Simple Realistic Treatment of Nuclear Direct-Interaction Processes*

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Physical arguments are used to predict qualitatively the effect on direct-interaction differential cross sections of the distortion of the wave functions of the scattered particle. These qualitative predictions are confirmed by calculations using a simple but fairly realistic model for the wave function distortion in (α,α') scattering. The model used is based on examination of the properties of optical model wave functions. Good fits to experimental data are found using the model for (α,α') scattering in the energy range 20–40 Mev for scattering angles less than 90°. Features of direct-interaction processes involving nucleons are interpreted in terms of a focus in the optical model wave functions for these particles, but detailed calculations are not presented.

1. INTRODUCTION

It is generally accepted that many nuclear reactions can be understood as direct interactions in which the incident particle excites only a few of the degrees of freedom of the target nucleus. Considerable success in describing these reactions has been achieved by assuming that the interaction proceeds via a two-body collision between the bombarding particle and a single nucleon in the target, and by using distorted-wave Born approximation.[1] However, it is probable that many direct-interaction processes involve collective rather than single-particle excitations of the target,[2,3] and Drozdov[4] and Inopin[5] have given a simple approximation for (α,α') processes involving collective excitations, which has been applied successfully by Blair.[6] It is also obviously possible to formulate a distorted-wave Born approximation for direct interactions whether the nuclear excitation is collective or single-particle. Since there is general agreement that the most important factors controlling angular distributions are the angular momentum transfer and the distortion of the wave functions of the bombarding particle, these angular distributions should be largely insensitive to the nature of the nuclear excitation for direct scattering processes (in which the wave functions for the ingoing and outgoing channels appear in the matrix element evaluated at the same point).

In this paper, we discuss the angular distributions in direct interactions, in particular for (α,α') and (p,p') processes, on the basis of distorted-wave Born approximation and a simple empirical but realistic approximation for the distorted wave functions of the entrance and exit channels in the neighborhood of the nuclear surface. As in the oversimplified Butler[7,8] theory and also in Blair's[6] theory, the positions of the maxima and minima in the angle dependence of the cross section are controlled, for a given angular momentum transfer, by a single parameter, which may be thought of as the nuclear radius. Our theory for (α,α') scattering, however, also has a parameter, related to the spread of the distorted α-particle wave functions round the nuclear surface, which controls the peak-to-valley ratio in the angular distribution, and a parameter related to the effective thickness of the nuclear surface, which controls the ratio of the cross section for large angles to that for forward scattering. We are also able to understand physically, by using the uncertainty principle, why various types of approximation fail in the way they are observed to do.

For all particles, the nucleus focusses the entrance- and exit-channel wave functions,[9] but for strongly absorbed particles such as α particles the foci are sufficiently attenuated to be unimportant. For protons, however, the intensity at the foci is large, and they cannot be ignored. The effect of these foci is particularly important when they lie in the nuclear surface. They then lead to a large uncertainty in the effective momentum transfer, which in turn smooths out the maxima and minima in the angular distribution and can also lead to a large amount of forward scattering. The theory for (p,p') scattering, to be realistic, needs more parameters than the theory for (α,α') scattering, but once these parameters are fitted it should be possible to use the same distorted wave functions in the description of other direct-interaction processes involving protons.

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2. MATHEMATICAL FORMULATION AND PHYSICAL INTERPRETATION

We shall confine ourselves in this paper to a discussion of direct (rather than exchange) inelastic scattering, although our approximations could be adapted to other processes. We shall also ignore internal structure of the bombarding particle, such as the spin of a nucleon or the structure of an alpha particle as a bound state of four nucleons. Our discussion is based on the Born approximation matrix element

$$\mathfrak{M}=\sum_{j=1}^{A}\int\cdots\int d^3r_1\cdots d^3r_A d^3r'\,\Phi_f^*(\mathbf{r}_1,\cdots,\mathbf{r}_j,\cdots,\mathbf{r}_A)\times\Phi_i(\mathbf{r}_1,\cdots,\mathbf{r}_j,\cdots,\mathbf{r}_A)v(\mathbf{r}_j,\mathbf{r}')\psi_f^*(\mathbf{r}')\psi_i(\mathbf{r}'),\quad(1)$$

where Φ_i and Φ_f are the initial and final nuclear wave functions, ψ_i and ψ_f are the distorted wave functions for the incoming and outgoing particles, and v is the interaction between the bombarding particle and the target nucleons. This form of matrix element will be valid whether the excitation of the final nucleus is single particle or collective, although to find the absolute magnitude of the matrix element in the latter case one would have to solve the problem of describing collective degrees of freedom in terms of particle coordinates.

