Time-Ordered Green's Functions and Perturbation Theory*

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A formulation of field theories based on the generalized unitarity condition and parametric dispersion relations is presented. In the perturbation theory we discuss the connection between the present scheme and the Lagrangian theory and derive the renormalizability condition in our formulation. Finally we show for typical processes in the first, second, third, and fourth orders that our theory can reproduce the renormalized Feynman perturbation theory.

I. INTRODUCTION

IN the conventional renormalizable field theories we start from a symbolic Hamiltonian involving divergent counter-terms such as the self-energy to cancel divergences inherent in the theory, and by the application of the renormalization prescription we get convergent expressions for the scattering matrix and expectation values of observable quantities. It is, however, unavoidable that we meet various divergences in the course of the calculation. For this reason we reformulate renormalizable field theories in such a way that one can reproduce the renormalizable field theories without encountering any divergences in the course of the calculation. One possible way to develop this idea is to find and exhaust all possible relationships among finite renormalized expressions. The essential part of this program has been undertaken in a previous paper by one of the authors,[1] and we shall show in this paper that we can really reproduce the renormalized Feynman calculation in our scheme.

Our formulation is, as we shall see later, essentially a kind of S-matrix approach proposed by Heisenberg.[2] When Heisenberg first discussed the properties of the S matrix, he gave as the fundamental properties of the S matrix (1) the unitarity and (2) the Lorentz invariance. Although all the S matrices satisfy these two properties, they do not exhaust all the properties of the S matrix that we need. Recently it has been suggested by Mandelstam[3] that the combination of the dispersion relations (or the analyticity properties) and unitarity would determine the dynamical structure of the S-matrix elements in the lower configurations. This is indeed a very powerful approach for practical calculations as stressed by Chew,[4] but we propose here a different approach for the following reasons: (1) In order to fix the Born terms in the dispersion relations we need a more fundamental theory than the combination of unitarity and dispersion relations. (2) It is very hard to exhaust all the dispersion relations that are needed to determine the complete dynamics.

In order to overcome these difficulties we generalize the S matrix so as to include the matrix elements off the mass shell. Then it is possible to find a simple set of dispersion relations, and the virtues of this approach are (1) that we need not assume Born terms, and (2) that we can always write down a dispersion relation for the S matrix element in an arbitrary configuration. Furthermore, under certain conditions unitarity and this new set of dispersion relations seem to *exhaust* all the possible relationships among the renormalized S-matrix elements. This can be verified by reproducing the renormalized perturbation theory from our scheme.

We shall first discuss the two fundamental postulates of our theory on which we base our calculations. First in Sec. II we introduce the unitarity condition which is generalized to accommodate those matrix elements which are off the mass shell. In Sec. III we introduce the parametric dispersion relations which determine the dynamics of the system. In Sec. IV we discuss the physical meaning of the subtractions in the parametric dispersion relations and the renormalizability condition. Finally, in Sec. V we shall discuss the reproduction of the Feynman perturbation theory. It will be shown by a direct calculation that the conventional perturbation theory can be reproduced to the fourth order.

II. GENERALIZED UNITARITY CONDITION

The unitarity condition is certainly one of the most important properties of the S matrix in any theory, and in this paper we try to further generalize the unitarity condition in order to exhaust all the available properties of the S matrix.

Put $S=1+T$, then the unitarity condition is given by

$$T+T\dagger+TT\dagger=0. \tag{2.1}$$

To be more precise, let us consider the elastic scattering of two particles below the threshold energy for other inelastic channels; then Eq. (2.1) is written more precisely as

$$T(p'q':pq)+T\dagger(p'q':pq)+\sum_{p''q''}T(p'q':p''q'')T\dagger(p''q'':p'q')=0, \tag{2.2}$$

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[1] K. Nishijima, Phys. Rev. **119**, 485 (1960). This paper will be referred to as A hereafter.

[2] W. Heisenberg, Z. Physik **120**, 513, 673 (1943); Z. Naturforsch. **1**, 608 (1946). See also C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **23**, No. 1 (1945); **24**, No. 19 (1946).

[3] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741, 1752 (1959).

[4] G. F. Chew, to be published).

where the four-momenta p's and q's satisfy the mass shell conditions

$$p^2+m^2=p'^2+m^2=p''^2+m^2=0,\quad q^2+\mu^2=q'^2+\mu^2=q''^2+\mu^2=0. \tag{2.3}$$

We can generalize Eq. (2.2) by lifting the mass shell conditions for four momenta occurring in the initial and final states but still retaining this condition for intermediate states, i.e.,

$$p''^2+m^2=q''^2+\mu^2=0. \tag{2.4}$$

We call this modified condition the generalized unitarity condition. Since the generalized unitarity condition is one of our fundamental postulates, we shall briefly show that it is always satisfied in the conventional theory.[5]

First, the S-matrix elements either on the mass shell or off the mass shell are given by the Fourier transform of a set of functions of the form

$$(-i)^n K_{x_1}\cdots K_{x_n}\langle 0|T[\varphi(x_1)\cdots\varphi(x_n)]|0\rangle, \tag{2.5}$$

where K is the Klein-Gordon operator for the field φ. We have taken for simplicity the neutral scalar field.

If we define the generalized S-matrix elements in this way, we can prove the generalized unitarity condition from the asymptotic condition. For this proof we start from an algebraic identity

$$\sum_{\text{comb}}(-i)^k i^{n-k}T[\varphi(x_1')\cdots\varphi(x_k')]\times\tilde{T}[\varphi(x_{k+1}')\cdots\varphi(x_n')]=0, \tag{2.6}$$

where $\tilde{T}$ is the antitime-ordered product symbol, and we sum over all possible combinations to divide n variables $x_1, x_2, \cdots, x_n$ into two groups $x_1'\cdots x_k'$ and $x_{k+1}'\cdots x_n'$.

For the sake of completeness we shall give the derivation of Eq. (2.6). We first define a functional U by

$$U=T\exp\left(-i\int\varphi(x)Q(x)d^4x\right), \tag{2.7}$$

where Q is a real c-number source. Then U is a generating functional of the T products and we get

$$\left(\frac{\delta^n U}{\delta Q(x_1)\cdots\delta Q(x_n)}\right)_{Q=0}=(-i)^n T[\varphi(x_1)\cdots\varphi(x_n)]. \tag{2.8}$$

U is unitary and its Hermitian conjugate $U^\dagger$ is given by

$$U^\dagger=\tilde{T}\exp\left(i\int\varphi(x)Q(x)d^4x\right). \tag{2.9}$$

Differentiating the unitarity equation,

$$UU^\dagger=1, \tag{2.10}$$

with respect to $Q(x_1), \cdots, Q(x_n)$ and putting $Q=0$ enables us to get Eq. (2.6).

