Mandelstam Representation with Anomalous Thresholds

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It is proved that fourth-order diagrams provide necessary and sufficient conditions for the Mandelstam representation to be valid for every finite order in perturbation theory.

INTRODUCTION

We have previously shown[1,2] that a sufficient condition for the validity of the Mandelstam representation in every order of perturbation theory is the absence of anomalous thresholds. In this paper we prove that the representation is valid in every order if it is valid for the lowest order diagrams. This includes some processes which have anomalous thresholds. These conditions are both necessary and sufficient for the Mandelstam representation to apply within the framework of renormalized perturbation theory.

From our result, in conjunction with the known properties of the fourth order,[3] it follows that the Mandelstam representation applies, for example, to the scattering processes[4]

$$\pi+X\to\pi+X,$$

where X can be a Λ, Σ, or Ξ particle. It also applies for the processes

$$\begin{aligned}
&\pi+\Lambda\to K+p, \quad \text{or} \quad \pi+\Sigma, \quad \text{or} \quad K+\Xi,\\
&\pi+\Sigma\to K+p, \quad \text{or} \quad K+\Xi,\\
&\pi+\Xi\to K+\Lambda, \quad \text{or} \quad K+\Sigma,\\
&\pi+d\to p+n.
\end{aligned}$$

All the above processes have anomalous thresholds but they do not lead to complex singularities on the physical sheet. Examples of scattering processes which do lead to complex singularities and for which the Mandelstam representation does not apply are given by

$$\begin{aligned}
\Sigma^++\Sigma^- &\to \Sigma^++\Sigma^-,\\
d+d &\to d+d,
\end{aligned}$$

or the scattering of any pair of compound particles. For each reaction, when drawing the perturbation diagrams, one must of course take account of conservation laws.

The method in this paper is based on analytic continuation in the external masses from values for which there are no anomalous thresholds. We show firstly that no complex singularities can appear in the physical sheet until a Landau curve of singularities has a point at which it is degenerate. This point will in general be a double point of the Landau curve in the real (s,t) plane. The second step is to prove using dual diagrams that, as the external masses are increased a double point will arise in a diagram of low order before it arises in higher orders. This completes the proof.

2. FOURTH-ORDER DIAGRAM

The physical sheet for a scattering amplitude $A(z_1,z_2)$ consists of a product of cut planes in the complex variables z_1, z_2, z_3 that correspond to the real Mandelstam variables s, t, u. The physical sheet is defined so that in a suitable limit on its boundary in a physical scattering region the amplitude $A(s,t)$ is the Feynman amplitude. We shall use the general term *Landau curve*[5] for the manifold in the complex (z_1,z_2) space given by a solution of the equations,

$$\textit{either} \qquad \partial D(\alpha,z_1,z_2)/\partial\alpha_i=0, \tag{2.1}$$

$$\textit{or} \qquad \alpha_i=0, \tag{2.2}$$

for each i, where D is the denominator of the Feynman parametrization of the amplitude,

$$A(z_1,z_2)=c_1\int_0^1 d\alpha_1\cdots d\alpha_n\frac{n(\alpha)\delta(1-\sum\alpha_i)}{D(\alpha,z_1,z_2)^p}. \tag{2.3}$$

The function D is homogeneous in the α variables so Eqs. (2.1) and (2.2) imply also that D is zero.

These equations define the two-dimensional manifolds in complex (z_1,z_2) space on which the amplitude may have singularities in the physical sheet and on which the amplitude is indeed singular on many other

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[1] R. J. Eden (to be published); and Phys. Rev. Letters **5**, 213 (1960).

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[4] A table of the numerical values of angles required to decide whether a given process satisfies the Mandelstam representation has been given by L. B. Okun and A. P. Rudik, Nuclear Phys. **15**, 261 (1960).

[5] L. D. Landau, Nuclear Phys. **13**, 181 (1959).