$\mathfrak{M}$ may now be rewritten

$$\mathfrak{M}=\int d^3r\,F(\mathbf{r})\rho(\mathbf{r})e^{iS(\mathbf{r})},\quad(2)$$

where

$$F(\mathbf{r})=\sum_{j=1}^{A}\int d^3r_1\cdots d^3r_A\,\Phi_f^*(\mathbf{r}_1,\cdots,\mathbf{r}_A)\times\Phi_i(\mathbf{r}_1,\cdots,\mathbf{r}_A)\delta(\mathbf{r}_j-\mathbf{r}),\quad(3)$$

and

$$\rho(\mathbf{r})e^{iS(\mathbf{r})}=\int d^3r'\,v(\mathbf{r},\mathbf{r}')\psi_f^*(\mathbf{r}')\psi_i(\mathbf{r}').\quad(4)$$

Thus F, which may be complex, is related only to properties of the initial and final nuclear states whereas $\rho\exp(iS)$, which for future convenience has been written as a product of amplitude and phase factors, describes properties of the bombarding particle and of the interaction

In those cases where direct-interaction theory has had a success, it has been possible to assume that the properties of the nuclear states determine the angular momentum transfer uniquely. A particularly important case occurs when the ground state of the target nucleus has zero spin—in this case the angular momentum transfer must just be the spin of the excited state. The examples of our theory which we display are all of this type. If the angular momentum transfer is L, then $F(\mathbf{r})$ must have the form

$$F(\mathbf{r})=R(r)Y_L{}^M(\theta,\varphi).\quad(5)$$

For a single-particle excitation, R will be proportional to the product of the radial functions for the single-particle orbitals which differ in the two states. For collective excitations, R will have a different meaning, but nevertheless F will still have the form given by (5).

Our procedure will be to make an empirical approximation for $\psi_f^*(\mathbf{r}')\psi_i(\mathbf{r}')$ based on optical model calculations, and to assume that $v(\mathbf{r},\mathbf{r}')$ is of sufficiently short range to allow us to use our approximation for $\psi_f^*\psi_i$ as an approximation for $\rho\exp(iS)$. This last assumption appears to be justified at least for calculations of the angular distribution, since Levinson and Banerjee[1] and also Glendenning[10] have found the results of their more detailed calculations to be largely insensitive to the range of the interaction. This topic is discussed in more detail in a recent paper by Austern.[11]

The discussion of the actual approximations we make will be postponed until Secs. 3 and 4. The remainder of this section will be devoted to a description of a physical picture based on the uncertainty principle which allows us to understand the reasons for the various effects which we predict.

First consider simple Born approximation, with a zero-range interaction. In this case, $\rho\exp(iS)$ is proportional to $\exp(i\mathbf{K}\cdot\mathbf{r})$, where $\mathbf{K}=\mathbf{k}_i-\mathbf{k}_f$, the difference of the wave number vectors for the initial and final projectile states; $\mathbf{K}\hbar$ is the momentum transfer in the complete scattering process. The matrix element $\mathfrak{M}$ is then proportional to the probability amplitude that the initial and final nuclear states have momenta differing by $\mathbf{K}\hbar$.

