio of the following two expressions:

$$\lim_{|\xi|\to 0}\xi^2\lim_{\xi_0\to 0}\langle|T[\varphi_a(x+\tfrac{1}{2}\xi),\varphi_b(x-\tfrac{1}{2}\xi)]|\rangle,$$

[19] G. C. Wick, Phys. Rev. **96**, 1124 (1954); R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954); M. Ida, Progr. Theoret. Phys. (Kyoto) **23**, 1156 (1960). Here we adopted the result of the last author.

and

$$\lim_{|\xi|\to 0} \xi^2 \lim_{\xi_0\to 0} f(\xi,P). \tag{A.4}$$

Or, in order to maintain the formal relativistic invariance in an explicit manner we can take the ratio of

$$\lim_{|\xi|\to 0} \lim_{\xi_0\to 0} \xi^2\langle|T[\varphi_a(x+\tfrac{1}{2}\xi), \varphi_b(x-\tfrac{1}{2}\xi)]|\rangle$$

to

$$\lim_{|\xi|\to 0} \lim_{\xi_0\to 0} \xi^2 f(\xi,P), \tag{A.5}$$

where $\xi^2=\boldsymbol{\xi}^2-\xi_0^2$.

Similarly in the second case we take the ratio of

$$\lim_{|\xi|\to 0} \lim_{\xi_0\to 0} \xi_\mu\frac{\partial}{\partial\xi_\mu}\langle|T[\varphi_a(x+\tfrac{1}{2}\xi), \varphi_b(x-\tfrac{1}{2}\xi)]|\rangle$$

to

$$\lim_{|\xi|\to 0} \lim_{\xi_0\to 0} \xi_\mu\frac{\partial}{\partial\xi_\mu} f(\xi,P). \tag{A.6}$$

The double limiting procedure can be carried out in the following fashion: Assume the Fourier representation of a function $f(\xi)$ to be

$$f(\xi)=\frac{1}{(2\pi)^4}\int d^4p\, e^{ip\xi}g(p); \tag{A.7}$$

then the limit is given by

$$\lim_{|\xi|\to 0} \lim_{\xi_0\to 0} f(\xi)=\frac{1}{(2\pi)^4}\int d^3p\left(\int dp_0 g(p)\right). \tag{A.8}$$

Therefore, if $g(p)$ has a representation of the type (A.2) or generally a Feynman type denominator, the problem is reduced to the evaluation of a Feynman integral.

(1) $N=1$. Instead of multiplying ξ^2 by $f(\xi)$, one can apply the differential operator $-(\partial/\partial p_\mu)^2$ on $g(p)$ and utilize

$$\begin{aligned}\int d^4p\left(\frac{\partial}{\partial p_\mu}\right)^2\frac{1}{[(p+a)^2+m^2-i\epsilon]^N}&=-2i\pi^2, \quad \text{for } N=1,\\ &=0, \quad \text{for } N>1. \end{aligned}\tag{A.9}$$

(2) $N=2$. The operator $\xi_\mu(\partial/\partial\xi_\mu)$ on $f(\xi)$ can be replaced by $(\partial/\partial p_\mu)p_\mu$ on $g(p)$, and one can utilize

$$\begin{aligned}\int d^4p\frac{\partial}{\partial p_\mu}\left[\frac{p_\mu}{[(p+a)^2+m^2-i\epsilon]^N}\right]&=2i\pi^2, \text{ for } N=2,\\ &=0, \quad \text{for } N>2,\\ &=\infty, \quad \text{for } N=1. \end{aligned}\tag{A.10}$$

For $N\geq 3$ we utilize

$$\int\frac{d^4p}{[(p+a)^2+\Lambda-i\epsilon]^N}=\frac{i\pi^2}{(N-1)!(\Lambda-i\epsilon)^{N-2}}. \tag{A.11}$$

In any case it will be clear that the results no longer depend on the direction of P but generally only on $P^2=-m_c^2$. Thus the independence of the operator $\varphi_c(x)$ on the choice of the vector P, or in other words the Lorentz invariance of $\varphi_c(x)$, has been established.