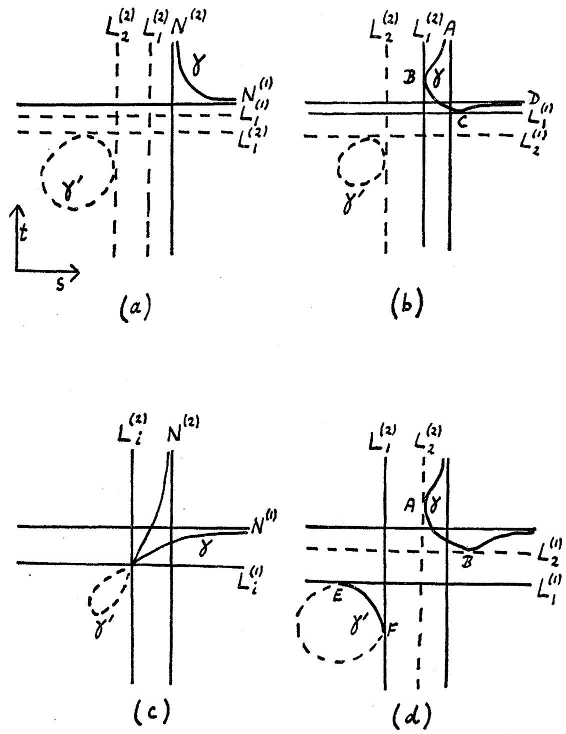

FIG. 1. Part of the real section of the Landau curve for various external masses: (a) no anomalous thresholds; (b) anomalous thresholds but no complex singularities; (c) the "transition point" at which the Landau curve has a double point; (d) Landau curve with complex singularities. (The parts drawn in broken lines are not singular on the boundary of the physical sheet.) *Note.* In Fig. 1(a), the right-hand side, the bottom notation should read $L_2^{(1)}$.

Riemann sheets.[6] The points of these manifolds for which the variables s and t are real define real curves and lines but again these are not necessarily singularities of the amplitude on the physical sheet. The remaining points of the manifolds are complex surfaces that join these real curves.

The key question is not where the Landau curves are located, but in what circumstances they give complex singularities on the physical sheet. We shall show that as the external masses are increased the transition point beyond which part of the complex surface of the Landau curve becomes singular on the physical sheet can be related to the occurrence of a double point in the real section of the Landau curve lying on the boundary of the physical sheet. We shall illustrate this by first considering the fourth-order diagram as the external masses are varied. This discussion of the fourth-order term is contained entirely in the work of previous authors,[3] but we repeat it here in order to note those aspects that we are able to consider in the general term in the next section.

When there is no anomalous threshold, part of the real section of the leading Landau curve has the form γ and γ' illustrated in Fig. 1(a). [The leading Landau curve for any diagram is that for which Eq. (2.1) rather than (2.2) is satisfied for each i.] There are no anomalous thresholds when the external masses are sufficiently small. The two arcs γ and γ' lie in the real s, t plane. The curve γ is on the boundary of the physical sheet and represents points at which the amplitude $A(z_1,z_2)$ is singular in each of the following limits taken from its continuation on the physical sheet,

$$z_1=s+i\epsilon\to s,\quad z_2=t+i\epsilon\to t, \tag{2.4a}$$

$$z_1=s-i\epsilon\to s,\quad z_2=t-i\epsilon\to t. \tag{2.4b}$$

The curve γ' is not singular in any limit from the physical sheet.

The real arcs γ and γ' are joined by a complex surface in (z_1,z_2) space. By considering the intersection of this connecting surface with a real searchline it can be seen that on the surface z_1 and z_2 have imaginary parts of opposite sign. The limit onto the real section of the Landau curve that is defined by giving z_1, z_2 small imaginary parts whose relative sign is *the same as that which they take on the attached surface* will be called the *appropriate limit*. When their relative sign is opposite to that taken on the attached surface, it will be called the *inappropriate limit*. If the complex surface is not singular on the physical sheet, then the appropriate limit cannot be singular. For the arc γ in Fig. 1(a), the limits (2.4) are inappropriate limits. The surface joined to γ,γ' is singular on the unphysical sheet reached from the physical sheet by going through a branch cut either in z_1 or in z_2.