If the simple Born approximation is valid, the whole of the momentum transfer takes place in the inelastic episode in the scattering, which in this case is the complete scattering process. In distorted-wave Born approximation, however, part of the momentum transfer will be caused by the elastic and absorptive processes which distort the wave functions, and this will show up in the deviation of $\rho\exp(iS)$ from $\exp(i\mathbf{K}\cdot\mathbf{r})$. In more detail, $\mathfrak{M}$ can be written as $\mathfrak{M}=\int d^3K'\,\mathfrak{M}_B(\mathbf{K}')P(\mathbf{K}-\mathbf{K}')$, where $\mathfrak{M}_B(\mathbf{K}')$ is the simple Born approximation matrix element for momentum transfer $\mathbf{K}'\hbar$ (and zero-range interaction), and may be interpreted as the probability amplitude that the momentum transfer in the inelastic episode in the collision is $\mathbf{K}'\hbar$, while $P(\mathbf{K}-\mathbf{K}')=(2\pi)^{-3}\int d^3r\,\rho\exp[i(S-\mathbf{K}'\cdot\mathbf{r})]$ may be interpreted (provided the interaction has zero range) as the probability amplitude that the momentum transfer in the elastic and absorptive processes which distort the wave functions is $(\mathbf{K}-\mathbf{K}')\hbar$. (An analogous interpretation is possible even if the direct interaction does not have zero range.) Quite generally, the factor $\rho\exp(iS)$ in the integrand of the matrix element may be regarded as describing the properties of a probe which measures some of the ways in which the nuclear state changes in the inelastic episode of the complete scattering process. We have just seen that in simple Born approximation

[10] N. K. Glendenning, Phys. Rev. **114**, 1297 (1959).
[11] N. Austern, preprint (1960).

the inelastic episode is the complete scattering process, and the probe measures the momentum transfer. We shall now consider how this interpretation must be modified in distorted-wave Born approximation on account of deviations of ρ from a constant and of S from $\mathbf{K}\cdot\mathbf{r}$. The first type of deviation we term *amplitude distortion*, and the second *phase distortion*.

We consider amplitude distortion first. If ρ is nonuniform in space, it implies that the probe is in fact making a simultaneous position and momentum measurement. Because of the uncertainty principle, the resolution for momentum transfer will therefore be reduced. We give two examples. The simple Butler[8] theory of direct interactions assumes that the interaction is confined to distances greater than some effective nuclear radius R from the center of the target. In a more extreme approximation, it is often assumed that the reaction is confined to the shell of radius R. In this case, there will be no resolution at all for the radial component of momentum transfer,[12] because of the uncertainty principle and the fact that the probe has perfect resolution for measuring radial position. This can be seen alternatively, since the radial target wave functions enter the matrix element evaluated at a single radius, and knowledge of a function at just one point gives no information whatever about its Fourier components. Since no angular localization is assumed over the nuclear surface, however, resolution of the angular momentum transfer should be a maximum. To exploit this, we must expand $\exp(iS)=\exp(i\mathbf{K}\cdot\mathbf{r})$ in partial waves, when only that one corresponding to the actual angular momentum transfer L contributes to the matrix element and as is well-known the angular distribution for the scattering is given by $|j_L(KR)|^2$. The sharpness of the maxima and minima in the differential cross section predicted by this expression, together with the fact that the minima actually reach zero, are associated with the maximal resolution for angular momentum.

To check that this is indeed so, we notice that a localization to just part of the nuclear surface will cause contributions to the matrix element from different partial waves in the expansion of $\exp(i\mathbf{K}\cdot\mathbf{r})$, as a result of which the matrix element will in general be complex, and only in very exceptional circumstances will the matrix element vanish for any real value of K. On the other hand, an unsharpness of the effective nuclear radius (i.e., a "thick surface") will merely replace $j_L(KR)$ by $g_L(K)=\int_0^\infty f(r)j_L(Kr)dr$, where f $[\equiv R(r)\rho(r)r^2]$ is a *real* function, and only for exceptional and rather implausible functions f will $g_L(K)$ have no zeros, so that the inclusion of a radial integration will not in general spoil the feature of the differential cross section which we associate with maximal resolution for angular momentum.

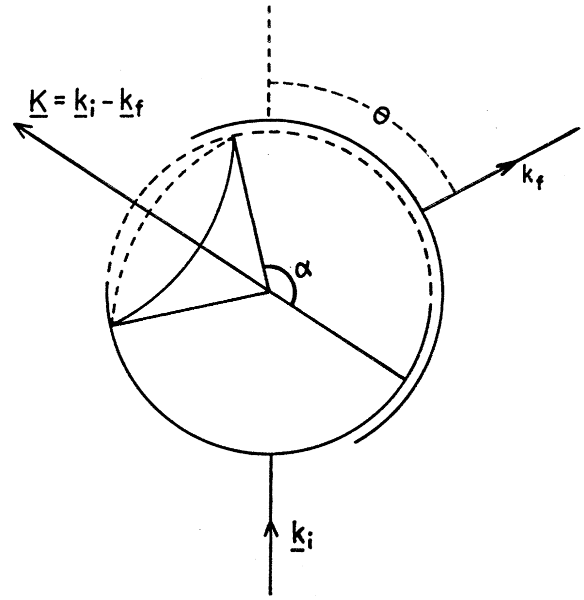

FIG. 1. "Bright" and "dark" regions of the nuclear surface for entrance and exit channels, and region of maximum overlap.