If we take the vacuum expectation value of Eq. (2.6), we get

$$(-i)^n\langle 0|T[\varphi(x_1)\cdots\varphi(x_n)]|0\rangle+i^n\langle 0|\tilde{T}[\varphi(x_1)\cdots\varphi(x_n)]|0\rangle+\sum_{\text{comb}}{}'\sum_{\alpha}(-i)^k i^{n-k}\langle 0|T[\varphi(x_1')\cdots\varphi(x_k')]|\alpha\rangle\times\langle\alpha|\tilde{T}[\varphi(x_{k+1}')\cdots\varphi(x_n')]|0\rangle=0, \tag{2.11}$$

where $\sum'$ denotes summation over all possible combinations excluding $k=0$ and $k=n$. Next we use the asymptotic condition in order to express the vacuum $-\alpha$ element of T products in terms of the vacuum expectation values.[6] Namely we make use of

$$\langle 0|T[\varphi(x_1)\cdots\varphi(x_n)]|p_1, p_2, \cdots, p_m, +\rangle=(-i)^m\int d^4z_1\cdots d^4z_m\langle 0|\varphi(z_1)|p_1\rangle\cdots\langle 0|\varphi(z_m)|p_m\rangle\times K_{z_1}\cdots K_{z_m}\langle 0|T[\varphi(x_1)\cdots\times\varphi(x_n)\varphi(z_1)\cdots\varphi(z_m)]|0\rangle. \tag{2.12}$$

A similar relation for $\tilde{T}$ products is obtained by taking the complex conjugate of the above equation.

We now define the τ functions by

$$\tau(x_1,\cdots,x_n)=(-i)^n K_{x_1}\cdots K_{x_n}\langle 0|T[\varphi(x_1)\cdots\varphi(x_n)]|0\rangle; \tag{2.13}$$

then the Fourier transform of a τ function expresses an S-matrix element. Now, inserting Eq. (2.12) into Eq. (2.11), we find a set of coupled equations for τ functions:

$$\tau(x_1\cdots x_n)+\tau^*(x_1\cdots x_n)+\sum_{\text{comb}}{}'\sum_{l=0}^{\infty}\frac{i^l}{l!}\int(du)(dv)\tau(x_1'\cdots x_k'u_1\cdots u_l)\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\times\tau^*(x_{k+1}'\cdots x_n'v_1\cdots v_l)=0, \tag{2.14}$$

where $(du)=d^4u_1\cdots d^4u_l$, $(dv)=d^4v_1\cdots d^4v_l$, and τ^* is the complex conjugate of τ, and

$$i\Delta^{(+)}(u-v)=\sum_p\langle 0|\varphi(u)|p\rangle\langle p|\varphi(v)|0\rangle=\frac{1}{(2\pi)^3}\int d^4p\, e^{ip(u-v)}\theta(p_0)\delta(p^2+m^2). \tag{2.15}$$

Equation (2.14) stands precisely for the generalized unitarity condition. If we take its Fourier transform and put all the four-momenta on the mass shell we find the ordinary unitarity condition, but if this mass shell con-

[5] A similar condition was discussed by Cutkosky based on the graphical technique. R. E. Cutkosky, J. Math. Phys. (to be published); Phys. Rev. Letters 4, 624 (1960).

[6] K. Nishijima, Phys. Rev. 111, 995 (1958). See also reference 1.

dition is lifted we obtain the generalized unitarity condition.

The generalization of this condition to systems consisting of many kinds of particles is obvious. In such a case we give a field operator to each stable particle no matter whether that particle is elementary or composite. In the latter case we understand the field operator in the sense of Zimmermann,[7] Haag,[8] and one of the authors.[6]

III. PARAMETRIC DISPERSION RELATIONS

The generalized unitarity condition is certainly useful and fundamental, but it is kinematical in nature and does not allow one to determine the dynamical structure of the S matrix if there were no other conditions. As a matter of fact, if we know the S matrix up to a certain order in the coupling constant, the unitarity condition would allow one to determine the absorptive part of the S matrix in the next order. In order to determine the dispersive part of the S matrix in this order, however, we need another condition, i.e., dispersion relations. We study here what kind of dispersion relations would be sufficient for this purpose. A universal kind of dispersion relations is obtained by extending the S matrix to those values of four-momenta which are off the mass shell.

In the previous paper A, an auxiliary set of functions was introduced in addition to the τ functions. Namely, the Feynman diagrams contributing to a τ function are in general disconnected, and we call the contributions from connected Feynman diagrams the ρ function.

The relation between two sets is given by

$$\tau(xx_1\cdots x_n)=\rho(xx_1\cdots x_n)+\sum_{\text{comb}}\rho(xx_1'\cdots x_k')\tau(x_{k+1}'\cdots x_n'), \quad (3.1)$$

where the summation is extended over all possible ways to divide $x_1\cdots x_n$ into two groups excluding $k=n$. This formula enables us to express τ in terms of ρ's and *vice versa*. Thus it is possible to write down the generalized unitarity condition in terms of ρ functions as discussed in A. In the case of a free field, ρ's are given by

$$\begin{aligned}\rho(x_1x_2)=\tau(x_1x_2)&=(-i)^2K_{x_1}K_{x_2}\Delta_F(x_1-x_2)\\&=-iK_{x_1}\delta(x_1-x_2),\end{aligned} \quad (3.2)$$

$\rho(x_1x_2\cdots x_n)=0,\quad \text{for}\quad n>2.$

The unitarity condition in terms of the ρ's looks like

$$\begin{aligned}&\rho(x_1\cdots x_n)+\rho^*(x_1\cdots x_n)\\&\quad+\sum_{\text{comb}}\sum_{l=1}^{\infty}\frac{i^l}{l!}\int(du)(dv)[\tau(x_1'\cdots x_k'u_1\cdots u_l)\\&\quad\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\\&\qquad\times\tau^*(x_{k+1}'\cdots x_n'v_1\cdots v_l)]_{\text{conn}}=0,\end{aligned} \quad (3.3)$$

where the subscript "conn" means to omit all the contributions to the vacuum expectation value arising from disconnected Feynman diagrams. Now let us study how one can solve this equation in the perturbation theory. If we know the ρ functions up to a certain order in the perturbation expansion one can immediately calculate $\rho+\rho^*$ to the next order. So the determination of ρ in this order can be gotten by giving a prescription of how to calculate $\mathrm{Im}\rho$ when $\mathrm{Re}\rho$ is known. The solution of this problem was studied in a previous paper A. First, introduce the Fourier transform of ρ by[9]

$$\begin{aligned}\rho(x_1\cdots x_n)&=\frac{-i}{(2\pi)^{4(n-1)}}\int d^4p_1\cdots d^4p_n\,\delta(p_1+\cdots+p_n)\\&\qquad\times e^{i(p_1x_1+\cdots+p_nx_n)}\mathcal{G}(p_1\cdots p_n).\end{aligned} \quad (3.4)$$