The lines N in Fig. 1(a) denote normal thresholds at which the amplitude is singular on the physical sheet, and are the Landau curves for the reduced diagrams which represent self-energy parts. The lines L arise from vertex parts corresponding to reduction of one line in the fourth-order diagram. These are assumed in Fig. 1(a) not to give singularities on the physical sheet, but are singular when reached by taking z_1 or z_2 through the branch cut that starts at a normal threshold. Thus there are no anomalous thresholds, and also there are no complex singularities on the physical sheet.

Now increase the external masses continuously, and at each set of values consider the form of the Landau curve shown in Fig. 1(a). The lines $L_1^{(1)}$ and $L_1^{(2)}$ move towards $N^{(1)}$ and $N^{(2)}$. When $L_1^{(i)}$ meets $N^{(i)}$ it becomes singular on the physical sheet and as the external masses are further increased it moves back again, remaining singular on the physical sheet. It is then an anomalous threshold. When $L_1^{(i)}$ coincides with $N^{(i)}$ the arc γ touches both lines at infinity. As $L_1^{(i)}$ moves away the curve γ continues to touch $L_1^{(i)}$ and the point of contact moves in from infinity to finite values as shown in Fig. 1(b). The part of γ between B and C is singular in the inappropriate limit taken on the boundary of the physical sheet, and the parts AB, CD are singular in *their* inappropriate limit on the physical sheet. The part of the

[6] J. C. Polkinghorne and G. R. Screaton, Nuovo cimento **15**, 289 and 925 (1960).

surface connecting γ' to the part BC of γ is still singular only in unphysical sheets, but now it can be reached by going through the cut attached to $L_1^{(1)}$ or through the cut attached to $L_1^{(2)}$. Similarly the parts of the complex surface connected to the other parts of γ are not singular on the physical sheet.

When the external masses are further increased, there comes a point where $L_1^{(1)}$ coincides with $L_2^{(1)}$ and $L_1^{(2)}$ simultaneously coincides with $L_2^{(2)}$. Although geometrically coincident with $L_1^{(i)}$ the line $L_2^{(i)}$ will still not be singular in the physical sheet. At the point of coincidence the curve has a point of degeneracy at the intersection of the anomalous thresholds, as shown in Fig. 1(c).

A further increase in the external masses leads to the situation shown in Fig. 1(d). Now the arc γ' is singular on the physical sheet from any limit. It is attached to a complex surface which is singular in the physical sheet and which connects it to the part AB of γ in Fig. 1(d). This complex surface leads to a breakdown of the Mandelstam representation.

The significant point is that in changing from a surface that is nonsingular in the physical sheet to a surface that is singular in the physical sheet the part of the complex surface in question must shrink to zero. This requires that the Landau curve be degenerate at the transition. Further, the point of degeneracy occurs at the intersection of two anomalous threshold lines. We shall show in the next section that this is true also for a general diagram.

The condition on the masses for the degeneracy to occur in fourth order has been given.[3] Labeling the masses as in Fig. 2, angles θ_i are defined by

$$\cos\theta_i = \frac{m_i^2 + m_{i+1}^2 - M_i^2}{2m_i m_{i+1}}, \quad m_5 = m_1. \tag{2.5}$$

Degeneracy occurs when

$$\sum_{i=1}^{4} \theta_i = 2\pi. \tag{2.6}$$

3. GENERAL CASE

In considering the general Feynman diagram it is supposed that all lower order diagrams satisfy the Mandelstam representation. If this were not so, then it would be necessary to consider instead how one of these lower order diagrams had come to have complex singularities in the physical sheet. We consider the manner in which Landau curves change as the squares of the external masses are increased through real values from an initial set of values for which the Mandelstam representation is known to hold. Such a set could be any that ensured there were no anomalous thresholds.[1,2]