Our second example illustrates the results on the effect of an angular localization of the interaction. In (α,α') scattering it is certain that the wave function of the incident particle will be considerably reduced on the shadow side of the nucleus compared to the side facing the incoming beam, and a similar effect will be true for the outgoing particle. The overlap of the wave functions will therefore be greatest at the side of the nucleus opposite to the direction of the momentum transfer. This is illustrated in Fig. 1. Although the approximation to be described in the next section is more realistic, we shall here consider the consequences of assuming that that part of the surface, axially symmetric about the direction of the momentum transfer, cut off by a cone of semiangle α (as illustrated in Fig. 1) contributes uniformly to the matrix element, and the remaining part of the surface does not contribute at all. In this case, the matrix element is proportional to

$$\sum_{L',l}\tfrac{1}{2}(2L'+1)i^{L'}j_{L'}(KR)(LL'00|LL'l0)^2 \times\int_{\cos\alpha}^{1}P_l(x)dx. \quad (6)$$

which reduces to $(2L+1)i^L j_L(KR)$ when $\alpha=\pi$, thus reproducing the Butler theory. However, when $\alpha<\pi$, (6) is a sum of terms $j_{L'}(KR)$ with varying coefficients. L represents the total angular momentum change of the nucleus, L' that part of the angular momentum change

[12] We use the terms "radial" and "tangential" components of momentum transfer in a rather loose way. The momentum transfer can be represented by the differential operator $-i\hbar\nabla$ operating on the nuclear overlap function $F(\mathbf{r})$. Now the gradient operator may be written $\nabla=\hat{r}(\hat{r}\cdot\nabla+r^{-1})+[\nabla-\hat{r}(\hat{r}\cdot\nabla+r^{-1})]$. These two terms we call the radial and tangential components of ∇, respectively. $-i\hbar(\hat{r}\cdot\nabla+r^{-1})$ is a Hermitian operator conjugate to r, and $-i\hbar\nabla+i\hbar\hat{r}(\hat{r}\cdot\nabla+r^{-1})$ commutes with r, these being the justification for the nomenclature. The two parts of ∇ do not commute with each other or with ∇ itself, and this is the source of some inexactness in our interpretation.

associated with the inelastic episode in the complete scattering process, and l that part involved in the (elastic) deformation of the initial and final α-particle wave functions which is necessary to produce the assumed localization of the direct interaction. The Clebsch-Gordan coefficients ensure that L' and l can indeed be added to give the true total angular momentum transfer L. The probability that the angular momentum transfer in the inelastic episode is L' will be proportional to

$$\left|\tfrac{1}{2}(2L'+1)i^{L'}\sum_{l}(LL'00|LL'l0)^2\int_{\cos\alpha}^{1}P_l(x)dx\right|^2, \quad (7)$$

which becomes negligible except for $L'=L$ as $\cos\alpha\rightarrow-1$. Of course since angular momentum is a discrete variable, the positions of the maxima and minima are controlled by the dominant term in (6), which will be that with $L'=L$ provided α is large enough: the effect of the poor resolution for the angular momentum transfer in the direct interaction is partially to fill the valleys between peaks.