Perhaps it will be instructive to give a simple example to illustrate the above statement. Since it is very hard, however, to illustrate this by a true composite particle c, we shall take an elementary particle as c. As has been discussed in reference 7, the formula (3.3) is applicable to a stable elementary particle as well as to a real composite particle. So we consider the interaction between a charged scalar nucleon field ψ and a neutral scalar meson field φ, and we shall check as a simple example the relation[20]

$$\begin{aligned}\lim_{|\xi|\to 0} \lim_{\xi_0\to 0}&\frac{\langle 0|T[\psi(x+\frac{1}{2}\xi)\psi^*(x-\frac{1}{2}\xi)\psi(y)\psi^*(z)]|0\rangle_{\text{conn}}}{(2q_0V)^{\frac{1}{2}}\langle q|T[\psi(\frac{1}{2}\xi)\psi^*(-\frac{1}{2}\xi)]|0\rangle}\\ &=\langle 0|T[\varphi(x)\psi(y)\psi^*(z)]|0\rangle, \end{aligned}\tag{A.12}$$

where $|q\rangle$ is a one-meson state with the energy-momentum q. As a model we shall take the interaction

$$H_{\text{int}}=g\psi^*\psi\varphi, \tag{A.13}$$

and check the relation (A.12) in the perturbation theory.

In the lowest order perturbation theory we get

$$\begin{aligned}&\langle q|T[\psi(x)\psi^*(y)]|0\rangle\\ &\quad=-i\int d^4z\langle q|\varphi(z)|0\rangle K_z^m\langle 0|T[\psi(x)\psi^*(y)\varphi(z)]|0\rangle\\ &\quad=-i\int d^4z\langle q|\varphi(z)|0\rangle g\Delta_F(x-z)\Delta_F(z-y), \end{aligned}$$

where $K_z^m=\Box_z-m^2$, and m is the meson rest mass.

Inserting $\langle q|\varphi(z)|0\rangle=e^{-iqz}/(2q_0V)^{\frac{1}{2}}$ into the above equation, one finds

$$\begin{aligned}\langle q|T[\psi(\tfrac{1}{2}\xi)\psi^*(-\tfrac{1}{2}\xi)]|0\rangle&=\frac{ig}{(2\pi)^4(2q_0V)^{\frac{1}{2}}}\int d^4k\\ &\times\frac{e^{ik\xi}}{[(k+\frac{1}{2}q)^2+M^2-i\epsilon][(k-\frac{1}{2}q)^2+M^2-i\epsilon]}, \end{aligned}\tag{A.14}$$

where M denotes the nucleon rest mass. This expression diverges logarithmically at the origin, and therefore

[20] *Note added in proof*. This equation is true only in the lowest order. In more general cases one applies the Klein-Gordon operator on the variable x and then puts the momentum canonically conjugate to x on the mass shell in order to get the generally valid relation.

we shall evaluate the following limit:

$$\lim_{|\xi|\to 0}\lim_{\xi_0\to 0}\xi_\mu\frac{\partial}{\partial\xi_\mu}(2q_0V)^{\frac{1}{2}}\langle q|T[\psi(\tfrac{1}{2}\xi)\psi^*(-\tfrac{1}{2}\xi)]|0\rangle$$
$$=\frac{-ig}{(2\pi)^4}\int d^4k\frac{\partial}{\partial k_\mu}\times\left[\frac{k_\mu}{((k+\frac{1}{2}q)^2+M^2-i\epsilon)((k-\frac{1}{2}q)^2+M^2-i\epsilon)}\right]$$
$$=\frac{2g\pi^2}{(2\pi)^4}. \tag{A.15}$$

This is the coefficient of $\ln|\xi|$ at the origin. Next we evaluate

$$\lim_{|\xi|\to 0}\lim_{\xi_0\to 0}\xi_\mu\frac{\partial}{\partial\xi_\mu}\times\langle 0|T[\psi(x+\tfrac{1}{2}\xi)\psi^*(x-\tfrac{1}{2}\xi)\psi(y)\psi^*(z)]|0\rangle_{\text{conn}}.$$