Then in general $\mathcal{G}$ is a function of scalar products of p's and has the following integral representation:

$$\mathcal{G}(p_\alpha p_\beta)=\int\frac{\sigma(c_{\alpha\beta},M^2)dc_{\alpha\beta}dM^2}{(\sum c_{\alpha\beta}p_\alpha p_\beta+M^2-i\epsilon)^N}, \quad (3.5)$$

where σ is a real weight function of real variables c's and a positive variable M^2. From this integral representation one can derive

$$\mathrm{Re}\mathcal{G}(p_\alpha p_\beta\cdot\xi)=\frac{P}{\pi}\left[\int_0^\infty-\int_{-\infty}^0\right]\frac{d\xi'}{\xi'-\xi}\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi'), \quad (3.6)$$

where ξ is a common scaling parameter to be multiplied into all the scalar products. We call this equation the parametric dispersion relation. The integral representation (3.5), however, is not the most general form and one has to add a polynomial of scalar products to (3.5). In such a case we have to make subtractions, i.e.,

$$\begin{aligned}&\left(\frac{d}{d\xi}\right)^s\mathrm{Re}\mathcal{G}(p_\alpha p_\beta\cdot\xi)\\&\qquad=s!\frac{P}{\pi}\left[\int_0^\infty-\int_{-\infty}^0\right]\frac{d\xi'}{(\xi'-\xi)^{s+1}}\mathrm{Im}\mathcal{G}(p_\alpha p_\beta\cdot\xi').\end{aligned} \quad (3.7)$$

As we shall see in the next section, the number of subtractions in each dispersion relation fixes the dynamics of the system under consideration.

We have shown that the two conditions, (1) generalized unitarity and (2) parametric dispersion relations, are the consequences of the present field theory, but from now on let us take these conditions as the fundamental postulates; that is we shall not try to establish these conditions based on another set of axioms. Then what we have to do is to show that these two postulates under certain conditions exhaust all possible relationships among finite renormalized expressions. This will be

[7] W. Zimmermann, Nuovo cimento **10**, 597 (1958).
[8] R. Haag, Phys. Rev. **112**, 669 (1958).
[9] From this definition the Fourier transform of $\mathrm{Im}\rho$ (or $\mathrm{Re}\rho$) is given by $\mathrm{Re}\mathcal{G}$ (or $\mathrm{Im}\mathcal{G}$).

illustrated by reproducing the renormalized perturbation theory from these two postulates.

Finally, let us briefly discuss the connection between the present postulates and the others. First, the generalized unitarity condition is the substitute for the asymptotic condition as is clear from its derivation. Next the Lorentz invariance and microscopic causality condition are already assumed implicitly in the assumption that $\mathcal{G}$ is a function of scalar products of p's, since ρ or τ would be an invariant function of the x's only when the T product can be defined independently of the choice of time axis. Therefore, in the derivation of our basic postulates we have already assumed the current postulates in the conventional axiomatic field theory. Once these two postulates are employed, however, one can forget the definition of the τ functions (2.13), through which our approach is related to the conventional field theory. If we do so, our theory turns out to be an S-matrix theory and we no longer have field operators.

IV. SIGNIFICANCE OF SUBTRACTIONS

The subtractions in the parametric dispersion relations are necessary in order to introduce interactions. In this section we shall discuss the correspondence between the subtractions and the Lagrangian theory. This problem has already been discussed in the previous paper A, but for the sake of completeness we shall recapitulate the results.

We first pick out terms linear in the coupling constant from the ρ equations. Then the generalized unitarity condition assures us of the equation

$$\mathrm{Re}\rho_1(x_1\cdots x_n)=0, \tag{4.1}$$

since the nonlinear terms vanish in this order due to the relation

$$\int d^4u\ \rho_0(x,u)\Delta^{(+)}(u-v)=-iK_x\Delta^{(+)}(x-v)=0.$$

In the momentum representation we get

$$\mathrm{Im}\mathcal{G}_1(p_1\cdots p_n)=0. \tag{4.2}$$

The nonsubtracted dispersion relation leads to

$$\mathrm{Re}\mathcal{G}_1(p_1\cdots p_n)=0, \tag{4.3}$$

and we cannot introduce interactions as far as perturbation theory is concerned.

In order to introduce interactions we have to use subtracted parametric dispersion relations for certain $\mathcal{G}$ functions. If we make one subtraction for the n-point $\mathcal{G}$ function, we get from Eqs. (3.7) and (4.2) the equation

$$\frac{d}{d\xi}\mathrm{Re}\mathcal{G}_1(p_\alpha p_\beta\cdot\xi)=0. \tag{4.4}$$

The solution of this equation is given by

$$\mathrm{Re}\mathcal{G}_1(p_\alpha p_\beta\cdot\xi)=g,\ \text{const},$$

and consequently

$$\mathcal{G}_1(p_\alpha p_\beta)=g. \tag{4.5}$$

There is another possible set of solutions like

$$\mathrm{Re}\mathcal{G}_1(p_\alpha p_\beta\cdot\xi)=c_{\alpha\beta}p_\alpha p_\beta\cdot\xi/d_{\alpha\beta}p_\alpha p_\beta\cdot\xi, \tag{4.6}$$

which is certainly independent of ξ and hence satisfies (4.4). We exclude, however, solutions like (4.6) for the reason to be stated below.

We get the same result as (4.5) from the conventional theory if we take an interaction of the form

$$(g/n!)\varphi(x)^n, \tag{4.7}$$

whereas the solution (4.6) can never be reproduced by the conventional local field theory. This means that the solution (4.6) certainly violates the microscopic causality condition and gives rise to certain unphysical singularities in the scattering amplitude. For this reason we employ only the simplest solution (4.5).

Thus we have introduced interactions through subtractions. In general, when higher order corrections are taken into account we determine this constant of integration by an appropriate boundary condition. For the three-point function, for instance, we use the boundary condition

$$\mathrm{Re}\mathcal{G}(p_1^2=-m^2,\ p_2^2=-m^2,\ p_3^2=-m^2)=g, \tag{4.8}$$

which defines the conventional renormalized coupling constant.

Next let us examine two subtractions, then we get from the equation

$$\frac{d^2}{d\xi^2}\mathrm{Re}\mathcal{G}_1(p_\alpha p_\beta\cdot\xi)=0, \tag{4.9}$$

the general solution

$$\mathrm{Re}\mathcal{G}_1(p_\alpha p_\beta\cdot\xi)=g+h_1\sum_{\alpha>\beta}p_\alpha p_\beta\cdot\xi+h_2\sum_\alpha p_\alpha^2\cdot\xi. \tag{4.10}$$

This solution corresponds, apart from trivial numerical factors, to the following interactions in the conventional field theory:

$$g\varphi^n,\ h_1\varphi^{n-2}(\partial\varphi/\partial x_\mu)^2,\ h_2\varphi^{n-1}\Box\varphi. \tag{4.11}$$

From these results we draw the conclusion that the subtractions in the parametric dispersion relations serve to introduce interactions and that the number of subtractions determines the type of interactions in the corresponding conventional field theory.