Localization of the interaction to part of the nuclear surface will tend also to enhance the cross section for large angles [since only for large momentum transfer do those $j_{L'}(KR)$ with large L' contribute]. However, the most important effect controlling the relative height of successive peaks is the thickness of the surface region which contributes to the matrix element. This thickness is controlled on the one side by the penetration of the bombarding particles into the nucleus, and for collective excitations probably also by the smallness of the radial nuclear overlap function $R(r)$ in the interior of the nucleus, and on the other by the falloff in the density of nuclear matter. The radial overlap function $R(r)$ of Eq. (5) will in general be oscillatory inside the nucleus: however, if the wavelengths characteristic of these oscillations are large compared to the effective surface thickness then their effect on the matrix element will only be slight. The validity of this assumption will be discussed in the next section. The result of the radial integration in the matrix element should then depend primarily on two parameters, the nuclear radius and the surface thickness. Decreasing the surface thickness will increase the number of high momentum components which contribute to the matrix element, and will therefore cause an increase in the large-angle scattering cross section (which requires large momentum transfer) relative to forward scattering. Similarly, increasing the surface thickness will depress the large-angle cross section. It should be remembered, however, that we cannot expect such a simple two-parameter description of the radial integration to be valid for momentum transfers greater than $\hbar/\lambda$, where λ is the surface thickness, for then finer details in the integrand of the matrix element will begin to be important.

We now summarize these results. We expect the sharpness of the maxima and minima in the differential cross section to be partially smoothed out by a localization of the interaction to just part of the nuclear surface (i.e., angular localization), and we expect this to be the main effect controlling the peak-to-valley ratio for the maxima and minima in the cross section. We expect both the angular localization and the radial localization to affect the ratio of the large-angle scattering cross section to the small angle scattering. The positions of the maxima and minima in the angular distribution will depend on the details of the radial integration, but for not too large momentum transfer we expect the positions and relative magnitudes of the peaks to be well described using a radial integral with just two parameters, an "interaction radius" and a "surface thickness." In the next section we shall see how these expectations are borne out in practice.

So far we have discussed amplitude distortion only. For (α,α') scattering it is plausible that phase distortion will not be very important (this will be discussed in more detail in the next section), but this is not true for processes, such as (p,p') scattering, which involve nucleons. The reason for this distinction is that the nuclear optical potential produces a focus in the incoming and outgoing wave functions, which is very intense for nucleons, but less so for more strongly absorbed particles. We find it possible to ignore the foci for (α,α') processes. It is obvious that phase distortion cannot be ignored in the region of the foci. Because of the complications of the foci, we do not yet have detailed results to report for nucleon-nucleon processes, but in Sec. 4 we outline the type of approximation we intend to make and describe why we believe it will lead to correct qualitative predictions.

Since phase distortion will be largely ignored in what follows, it is important to know in which direction our predictions are likely to be wrong. To understand this, we use the result proved in Appendix 1, namely that if a probe is described by $\rho\exp(iS)$ (see the earlier discussion) where ρ is a *known* function of position, and if the probe is interpreted as measuring momentum transfer, then the probe will have maximum resolution for detecting a momentum transfer $\mathbf{K}\hbar$, consistent with the uncertainty principle and the known localization ρ of the probe, if $S=\mathbf{K}\cdot\mathbf{r}$. Thus the effect of phase distortion should be to reduce still further the resolution for momentum transfer, and for angular momentum transfer. It follows that angular distributions predicted ignoring phase distortion, in those cases when it is important, should have more pronounced structure than if phase distortion were correctly treated.

3. (α,α') SCATTERING

In this section we shall describe a realistic approximation for the distortion of the ingoing and outgoing α-particle wave functions in the vicinity of the nuclear surface.

First of all, we notice that if the interaction is localized

on the surface, only the angle dependence of the wave functions matters. Now the divergence of the flux of particles incident on a nucleus is proportioned to $|\psi|^2$ multiplied by the absorptive part of the optical model potential, so that we can use the calculations of McCarthy *et al.*[9] as a guide in making an approximation for $|\psi|$. Reference to the diagrams in those papers shows (i) the dependence of $|\psi|$ on angle is similar at all radii in the neighborhood of the nuclear surface, (ii) the intensity is greatest on the front side of the nucleus (i.e., the side facing the incident beam) and falls off towards the back, except for the focus, (iii) there is a focus at the back of the nucleus, where the intensity may be large compared to that at the front for particles such as nucleons which are not too strongly absorbed, and comparable to the front intensity for more strongly absorbed particles such as α particles, (iv) the focal region does not deviate very seriously from spherical symmetry.