As the external masses are increased, anomalous thresholds will appear in the physical sheet. The first of these appears when part of the Landau curve for a single loop vertex diagram moves up towards a singular normal threshold and emerges through the branch point at the normal threshold onto the boundary of the physical sheet. It is then "reflected" from the normal threshold as the external masses are further increased and is a singular anomalous threshold. Other anomalous thresholds may arise in the same way or by lines corresponding to more complicated vertex diagrams being reflected off lower order anomalous thresholds that are already singular on the boundary of the physical sheet. The point through which a new singularity appears is always a singularity of a simpler (reduced) diagram. This is the theorem on continuity of singular curves that was used in previous papers.[1,2]

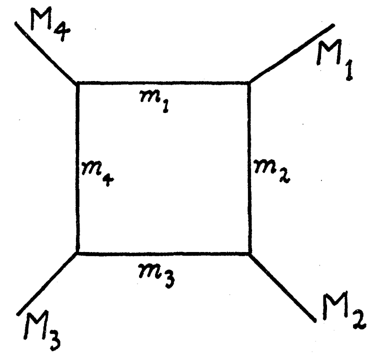

FIG. 2. Notation for the fourth-order diagram.

The appearance of anomalous threshold singularities on the physical sheet can give rise to two effects. The first concerns the curves of virtual singularities. A virtual singularity is an arc of the real part of a Landau curve that lies in a region of the real plane where cuts corresponding to two variables overlap, and that is singular when approached from the physical sheet in the inappropriate limit. It has been shown[1,2] that in the absence of anomalous threshold singularities these curves are arcs of negative slope (taking as variables the two variables that correspond to the two overlapping cuts), and they have the normal thresholds that generate the cuts as their asymptotes. In the presence of anomalous thresholds but in the continued absence of complex singularities, this behavior is modified. The curves can have parts of positive slope but their turning points can only occur where the Landau curve touches an anomalous threshold singularity in such a way that the point of contact corresponds to the same critical values of the Feynman parameters for both the Landau curve and line. This follows from the fact that the behavior in the inappropriate limit on one side of a turning point is the same as the behavior in the appropriate limit on the other side of the turning point unless a singularity generating one of the cuts occurs at the turning point. Except in this latter case it is possible to move along the complex surface of the Landau curve from the complex neighborhood of one side of the turning point to the complex

neighborhood of the other side of the turning point so that the path followed changes from one Riemann sheet to another in a way that just corresponds to the difference between appropriate and inappropriate limits. Since all appropriate limits are nonsingular when there are no complex singularities, the turning points of curves of virtual singularities must be at contacts with anomalous threshold singularities.

The second possible effect of the appearance of anomalous threshold singularities on the physical sheet is the development of complex singularities. The complex surface of the Landau curve is divided into a number of primitive sections[2] by its intersection with the cuts bounding the physical sheet. It is known that if a point of any primitive section is singular on the physical sheet then the whole primitive section is singular.[1,2] However, as the external masses are increased, the complex singularities must appear in the physical sheet in a continuous way. This gradual appearance can only be achieved by the gradual shrinking to a point of one of the initially nonsingular primitive sections which then re-emerges as a primitive section that is singular in the physical sheet.

A primitive section is the complex surface spanning two real arcs of the Landau curve whose end points are turning points.[6a] When the primitive section shrinks to a point and re-emerges in the physical sheet, these arcs must also shrink to the same point and then expand. The process must follow exactly the same pattern that is illustrated for the fourth order in Fig. 1(a)–(d). It follows that the critical situation, past which complex singularities appear, occurs when the real Landau curve has a double point in a manner similar to that already described for the fourth order case.

The complex singularities have entered the physical sheet from neighboring Riemann sheets. The curves of virtual singularities on the boundaries of the physical sheet represent the presence of complex singularities in neighboring Riemann sheets. Therefore one of the two coalescing arcs must be part of a curve of virtual singularities and hence the two tangents at the double point must always be singular anomalous threshold lines, as is the case in fourth order. This is confirmed by considering the arcs of positive slope that are joined to AB in Fig. 1(d). These arcs are singular in the inappropriate limit when AB bounds a complex surface in the physical sheet. If they were not singular in the same limit before the double point was reached, the whole of these curves would suddenly become singular when the double point appeared. This would contradict the fact that analytic properties change continuously as a continuation is made in the external masses.