In the lowest order perturbation theory this is given by[21]

$$\langle 0|T[\psi(x+\tfrac{1}{2}\xi)\psi^*(x-\tfrac{1}{2}\xi)\psi(y)\psi^*(z)]|0\rangle_{\text{conn}}$$
$$=(-ig)^2\int d^4u d^4v[\Delta_F(x+\tfrac{1}{2}\xi-u)\Delta_F(x-\tfrac{1}{2}\xi-u)\times D_F(u-v)\Delta_F(y-v)\Delta_F(z-v)+\Delta_F(y-u)\Delta_F(x-\tfrac{1}{2}\xi-u)D_F(u-v)\times\Delta_F(x+\tfrac{1}{2}\xi-v)\Delta_F(z-v)]. \tag{A.16}$$

One can check that only the first term contributes when we take the limits, and we get

$$\lim_{|\xi|\to 0}\lim_{\xi_0\to 0}\xi_\mu\frac{\partial}{\partial\xi_\mu}\langle 0|T[\cdots]|0\rangle_{\text{conn}}$$
$$=(-ig)^2\int d^4u\lim_{|\xi|\to 0}\lim_{\xi_0\to 0}\xi_\mu\frac{\partial}{\partial\xi_\mu}\times\Delta_F(x+\tfrac{1}{2}\xi-u)\Delta_F(x-\tfrac{1}{2}\xi-u)\times\int d^4v D_F(u-v)\Delta_F(y-v)\Delta_F(z-v). \tag{A.17}$$

[21] Δ_F denotes the nucleon propagator, and D_F refers to the meson.

Now we use the formula

$$\Delta_F(x+\tfrac{1}{2}\xi-u)\Delta_F(x-\tfrac{1}{2}\xi-u)=-\frac{1}{(2\pi)^8}\int d^4P d^4p\times\frac{e^{iP(x-u)+ip\xi}}{[(p+\frac{1}{2}P)^2+M^2-i\epsilon][(p-\frac{1}{2}P)^2+M^2-i\epsilon]},$$

then we find

$$\lim_{|\xi|\to 0}\lim_{\xi_0\to 0}\xi_\mu\frac{\partial}{\partial\xi_\mu}\Delta_F(x+\tfrac{1}{2}\xi-u)\Delta_F(x-\tfrac{1}{2}\xi-u)$$
$$=\frac{1}{(2\pi)^8}\int d^4P\, e^{iP(x-u)}\int d^4p\frac{\partial}{\partial p_\mu}\times\left[\frac{p_\mu}{((p+\frac{1}{2}P)^2+M^2-i\epsilon)((p-\frac{1}{2}P)^2+M^2-i\epsilon)}\right]$$
$$=\frac{1}{(2\pi)^8}\int d^4P\, e^{iP(x-u)}2i\pi^2$$
$$=\frac{2i\pi^2}{(2\pi)^4}\delta(x-u). \tag{A.18}$$

Inserting this result into (A.17), we get

$$\lim_{|\xi|\to 0}\lim_{\xi_0\to 0}\xi_\mu\frac{\partial}{\partial\xi_\mu}\langle 0|T[\cdots]|0\rangle_{\text{conn}}\Big/\frac{2g\pi^2}{(2\pi)^4}$$
$$=(-ig)^2\frac{2i\pi^2}{(2\pi)^4}\int d^4v\, D_F(x-v)\times\Delta_F(y-v)\Delta_F(z-v)\Big/\frac{2g\pi^2}{(2\pi)^4}$$
$$=-ig\int d^4v\, D_F(x-v)\Delta_F(y-v)\Delta_F(z-v)$$
$$=\langle 0|T[\varphi(x)\psi(y)\psi^*(z)]|0\rangle.$$

Thus Eq. (A.12) has been verified.

In a similar way the statement on the W-T equations in Sec. III can be checked in the perturbation theory when c is an elementary particle.