Our basic approximation for ψ_i, which is a compromise between complete realism and a form leading to simple analytical results for the matrix element, is

$$\psi_i = A(r)\exp(i\mathbf{k}_i\cdot\mathbf{r} - \gamma\hat{k}_i\cdot\hat{r}) + B\exp[iS_i(\mathbf{r}) - (\mathbf{r} - a\hat{k}_i)^2/2\sigma^2], \quad (8)$$

where the first term represents the over-all surface intensity and the second term represents the focus. $\mathbf{k}_i$ is the incident wave vector, and $\hat{k}_i$, $\hat{r}$ are unit vectors in the directions of $\mathbf{k}_i$ and $\mathbf{r}$. For the outgoing particles, a similar form of wave function is chosen, but with $\mathbf{k}_i$ replaced by $\mathbf{k}_f$ in the phase and by $-\mathbf{k}_f$ in the amplitude factors. For convenience we shall call the two terms in (8) the *surface term* and the *focus term*.

Let us first discuss the surface term. We shall postpone consideration of the radial dependence $A(r)$ of the amplitude. To determine whether the assumed angle dependence of the amplitude is reasonable, we plot $\log_e|\psi|^2$ against $-\hat{k}\cdot\hat{r}$: this should give a straight line of slope 2γ, except near $\hat{k}\cdot\hat{r}=1$, where the focus contributes. Figure 2 shows such a plot, for 18-Mev α particles on an argon target, using results from McCarthy *et al.*[9] The plots are reasonably straight, and correspond to values of γ in the region of 0.8; furthermore the mean slope is indeed independent of radius, thus confirming the assumption that the angle dependence of the surface term is similar at all radii. The Butler approximation would be represented on Fig. 2 by a line of zero slope. We shall refer to γ as the *anisotropy parameter*, since it is a measure of the lack of uniformity of the wavefunction over the nuclear surface. The anisotropy parameter does not appear to be very sensitive to the energy or to the nature of the incident particles.

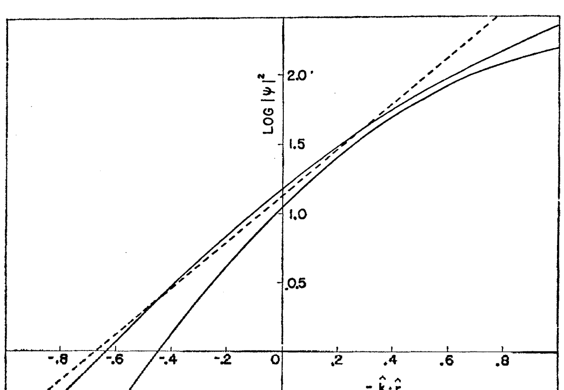

FIG. 2. Plot of $\log_e|\psi|^2$ against $-\hat{k}\cdot\hat{r}$ for 18-Mev α-particles incident on argon, corresponding to the optical model parameters α_1 of reference 9. Curve A corresponds to the radius with nuclear density 90% of maximum, curve B to 10% nuclear density. The dashed line corresponds to anisotropy parameter $\gamma=0.9$.

Melkanoff and Dyer[13] have also recently made calculations of optical model wave functions, and their preliminary results are similar to those displayed in Fig. 2. Figure 3 shows a plot of $\log_e|\psi|^2$ against $-\hat{k}\cdot\hat{r}$ for 40-Mev. α-particles incident on an aluminum target, using optical model parameters from Igo and Thaler.[14] The calculations of Melkanoff and Dyer also give the phase of the wave function, which is shown plotted against $\hat{k}\cdot\hat{r}$ in Fig. 4. To a remarkably good approximation, the phase at all radii is seen to be a linear function of $\hat{k}\cdot\hat{r}$. Of course, this does not mean that there is no phase distortion; rather, Fig. 4 shows only that the phase of the surface term in the wave function in and near the nucleus has the form $\alpha(r)\hat{k}\cdot\hat{r}+\beta(r)$. $\beta(r)$, which is the value of this phase when $\hat{k}\cdot\hat{r}=0$ and should be constant if there were no phase distortion, varies quite rapidly in the vicinity of the nuclear surface. Also, although $\alpha(r)$ does appear to be a linear function of r except for small radii, it is not just kr but rather has the form $\alpha(r)=ar+b$, where a is considerably smaller than k. (These remarks are based on a study by Mr. T. Menne of the wave functions computed by Melkanoff and Dyer.)