The foregoing argument establishes that if the Mandelstam representation is valid for certain values of the squares of the external masses, it remains valid as these squares of masses are varied through real values until one or more curves of virtual singularities acquire a double point. Since a double point can occur only when there are anomalous thresholds in two directions (at least), this discussion enables us to deduce the validity of the Mandelstam representation for all values of the external masses in the absence of anomalous thresholds provided it holds for some range of these variables. We use here the fact that any range of the external masses not giving anomalous thresholds is accessible from any other such range by continuous variation without passing through a region in which there are anomalous thresholds.

In the next section we show that the onset of complex singularities in the physical sheet, which is heralded by a double point in the relevant Landau curve, normally occurs *first* in the fourth-order diagram as the external masses are increased. In cases where selection rules prohibit the fourth-order diagram it may be necessary to consider more than one of the lowest order diagrams.

4. APPEARANCE OF A DOUBLE POINT

In this section we show that there will be a double point on the relevant Landau curve in lowest order for values of the external mass that are never larger than those that are required to give a double point in higher orders. There are, of course, many different values of the external masses at which a double point will appear. Our method of increasing the external masses is to vary them one at a time up to the values in which we are interested. Our terms "smaller" and "greater" apply to the single external mass that is being varied when the transition through a double point takes place.

The method is based on the dual diagram analysis[5,7] of the singularities of a Feynman amplitude. The dual diagram is essentially the vector diagram for the internal and external four-momenta of the corresponding Feynman diagram, each momentum being on the mass shell. It may conveniently be drawn in a complex Euclidean space. We consider first the occurrence of anomalous thresholds for the vertex parts given by reducing scattering diagrams. (Anomalous thresholds do not arise from reduction to generalized self-energy parts.[8]) A vertex dual diagram must be drawn in a plane in order to give an anomalous threshold,[8] (or, more accurately, dual diagrams not in a plane do not give any new anomalous threshold). It follows[7] that for fixed masses the anomalous threshold (if any) from the lowest order vertex occurs at a smaller value of the appropriate variable (s, t, or u) than any such threshold from a higher order vertex part. This result comes from the fact that for an anomalous threshold the dual diagram must be real and all its internal lines must lie inside the

[6a] *Note added in proof.* This statement requires modification if cusps or double points appear. This is found to happen in a particular sixth order contraction but only for values of the external masses well above the fourth order limit. This will be discussed in a further paper.

[7] J. C. Taylor, Phys. Rev. **117**, 261 (1960).
[8] P. V. Landshoff, Nuclear Phys. **20**, 129 (1960).

triangle formed by the external momenta. If the lowest-order diagram does not yield an anomalous threshold nor does any other.[2]

The dual diagram for a general scattering diagram is drawn in three dimensions. It gives the relation between s and t which is the equation of the Landau curve. The external momenta together with $\sqrt{s}$ and $\sqrt{t}$ form the framework of a tetrahedron. Within this tetrahedron are four parts each consisting of one triangular face and connected internal lines that will also appear in a similar manner in the dual diagram for the vertex part formed by reducing lines in the scattering diagram. However, for the scattering dual diagram each of these parts does not in general lie in a plane as it would for a vertex part. Exceptionally for particular values of s and t one of these parts may lie in a plane. This means that for these values of s and t the Landau curve for the scattering diagram touches the Landau curve for the corresponding vertex part, giving the same values of the parameters α on each curve. Examples of such points of contact are given in Fig. 1(b) by the points B and C.