In particular, for two-point functions it is clear that we have to make two subtractions since the free Lagrangian is quadratic and involves second order derivatives. This is seen from (4.11) with $n=2$. This same conclusion is drawn from the Källén-Lehmann

representation.[10] The boundary condition in this case is

$$\lim_{p^2+m^2\to 0}\frac{\mathcal{G}(p^2)}{p^2+m^2}=-1. \tag{4.12}$$

So far we treated only scalar theory and it would be instructive to give a simple example of interacting fields with spin. For this purpose let us take a nucleon interacting with a neutral pseudoscalar field.

We first introduce the three-point ρ function by

$$\rho(x,y,z)=(-i)K_x\cdot iD_y \cdot(-i)\tilde{D}_z\langle 0|T[\varphi(x)\psi(y)\bar{\psi}(z)]|0\rangle, \tag{4.13}$$

where $D_y=\gamma\partial_y+m$, $\tilde{D}_z=\gamma^T\partial_z-m$, and m is the nucleon mass. The first thing we do is to expand ρ into a sum of all possible invariants[11]:

$$\rho(xyz)=i\gamma_5\rho_a(xyz)+i\gamma_5\gamma_\mu\frac{\partial}{\partial x_\mu}\rho_b(xyz) +i\gamma_5\sigma_{\mu\nu}\frac{\partial^2}{\partial y_\mu\partial z_\nu}\rho_c(xyz). \tag{4.14}$$

Now introduce the Fourier transforms of ρ functions, $\mathcal{G}_a$, $\mathcal{G}_b$, and $\mathcal{G}_c$. If we make one subtraction for $\mathcal{G}_a$ and fix the constant of integration by

$$\mathcal{G}_a(q^2=-\mu^2,\ p_1{}^2=-m^2,\ p_2{}^2=-m^2)=f, \tag{4.15}$$

this corresponds to the pseudoscalar coupling $if\,\bar{\psi}\gamma_5\psi\varphi$. On the other hand, one subtraction for $\mathcal{G}_b$ with the boundary condition

$$\mathcal{G}_b(q^2=-\mu^2,\ p_1{}^2=-m^2,\ p_2{}^2=-m^2)=-g \tag{4.16}$$

will introduce the pseudovector coupling $ig\bar{\psi}\gamma_5\gamma_\mu\psi\cdot\partial_\mu\varphi$.

In this way we can distinguish between different types of interactions through the subtracted parametric dispersion relations.[12] This result refines the conclusion drawn above.

We have clarified the correspondence between the present approach and the Lagrangian theory and are going to push forward this program. In the Lagrangian theory we have a criterion as to whether a theory is renormalizable or not, so that we shall try to find out a corresponding criterion in the present scheme. For this purpose we first have to study the rôles played by the subtractions. As we have seen, the subtractions serve to introduce interactions on one hand, and once interactions are introduced they serve to eliminate divergences on the other hand.

When we proceed to higher order calculations we combine the generalized unitarity condition with the parametric dispersion relations. In the former equation there occur no divergences, but in the latter the dispersion integrals do not converge sometimes and imply subtractions in order to make the integrals converge. This subtraction procedure corresponds to the renormalization procedure in the conventional field theory. If we need more subtractions than are assumed in the beginning, the theory is called unrenormalizable, otherwise the theory is renormalizable. In what follows we shall study this condition in the perturbation theory, and for the sake of simplicity we take the neutral scalar theory here.

First we can readily notice that the function $\rho(x_1x_2)$ is decoupled from all others. Namely the unitarity condition for three- or more-point functions do not involve the two-point function. Once all other ρ functions are known, one can calculate $\rho(x_1x_2)$ from the unitarity. Therefore in this scheme, unlike in the conventional theory, it is not necessary to know $\rho(x_1x_2)$ to solve all other functions.

In order to verify this statement, we shall refer to the following formula:

$$\int K_xK_u\Delta_F'(x-u)\cdot\Delta^{(+)}(u-v)d^4u=0,$$

or

$$\int\rho(x-u)\Delta^{(+)}(u-v)d^4u=0. \tag{4.17}$$

For this reason we shall not be worried about the two-point function and study only three- or more-point functions. The problem now is concerned with the convergence of the dispersion integrals in the perturbation theory. Take the n-point function $\mathcal{G}^{(n)}(p_\alpha p_\beta)$. We are interested in the behavior of $\mathcal{G}^{(n)}$ for large values of the p's. As far as the parametric dispersion relations are concerned, we need not distinguish between different p's.

We assume that the singularity of $\mathcal{G}^{(n)}$ at $p=\infty$ is simply given by a certain power of p, i.e.,

$$\mathcal{G}^{(n)}(p_\alpha p_\beta)\sim p^\alpha. \quad (p\to\infty) \tag{4.18}$$

The singularity $\ln p$ is counted as p^0. This assumption is valid in every order of the perturbation expansion. We denote this power α as $c(n)$ since this power depends on the number of variables n. When we need subtractions the power of the dispersive part would in general be higher than that of the absorptive part. In order to determine the powers we utilize the unitarity condition. Take Eq. (3.3) and expand τ's into ρ's, then the nonlinear part would consist of bilinear terms in ρ, trilinear terms and so on. We first retain only bilinear terms and compare the powers of the equation.

$$c(n)+4(n-1)\geq \operatorname*{Max}_{\substack{k+l>2\\ n-k+l>2}}[c(k+l)+c(n-k+l) +4(n+2l-2)]-6l,\quad (n>2). \tag{4.19}$$

[10] G. Källén, Helv. Phys. Acta 25, 417 (1952); H. Lehmann, Nuovo cimento 11, 342 (1954); M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[11] Due to the translational invariances, i.e., $(\partial/\partial x_\mu+\partial/\partial y_\mu+\partial/\partial z_\mu)\rho(xyz)=0$, only two of the derivatives are linearly independent.

[12] In the S-matrix approach in which only matrix elements on the mass shell are considered, one cannot distinguish between pseudoscalar and pseudovector couplings. On the mass shell, two invariants are no longer independent.

The inequality sign $\geq$ is the consequence of the fact that the power of the absorptive part given by the unitarity is in general lower than the dispersive part which is involved in the n-point Green's function $\mathcal{G}^{(n)}$. $4(n-1)$ comes from the definition of $\mathcal{G}$ from ρ, Eq. (3.4). "Max" means that we have to pick up the highest power from the sum over the various k's, excluding $k+l=2$, $n-k+l=2$. $6l$ comes from du, dv and $\Delta^{(+)}(u-v)$.

If we take trilinear terms, we get another inequality involving three c's on the right-hand side, but this inequality is always satisfied provided that (4.19) is satisfied. Therefore we shall discuss only (4.19).