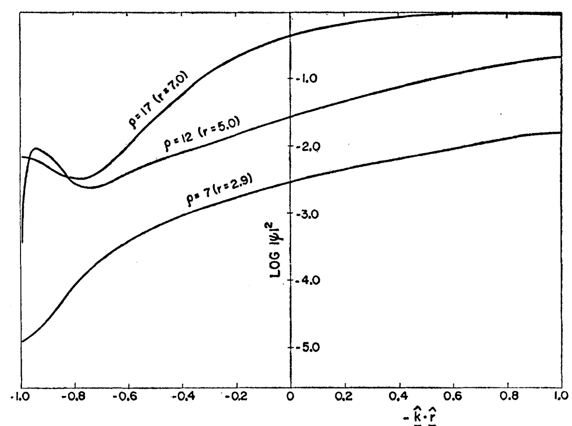

FIG. 3. Plot of $\log_e|\psi|^2$ against $-\hat{k}\cdot\hat{r}$ for 40-Mev α-particles incident on Al^{13}, using optical model parameters from Igo and Thaler.[14] The nuclear radius is approximately 5.5 f.

[13] M. Melkanoff and J. Dyer (private communication).

[14] G. Igo and R. M. Thaler, Phys. Rev. **106**, 126 (1957).

Despite these facts, however, it is reasonable to investigate the consequences of using an undistorted phase as in (8) before trying to study anything more complicated. Indeed, for the special case in which the incoming and outgoing particles are the same and for which the intensity of the focus is small enough for the focus to be ignored, as for example in (α,α') scattering, we can see that despite phase distortion we can use (8) with very little error provided only that we change our interpretation of the meaning of r. This is because the $\beta(r)$ terms in the phase cancel between the incoming and outgoing wave functions and the angle-dependent terms can be brought to the form $\mathbf{k}\cdot\mathbf{r}'$ by the change of radial variable $r'=k^{-1}\alpha(r)$; thus the r in (8) should really be interpreted as the r' just defined rather than as the true radial coordinate of the particle. Of course the cancellation of the $\beta(r)$ terms, and the simultaneous transformation of the angle-dependent parts of the phases to undistorted form for both incoming and outgoing particles, is exact only if $k_f=k_i$, but since the distortions are produced by a potential of strength much greater than any probable energy loss in a direct-interaction scattering, the error involved in using the undistorted phase in (8) should be small.

For the focus term, we assume a spherically symmetric dependence for the amplitude, centered about a point distant a from the center of the nucleus. The spherical symmetry appears from McCarthy's work to be a reasonable first approximation, and the Gaussian shape reproduces the angle dependence of the intensity at the focus extremely well. For the focus, it is certainly invalid to ignore phase distortion, and in (8) we have for the moment left the phase factor unspecified as $\exp[iS(r)]$.

For nucleons, the focus is intense, and is therefore very important, but for α particles the intensity at the focus is not significantly greater than at the brightest part of the front surface, and since the focus covers a comparatively small part of the nuclear surface we assume that as a first approximation it may be ignored. This approximation is probably invalid for small-angle scattering, when the overlap of the surface terms of ψ_i and ψ_f is small. Nevertheless the effect of the focus will be ignored in the rest of this section, but will be discussed qualitatively in the next section.

For (α,α') scattering, therefore, our approximation is

$$\begin{aligned}\psi_i&=A(r)\exp(i\mathbf{k}_i\cdot\mathbf{r}-\gamma\hat{k}_i\cdot\hat{r}),\\ \psi_f&=A(r)\exp(i\mathbf{k}_f\cdot\mathbf{r}+\gamma\hat{k}_f\cdot\hat{r}).\end{aligned}\tag{8}$$

The matrix element (2) then becomes

$$\mathfrak{M}=\int d^3\mathbf{r}\,R(r)Y_L{}^M(\theta,\varphi)[A(r)]^2\times\exp[i\mathbf{K}\cdot\mathbf{r}-\gamma(\hat{k}_i-\hat{k}_f)\cdot\hat{r}],\tag{9}$$

where $\mathbf{K}=\mathbf{k}_i-\mathbf{k}_f$, and we have made use of (5). The angle integration can be done exactly, and yields