For the Landau curve simultaneously to have such contact with two perpendicular anomalous thresholds as in Fig. 1(c), the corresponding two parts of the dual diagram must be simultaneously coplanar. Further, these two parts share at least one external line and one internal point so that in fact they must lie in the same plane. This situation is illustrated for the fourth order (Fig. 2) by the dual diagram drawn in Fig. 3. Here AB, BC, CD, DA represent the external lines of the Feynman graph, while the lines converging at O represent the internal lines. The values of s and t are given by the squares of the lengths AC, BD. The point (s,t) at which the Landau curve has contact with the Landau curve for the vertex diagram obtained by reducing the line corresponding to OD in the scattering diagram is obtained by requiring $OABC$ to be coplanar. If also $OBCD$ is coplanar the whole dual diagram will lie in a plane. The two vertex diagrams in question are actually singular if O lies within the triangles ABC, BCD. The condition for this is that the sum of the angles at O is 2π, which is just the known condition [Eq. (2.6)] for the Mandelstam representation to be about to break down.

Now consider the addition of internal lines to the scattering diagram, beginning from a fourth-order diagram which has just reached this critical point, as indicated by taking Fig. 3 to lie in a plane. We keep the external masses M_1, M_2, M_3 fixed; this means AB, BC, CD are of fixed length. Then the addition of internal lines such that ABC and the related internal lines still correspond to an anomalous threshold cannot shorten the length AC. Similarly, if on adding internal lines BD is still to give an anomalous threshold with dual diagram BCD, this length also will not be decreased. If, in addition, we require the more complicated diagram to lead to a Landau curve with a double

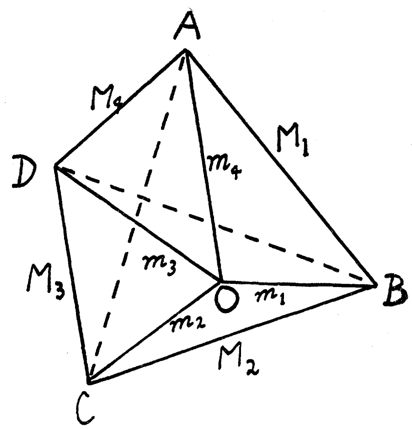

FIG. 3. The dual diagram for fourth-order scattering, drawn in a plane when a double point occurs in the Landau curve.

point, $ABCD$ must lie in the same plane. An increase of AC and an increase of BD with given AB, BC, CD can only increase the length of AD. This length defines the mass M_4 and the increase is what we set out to prove. Thus only by an increase of an external mass beyond the degeneracy for the lowest order diagram can we obtain a degeneracy for a higher order diagram.

5. REPRESENTATION WITH ANOMALOUS THRESHOLDS

We have seen that under the condition obtained from Eq. (2.6) there is cut-plane analyticity for every term in the perturbation series for the scattering amplitude. Thus a repeated application of Cauchy's theorem gives the Mandelstam representation. The weight functions ρ_1, ρ_2, ρ_3 appearing in it have the form,

$$\rho_1 = \lim_{\epsilon,\epsilon'\to 0}\{[A(s+i\epsilon,\,t+i\epsilon')-A(s-i\epsilon,\,t+i\epsilon')] \\ -[A(s-i\epsilon,\,t+i\epsilon')-A(s-i\epsilon,\,t-i\epsilon')]\}, \quad (5.1)$$

with two analogous expressions for ρ_2, ρ_3.

The region in which ρ_1 is nonzero must be bounded by a curve on which the right-hand side of Eq. (5.1) is singular, and therefore must be composed of portions of Landau curves that represent singularities on the boundaries of the physical sheet in one of the limits shown in Eq. (2.4) or in the other limits. In fact, since each of the square brackets in Eq. (5.1) contains the difference between the two types of limit the boundary of the region must be singular in only one type of limit. It is therefore composed of pieces of Landau curves which are singular in the inappropriate limit, that is by arcs of virtual singularity. (At the normal and the anomalous thresholds, in s for example, the value of z_2 does not affect the singularity. Thus normal and anomalous thresholds are singular in *all* limits and cannot form part of the boundary of the region in which the spectral function is nonzero.)