Put

$$d(n)=c(n)+n-4, \tag{4.20}$$

then (4.19) is transformed into

$$d(n)\geq \mathrm{Max}[d(k+l)+d(n-k+l)]. \tag{4.21}$$

The solution is easily obtained as

$$c(3)\leq 1,\ c(4)\leq 0,\ c(5)\leq -1,\ \cdots, \tag{4.22}$$

This means that one subtraction for either one of $\mathcal{G}^{(3)}$ and $\mathcal{G}^{(4)}$ or both is sufficient to eliminate divergences if there is a consistent solution in perturbation theory. This also means that one cannot introduce additional interactions through subtractions for five- or more-point functions. Furthermore if such a solution exists one can prove $c(2)=2$ and consequently two subtractions for $\mathcal{G}^{(2)}$ are sufficient. This result is consistent with the conclusion of the conventional theory that only φ^3 and φ^4 interactions are renormalizable. The theorem proved here is essentially equivalent to Dyson's power counting theorem.[13]

V. PERTURBATION THEORY

In this section we shall show how the Feynman perturbation theory can be reproduced based on the two fundamental postulates: (1) generalized unitarity condition, and (2) parametric dispersion relations. We understand that the Lorentz invariance and local commutativity are implicitly involved in the latter. The purpose of this section is to show that these two postulates really exhaust all possible relationships among the finite renormalized expressions under the assumed expandability of the S-matrix elements in powers of the coupling constant. In this section we shall discuss a simplified model of the meson-nucleon system. We assume the nucleon is a charged scalar particle and the meson is a neutral scalar particle, and they are denoted by Φ and φ, respectively.

We start from a nontrivial order.

Meson Propagator

From the unitarity condition we find in the second order

$$\mathrm{Re}K_xK_y\langle 0|T[\varphi(x)\varphi(y)]|0\rangle =g^2\,\mathrm{Re}[i\Delta^{(+)}(r-y,M)\cdot i\Delta^{(+)}(x-y,M)], \tag{5.1}$$

where we have assumed the simplest coupling $g\Phi^*\Phi\varphi$ in the sense discussed in the previous section.[14] M denotes the nucleon rest mass. If we utilize the formula

$$i\Delta^{(+)}(x,m_1)\cdot i\Delta^{(+)}(x,m_2) =\frac{1}{(4\pi)^2}\int_{(m_1+m_2)^2}^{\infty}d\kappa^2\, i\Delta^{(+)}(x,\kappa) \times\left\{\left[1-\left(\frac{m_1+m_2}{\kappa}\right)^2\right]\left[1-\left(\frac{m_1-m_2}{\kappa}\right)^2\right]\right\}^{\frac{1}{2}}, \tag{5.2}$$

we find

$$\mathrm{Re}K_xK_y\langle 0|T[\varphi(x)\varphi(y)]|0\rangle =\frac{g^2}{(4\pi)^2}\int_{4M^2}^{\infty}d\kappa^2\,\mathrm{Re}(i\Delta^{(+)}(x-y,\kappa)) \cdot\left[1-\left(\frac{2M}{\kappa}\right)^2\right]^{\frac{1}{2}}. \tag{5.3}$$

The proof of Eq. (5.2) is given in Appendix A. By the formula

$$i\Delta^{(+)}(x)=\tfrac{1}{2}[\Delta^{(1)}(x)+i\Delta(x)], \tag{5.4}$$

the right-hand side of Eq. (5.3) turns out to be

$$\frac{g^2}{(4\pi)^2}\int_{4M^2}^{\infty}d\kappa^2\,\tfrac{1}{2}\Delta^{(1)}(x-y,\kappa)\left[1-\left(\frac{2M}{\kappa}\right)^2\right]^{\frac{1}{2}}. \tag{5.5}$$

Now we assume the Källén-Lehmann representation,

$$\langle 0|T[\varphi(x)\varphi(y)]|0\rangle =\Delta_F(x-y,m)+\int_{4M^2}^{\infty}d\kappa^2\,\Delta_F(x-y,\kappa)\sigma(\kappa^2). \tag{5.6}$$

From this representation we find

$$\mathrm{Re}K_xK_y\langle 0|T[\varphi(x)\varphi(y)]|0\rangle =\int_{4M^2}^{\infty}d\kappa^2\,\tfrac{1}{2}\Delta^{(1)}(x-y,\kappa)(\kappa^2-m^2)^2\sigma(\kappa^2), \tag{5.7}$$

where m is the meson rest mass entering in K_x and K_y. By comparing Eq. (5.5) with Eq. (5.7), we find

$$\sigma(\kappa^2)=\frac{g^2}{(4\pi)^2}\frac{[1-(2M/\kappa)^2]^{\frac{1}{2}}}{(\kappa^2-m^2)^2}\theta(\kappa^2-4M^2). \tag{5.8}$$

Thus the propagation function is determined. Of course, we could use the subtracted parametric dispersion relations, but the above method yields the same result somewhat simpler. The two subtractions in the parametric dispersion relations correspond to mass and charge (or Z) renormalizations.

[13] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[14] $(-i)^3K_x{}^MK_y{}^MK_z{}^m\langle 0|T[\Phi(x)\Phi^*(y)\varphi(z)]|0\rangle =-ig\delta(x-z)\delta(y-z)$.

Finally if we transform the integral (5.6) by

$$\kappa^2 = M^2/\alpha(1-\alpha), \tag{5.9}$$

we find the Feynman expression for this propagator if α denotes the Feynman parameter to unify the denominators.

We can calculate the nucleon propagator in a similar way.

Second Order Scattering

As the simplest example of the use of the nonsubtracted parametric dispersion relation let us discuss the Green's function for nucleon-nucleon scattering

$$\rho(x_1x_2x_3x_4) = (-i)^4 K_{x_1}{}^M K_{x_2}{}^M K_{x_3}{}^M K_{x_4}{}^M \times\langle 0|T[\Phi(x_1)\Phi^*(x_2)\Phi(x_3)\Phi^*(x_4)]|0\rangle_{\text{conn}}, \tag{5.10}$$

where the subscript "conn" means to omit all the contributions to the vacuum expectation value arising from disconnected Feynman diagrams.

By picking up the second-order terms from the unitarity condition for $\rho(x_1x_2x_3x_4)$, we find

$$\rho(x_1x_2x_3x_4)+\rho^*(x_1x_2x_3x_4) = -g^2D^{(1)}(x_1-x_3)[\delta(x_1-x_2)\delta(x_3-x_4) +\delta(x_1-x_4)\delta(x_3-x_2)], \tag{5.11}$$

where $D^{(1)}$ refers to the contraction function for the exchanged meson. In the derivation of the above equation, use has been made of

$$i[D^{(+)}(x)+D^{(+)}(-x)] = D^{(1)}(x), \tag{5.12}$$

as well as our previous result for the first-order three-point ρ function.

Introducing the Fourier transform of ρ_2 by means of Eq. (3.4), we get after inserting the momentum representation of $D^{(1)}$ into Eq. (5.11)

$$\mathrm{Im}\mathcal{G}_2 = -\pi g^2[\delta((p_1+p_2)^2+m^2) +\delta((p_1+p_4)^2+m^2)]. \tag{5.13}$$

Then using the parametric dispersion relation

$$\mathrm{Re}\mathcal{G}_2(p_\alpha p_\beta\cdot\xi) = \frac{P}{\pi}\left[\int_0^\infty - \int_{-\infty}^0\right]\frac{d\xi'}{\xi'-\xi}\,\mathrm{Im}\mathcal{G}_2(p_\alpha p_\beta\cdot\xi'),$$

we find

$$\mathrm{Re}\mathcal{G}_2 = -g^2\left[\frac{1}{(p_1+p_2)^2+m^2}+\frac{1}{(p_1+p_4)^2+m^2}\right], \tag{5.14}$$

or

$$\mathcal{G}_2 = -g^2\left[\frac{1}{(p_1+p_2)^2+m^2-i\epsilon}+\frac{1}{(p_1+p_4)^2+m^2-i\epsilon}\right]. \tag{5.15}$$

Next going back to the position space, we get

$$\rho_2(x_1x_2x_3x_4) = -g^2D_F(x_1-x_3)[\delta(x_1-x_2)\delta(x_3-x_4) +\delta(x_1-x_4)\delta(x_3-x_2)]. \tag{5.16}$$

In a similar way we obtain the ρ function for meson-nucleon scattering in the second order

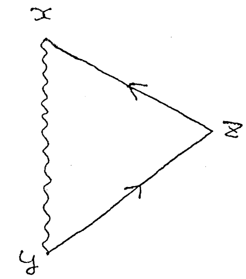

FIG. 1. Feynman diagram for the third order vertex function. The straight lines represent nucleon lines; the wavy line is a meson line.

$$(-i)^4K_{x_1}{}^MK_{x_2}{}^MK_{x_3}{}^mK_{x_4}{}^m \times\langle 0|T[\Phi(x_1)\Phi^*(x_2)\varphi(x_3)\varphi(x_4)]|0\rangle_{\text{conn}} = -g^2\Delta_F(x_1-x_2)[\delta(x_1-x_3)\delta(x_2-x_4 +\delta(x_1-x_4)\delta(x_2-x_3)]. \tag{5.17}$$

Third-Order Vertex Function

The next Green's function we are interested in is

$$\rho(xyz) = (-i)^3K_x{}^MK_y{}^MK_z{}^m \times\langle 0|T[\Phi(x)\Phi^*(y)\varphi(z)]|0\rangle. \tag{5.18}$$

The first-order expression for ρ is trivial and can be written down immediately by using the prescription given in the previous section. Here we try to find out the third-order expression of this function. In particular, we shall study the contribution of the Feynman graph given by Fig. 1. This contribution will be denoted by $\rho_a(xyz)$. If we know all the ρ functions up to second order, we can immediately write down an equation for the absorptive part of $\rho_a(xyz)$ by just picking up terms corresponding to this graph from the unitarity condition. The explicit form of this equation is given by

$$\rho_a(xyz)+\rho_a{}^*(xyz)+(ig^3\sum_{\text{cycl}}D_F(x-y)\Delta^{(+)}(x-z) \times\Delta^{(+)}(y-z)+\text{comp. conj.})=0, \tag{5.19}$$

or

$$\mathrm{Re}\rho_a(xyz) = g^3\,\mathrm{Im}\sum_{\text{cycl}}D_F(x-y)\Delta^{(+)}(x-z)\Delta^{(+)}(y-z),$$

where D refers to the meson field and Δ to the nucleon field. The Green's functions are defined and related to each other by

$$\begin{aligned}\Delta_F(x) &= \tfrac{1}{2}\Delta^{(1)}(x)-i\bar{\Delta}(x),\\ i\Delta^{(+)}(x) &= \tfrac{1}{2}[\Delta^{(1)}(x)+i\Delta(x)],\\ \bar{\Delta}(x) &= -\tfrac{1}{2}\epsilon(x_0)\Delta(x).\end{aligned} \tag{5.20}$$

From the above relations we find

$$\begin{aligned}\mathrm{Im}[D_F(x-y)\Delta^{(+)}(x-z)\Delta^{(+)}(y-z)] &= \tfrac{1}{4}[\bar{D}(x-y)\Delta^{(1)}(x-z)\Delta^{(1)}(y-z)]\\ &-\tfrac{1}{4}[\bar{D}(x-y)\Delta(x-z)\Delta(y-z)]\\ &-\tfrac{1}{4}[\Delta^{(1)}(x-z)\cdot\tfrac{1}{2}D^{(1)}(x-y)\Delta(y-z)]\\ &-\tfrac{1}{4}[\Delta^{(1)}(y-z)\cdot\tfrac{1}{2}D^{(1)}(x-y)\Delta(x-z)].\end{aligned} \tag{5.21}$$

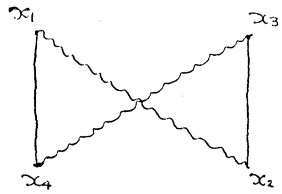

FIG. 2. A typical fourth-order Feynman diagram for the nucleon-nucleon scattering. The straight lines represent nucleon lines; the wavy lines are meson lines.

Therefore we find for the sum the expression

$$\sum_{\text{cycl}} \mathrm{Im} D_F(x-y)\Delta^{(+)}(x-z)\Delta^{(+)}(y-z)$$
$$=\tfrac{1}{4}\sum_{\text{cycl}} \bar{D}(x-y)\Delta^{(1)}(x-z)\Delta^{(1)}(y-z)$$
$$-\tfrac{1}{4}\sum_{\text{cycl}} \bar{D}(x-y)\Delta(x-z)\Delta(y-z). \quad (5.22)$$

The last two terms in Eq. (5.21) cancel each other when summed over the cyclic permutations. Furthermore, Eq. (5.22) can be written as

$$\sum_{\text{cycl}} \bar{D}(x-y)\cdot\tfrac{1}{2}\Delta^{(1)}(x-z)\cdot\tfrac{1}{2}\Delta^{(1)}(y-z)-\bar{D}(x-y)$$
$$\times\bar{\Delta}(x-z)\bar{\Delta}(y-z)\sum_{\text{cycl}}\epsilon(x_0-z_0)\epsilon(y_0-z_0). \quad (5.23)$$

If we use the relation

$$\epsilon(x_0-z_0)\epsilon(y_0-z_0)+\epsilon(y_0-x_0)\epsilon(z_0-x_0)$$
$$+\epsilon(z_0-y_0)\epsilon(x_0-y_0)=1, \quad (5.24)$$

it is easy to show that the expression (5.23) is equal to

$$-\mathrm{Im}(D_F(x-y)\Delta_F(y-z)\Delta_F(z-x)).$$

Thus we have proved

$$\mathrm{Re}\rho_a(xyz)$$
$$=-g^3\,\mathrm{Im}[D_F(x-y)\Delta_F(y-z)\Delta_F(z-x)]. \quad (5.25)$$

This means that our method gives the same absorptive part of $\rho_a(xyz)$ as that given by Feynman's method. Since in both theories the Fourier transform of ρ_a satisfied the same parametric dispersion relation with one subtraction, we have verified the equivalence between two theories in this case.