We have remarked in Sec. 3 that, when there are no anomalous thresholds, the arcs of virtual singularity

can have no turning points. However, when anomalous thresholds are present turning points may occur at contacts with the anomalous thresholds. Thus, for example, when there are anomalous thresholds in two directions (but no complex singularities on the physical sheet) the boundary of the region for ρ_1 for the fourth-order contribution is the arc γ, which includes parts of positive slope as well as of negative slope [Fig. 1(b)].

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Effects of K-Meson Interactions on the Nucleon Anomalous Magnetic Moment

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A previous calculation of the physical nucleon wave function in static source pion theory is extended by including the pseudoscalar K-meson interactions, the motivation being that this should increase the scalar part of the nucleon anomalous magnetic moment, thus improving the result of previous calculations. In constructing a trial function for the physical nucleon, the method of moments was used and terms containing up to three mesons were included. Calculation shows that a too strong K-meson coupling is detrimental to the vector part, and that the scalar part can be increased approximately by 10% if the K-meson interaction is made moderately low.

INTRODUCTION

In a recent paper[1] (hereafter referred to as I), the method of moments was applied to the problem of the physical nucleon in pion theory, using the Chew-Low-Wick static source Hamiltonian.[2] An approximate ground-state nucleon wave function was constructed, with terms containing as many as five virtual pions. One of the important results was that the average number of pions in the cloud is more than might be suggested by the success of the one-meson approximation. When the electromagnetic properties of the nucleon were calculated from the wave functions, however, hardly any significant improvement was achieved over the results obtained by the one-meson approximation.

On the other hand, we are aware of the possible contributions of the strange particles to the physical nucleon state, since these particles are known to interact strongly with nucleons and mesons. In particular, conservation of strangeness allows nucleons to emit a virtual K^+ but not K^-. Thus the inclusion of interactions involving pseudoscalar K mesons should have the consequence of increasing μ_S, the scalar part of the nucleon anomalous magnetic moment.[3] This implies an improvement because pseudoscalar pion theories invariably give too large a negative value for μ_S. This motivates us to extend the work of I by including strange particle interactions, even though a fixed-source approximation is less justified for K mesons than for pions.

Another difficulty involved here is that we still do not know definitely the relative parities or intrinsic spins of these particles. In this paper we assume that different baryons are different states of the same fermion, all having spin $\frac{1}{2}$ and the same intrinsic parity, and that the K mesons are pseudoscalar with respect to the baryons. The static version of the relativistic interactions[4] consistent with the Gell-Mann-Nishijima isospin assignments will be used.[5] In this scheme, the interaction Hamiltonian H_I contains eight coupling constants and as many cutoff functions. Even though there is no good reason to keep these parameters from changing for different processes, simplicity is of great importance in the present work since the number of terms contributing to the wave function increases quite rapidly as the order of approximation increases. In this spirit, the same cutoff function will be used uniformly for the pionic and K-mesonic contributions for all the baryons. Throughout this paper, we employ a square cutoff at $K=6$ ($\hbar=c=\mu=1$, where μ is the pion mass) so that comparison with I will be possible. In the same spirit, the square of the unrationalized, unrenormalized $NN\pi$ coupling constant will take the values 0.2, 0.4, and 0.6. The other coupling constants are assumed to satisfy the following relations;

$$(f_{NN\pi}{}^0)^2=(f_{\Sigma\Lambda\pi}{}^0)^2=(f_{\Sigma\Sigma\pi}{}^0)^2=(f_{\Xi\Xi\pi}{}^0)^2\equiv(f_\pi{}^0)^2, \quad (1a)$$

and

$$(f_{N\Lambda K}{}^0)^2=(f_{N\Sigma K}{}^0)^2=(f_{\Lambda\Xi K}{}^0)^2=(f_{\Sigma\Xi K}{}^0)^2\equiv(f_K{}^0)^2. \quad (1b)$$

One of the consequences of these simplifications is that invariance under rotations in this representation

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