There is one interesting remark in order, namely, if we carry out this calculation in momentum space we run in to a complication. The vertex function in momentum space is a many-valued function of p_1^2, p_2^2, q^2 and we have to find the correct Riemann sheet. This is due to the occurrence of anomalous thresholds in this problem.[15] The calculation in the position space as done here, however, shows that the result is unique and that the complication in the momentum space is not essential, and there certainly would be a correct way of handling this problem in momentum space.

[15] We have to study the vertex function for all possible real values of p_1^2, p_2^2, q^2. For certain values we encounter anomalous thresholds. See in this connection S. Mandelstam, Phys. Rev. Letters **4**, 84 (1960); R. Blankenbecler and L. F. Cook, Jr., Phys. Rev. **119**, 1745 (1960); Y. Nambu and R. Blankenbecler, Nuovo cimento (to be published).

Thus we have verified the equivalence of our scheme to that of Feynman for typical processes up to the third order. The result here also shows the significance of the position space in some problems. In the Appendix B we shall discuss some properties of ρ function in the position space.

Fourth Order Scattering

As a representative of the class of diagrams associated with $\rho(x_1x_2x_3x_4)$ defined by Eq. (5.10) in the fourth order, we consider the diagram shown in Fig. 2. Let us denote the contribution of this diagram to $\rho(x_1x_2x_3x_4)$ by $\rho_b(x_1x_2x_3x_4)$. Using the generalized unitarity condition we can immediately write down an equation for the absorptive part of the function $\rho_b(x_1x_2x_3x_4)$:

$$\mathrm{Re}\rho_b(x_1x_2x_3x_4)$$
$$=g^4\,\mathrm{Re}[D_F(x_1-x_2)\Delta^{(+)}(x_2-x_3)\Delta^{(+)}(x_1-x_4)D_F^*(x_3-x_4)$$
$$+\Delta_F(x_1-x_4)D^{(+)}(x_4-x_3)D^{(+)}(x_1-x_2)\Delta_F^*(x_3-x_2)$$
$$-D_F(x_1-x_2)\Delta_F(x_1-x_4)D^{(+)}(x_4-x_3)\Delta^{(+)}(x_2-x_3)$$
$$-D_F(x_1-x_2)\Delta_F(x_2-x_3)D^{(+)}(x_3-x_4)\Delta^{(+)}(x_1-x_4)$$
$$-D_F(x_3-x_4)\Delta_F(x_4-x_1)D^{(+)}(x_1-x_2)\Delta^{(+)}(x_3-x_2)$$
$$-D_F(x_3-x_4)\Delta_F(x_3-x_2)D^{(+)}(x_2-x_1)\Delta^{(+)}(x_4-x_1)$$
$$-\Delta^{(+)}(x_1-x_4)\Delta^{(+)}(x_3-x_2)$$
$$\times D^{(+)}(x_3-x_4)D^{(+)}(x_1-x_2)]. \quad (5.26)$$

The first two terms arise from that part of the unitarity equation which is of the form $\rho_2\rho_2^*$. The next four terms arise from $\rho_1\rho_3^*$ and $\rho_3\rho_1^*$,[16] and the last term has its origin in four disconnected first-order diagrams, i.e., the term of the form $\rho_1\rho_1\rho_1^*\rho_1^*$ as shown in Fig. 3.

We can show by a straightforward calculation that $\mathrm{Re}\rho_b$ is given by

$$\mathrm{Re}\rho_b(x_1x_2x_3x_4)=g^4\,\mathrm{Re}[D_F(x_1-x_2)D_F(x_3-x_4)$$
$$\times\Delta_F(x_2-x_3)\Delta_F(x_4-x_1)], \quad (5.27)$$

when use is made of identities (5.20) and

$$-\epsilon(x_1-x_2)\epsilon(x_2-x_3)\epsilon(x_3-x_4)\epsilon(x_4-x_1)$$
$$+\epsilon(x_1-x_4)\epsilon(x_2-x_3)+\epsilon(x_1-x_2)\epsilon(x_4-x_3)$$
$$+\epsilon(x_4-x_3)\epsilon(x_2-x_3)+\epsilon(x_4-x_1)\epsilon(x_4-x_3)$$
$$+\epsilon(x_1-x_2)\epsilon(x_1-x_4)+\epsilon(x_2-x_3)\epsilon(x_2-x_1)=1, \quad (5.28)$$

which is an analog of Eq. (5.24). $\epsilon(x)$ here denotes $\epsilon(x_0)$.

It must be stressed that the last term on the right-hand side of Eq. (5.26) as well as other four terms—except for the first two terms—can hardly be included in the ordinary unitarity condition on the mass shell and that these terms play an important rôle in the presence of an anomalous threshold.[17] In our method these terms

[16] In evaluating the third-order five-point function, we use the Feynman result. Here again the Feynman method and our methods can be shown to give identical results although we do not go into details about this point.

[17] Y. Nambu, Nuovo cimento **9**, 610 (1958); R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. **111**, 1187 (1958); **114**, 376 (1959); L. D. Landau, Nuclear Phys. **13**, 181 (1959); N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **22**, 128 (1959); **23**, 284 (1960). See also reference 15.

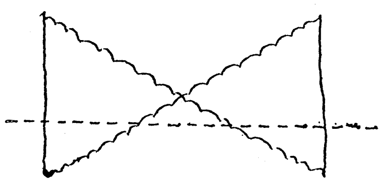

FIG. 3. The Feynman diagram for the last term in Eq. (5.26). The straight lines represent nucleon contraction $\Delta^{(+)}$; the wavy lines represent meson contraction $D^{(+)}$; the dashed line indicates the intermediate state.

can be included in the unitarity condition in a natural way.

Finally it will be clear that the combination of Eq. (5.27) with the nonsubtracted dispersion relation yields

$$\rho_b(x_1x_2x_3x_4)=g^4D_F(x_1-x_2)D_F(x_3-x_4)\times\Delta_F(x_2-x_3)\Delta_F(x_4-x_1). \quad (5.29)$$

Thus we have reproduced the Feynman result in this example.

ACKNOWLEDGMENT

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APPENDIX A. PROOF OF EQ. (5.2)

The $\Delta^{(+)}$ function is defined by (2.15) and we get

$$i\Delta^{(+)}(x,m_1)\cdot i\Delta^{(+)}(x,m_2)=\frac{1}{(2\pi)^6}\int e^{ikx}d^4k\int d^4q\,\theta(k_0-q_0)\theta(q_0)\times\delta[(k-q)^2+m_1^2]\delta(q^2+m_2^2).$$

We first carry out the q integration, i.e., we evaluate

$$I=\int d^4q\,\theta(k_0-q_0)\theta(q_0)\delta(k^2-2kq+m_1^2-m_2^2)\delta(q^2+m_2^2).$$

Since the integrand survives only for time-like k with $k_0>0$, we choose the direction of k as the time axis; then from the first δ function we find

$$q_0=-(k^2+m_1^2-m_2^2)/2k_0,$$

where $k^2=-k_0^2$. Consequently we get

$$\theta(k_0-q_0)=\theta\left(k_0+\frac{k^2+m_1^2-m_2^2}{2k_0}\right)=\theta(2k_0^2+k^2+m_1^2-m_2^2)=\theta(-k^2+m_1^2-m_2^2),$$

and

$$\theta(q_0)=\theta(-k^2-m_1^2+m_2^2).$$

Therefore we find an expression for $\theta\theta$ independent of q,

$$\theta(k_0-q_0)\theta(q_0)=\theta(k_0)\theta(-k^2-|m_1^2-m_2^2|).$$

Apart from the θ factors, the integral I is given by

$$\frac{1}{2k_0}\int d^3q\,\delta\left[\mathbf{q}^2-\left(\frac{k^2+m_1^2-m_2^2}{2k_0}\right)^2+m_2^2\right]=\frac{1}{2(-k^2)^{\frac{1}{2}}}4\pi\int_0^\infty q^2dq\,\delta\left[q^2+\frac{(k^2+m_1^2-m_2^2)^2+4m_2^2k^2}{4k^2}\right],$$

where $q=|\mathbf{q}|$. This integral is readily carried out to yield

$$\frac{\pi}{2(-k^2)}\{[k^2+(m_1+m_2)^2][k^2+(m_1-m_2)^2]\}^{\frac{1}{2}}.$$

Hence we finally get

$$I=\theta(k_0)\theta(-k^2-|m_1^2-m_2^2|)\times\frac{\pi}{2(-k^2)}\{[k^2+(m_1+m_2)^2][k^2+(m_1-m_2)^2]\}^{\frac{1}{2}}$$
$$=\frac{\pi}{2}\theta(k_0)\int_{(m_1+m_2)^2}^\infty d\kappa^2\delta(k^2+\kappa^2)\times\left\{\left[1-\left(\frac{m_1+m_2}{\kappa}\right)^2\right]\left[1-\left(\frac{m_1-m_2}{\kappa}\right)^2\right]\right\}^{\frac{1}{2}}.$$

Inserting this result into the original expression, one finds

$$i\Delta^{(+)}(x,m_1)\cdot i\Delta^{(+)}(x,m_2)=\frac{1}{(4\pi)^2}\int_{(m_1+m_2)^2}^\infty d\kappa^2\left\{\left[1-\left(\frac{m_1+m_2}{\kappa}\right)\right]^2\times\left[1-\left(\frac{m_1-m_2}{\kappa}\right)^2\right]\right\}^{\frac{1}{2}}\times\frac{1}{(2\pi)^3}\int d^4k\,e^{ikx}\theta(k_0)\delta(k^2+\kappa^2)$$
$$=\frac{1}{(4\pi)^2}\int_{(m_1+m_2)^2}^\infty d\kappa^2\,i\Delta^{(+)}(x,\kappa)\times\left\{\left[1-\left(\frac{m_1+m_2}{\kappa}\right)^2\right]\left[1-\left(\frac{m_1-m_2}{\kappa}\right)^2\right]\right\}^{\frac{1}{2}}.$$

APPENDIX B. PARAMETRIC DISPERSION RELATIONS IN POSITION SPACE

We start from the study of the integral

$$F(\lambda)=i\int_0^\infty \exp\left[-i\left(\alpha\lambda+\frac{m^2}{4\alpha}\right)\right]d\alpha.$$

This integral has the following interesting property:

$$\text{Im}F(\lambda)=0,\quad \text{for}\quad \lambda<0.$$

Next let us observe the function $F(\lambda\xi)$ as a function of ξ and denote it as $f(\xi)$. Then we clearly get

$$\mathrm{Im}f(\xi)=0, \quad \text{for} \quad \lambda\xi<0.$$

As a function of ξ, $f(\xi)$ is analytic in the lower half plane when $\lambda>0$, and we get

$$f(\xi)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi')d\xi'}{\xi'-\xi+i\epsilon}, \quad \text{for} \quad \lambda>0.$$

When $\lambda<0$, $f(\xi)$ is analytic in the upper half plane and we get

$$f(\xi)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi')d\xi'}{\xi'-\xi-i\epsilon}, \quad \text{for} \quad \lambda<0.$$

We can write these two equations in a unified manner as

$$f(\xi)=i\epsilon(\lambda)\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi')d\xi'}{\xi'-\xi}.$$

Now take the imaginary part of this equation; then we get

$$\epsilon(\lambda)\ \mathrm{Im}f(\xi)=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}f(\xi')d\xi'}{\xi'-\xi}.$$

From the property of $\mathrm{Im}f(\xi)$ discussed above, the left-hand side now may be written as $\epsilon(\xi)\ \mathrm{Im}f(\xi)$, or we may write

$$\mathrm{Im}f(\xi)=\epsilon(\xi)\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}f(\xi')d\xi'}{\xi'-\xi}.$$

This is the basis for the parametric dispersion relations for the ρ functions. The same argument can apply to a slightly more general class of functions defined by

$$f(\xi)=i\int_{0}^{\infty}\frac{d\alpha}{(i\alpha)^{n}}\exp\left[-i\left(\alpha\lambda\xi+\frac{m^2}{4\alpha}\right)\right].$$

Next we shall recall the integral representation of the Feynman propagator in the position space:

$$\Delta_F(x)=-\frac{i}{4\pi^2}\int_0^{\infty}\exp\left[i\left(\alpha x^2-\frac{m^2}{4\alpha}\right)\right]d\alpha.$$

If we insert this representation into the Feynman formula and use the integral formula,

$$\int\exp(iax^2)d^4x=\frac{i\pi^2}{a|a|},$$

then it is not difficult to show that the ρ functions obey an integral representation of the following form[18]:

$$\rho(x_1\cdots x_n)=i\int dc_{rs}dM^2\sigma(c_{rs},M^2)$$
$$\times\int\frac{d\alpha}{(i\alpha)^N}\exp\left[-i\left(\alpha\Lambda+\frac{M^2}{4\alpha}\right)\right],$$

where $\Lambda=-\sum_{r>s}c_{rs}(x_r-x_s)^2$, $c_{rs}\geq0$, and σ is a real weight function. N is an integer which depends on the structure of the Feynman diagram under consideration.

Comparison of this integral representation with $f_n(\xi)$ yields immediately the dispersion relation

$$\mathrm{Im}\rho(x_\alpha x_\beta\cdot\xi)=\epsilon(\xi)\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\rho(x_\alpha x_\beta\cdot\xi')}{\xi'-\xi}d\xi'.$$

ρ is a function of scalar products $x_\alpha x_\beta$ alone, and ξ is a common parameter to be multiplied into all the scalar products. In general one needs subtractions in the above dispersion integrals, but it is yet an unsettled question how to make the proper subtractions.

From the above integral representation, one can recognize that if all the n points lie on a space-like surface, then $\Lambda<0$ holds, and therefore we get

$$\mathrm{Im}\rho(x_1\cdots x_n)=0.$$

The dispersive part of a ρ function vanishes when all the n points are separated from each other by space-like distances. This is a manifestation of the microscopic causality condition.

[18] Y. Nambu, Nuovo cimento **9**, 610 (1958).