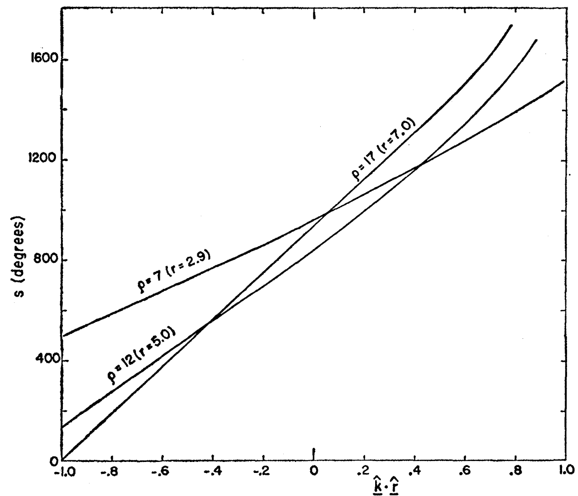

FIG. 4. Plot of phase of wave function against $\hat{k}\cdot\hat{r}$ for 40-Mev α-particles incident on Al.[13]

$$\mathfrak{M}=\int r^2dr\,R(r)[A(r)]^2\times4\pi i^Lj_L(\xi r)Y_L{}^M(\theta_\xi,\varphi_\xi),\tag{10}$$

where

$$\begin{aligned}\xi&=(\boldsymbol{\xi}\cdot\boldsymbol{\xi})^{\frac{1}{2}},\\ \boldsymbol{\xi}&=\mathbf{K}+i(\gamma/r)(\hat{k}_i-\hat{k}_f),\end{aligned}\tag{11}$$

and θ_ξ, φ_ξ are the polar angles of the complex vector $\boldsymbol{\xi}$.

To obtain the differential cross section, we must sum $|\mathfrak{M}|^2$ over values of M from $-L$ to L. This will lead us to

$$\sum_{M=-L}^{L}Y_L{}^M(\theta_\xi,\varphi_\xi)\,(-1)^MY_L{}^{-M}(\theta_{\xi'}{}^*,\varphi_{\xi'}{}^*)=\frac{2L+1}{4\pi}P_L(\cos\Theta),\tag{12}$$

where $\boldsymbol{\xi}'=\mathbf{K}+(i\gamma/r')(\hat{k}_i-\hat{k}_f)$, r' is the radial integration variable in $\mathfrak{M}^*$ (whereas r is the corresponding variable in $\mathfrak{M}$), and

$$\cos\Theta=\boldsymbol{\xi}\cdot\boldsymbol{\xi}'^*/\xi\xi'^*.$$

If $k_i\approx k_f$ (which we call the *quasi-elastic approximation*) then it is obvious that $\cos\Theta=1$. Even if the quasi-elastic approximation is not valid, however, it can be shown as in appendix 2 that $\cos\Theta\approx1$. Hence $P_L(\cos\Theta)$ in $|\mathfrak{M}|^2$ can be put equal to 1, and the radial dependence of $Y_L{}^M(\theta_\xi,\varphi_\xi)$ in (10) ignored. The differential cross section is then proportional to

$$\frac{d\sigma}{d\Omega}\propto\left|\int_0^\infty R(r)[A(r)]^2j_L(\xi r)r^2dr\right|^2.\tag{13}$$

We must now make some sort of approximation for the radial integral in (13). First of all, we realize that

because of the absorption of the α particles by the nucleus, $[A(r)]^2$ will fall off for small r. Also, because of the falloff of the nuclear wave functions outside the nucleus, there will be an upper cutoff to the integral. Hence the integral will be confined to a region fairly close to the nuclear surface. We now ask whether the oscillations of $R(r)$ within the nucleus are likely to be of significance. For a single-particle excitation, the shortest wavelength which one might expect to find associated with these oscillations is of order $2R/(n_i+n_f+1)$, where n_i and n_f are the numbers of radial nodes in the initial and final nuclear wave functions and R is the nuclear radius. For most direct interactions, this will not be less than about 2 f, while we shall find that the thickness of the surface region needed to give a reasonable fit to experiment is around 1 f. We therefore believe that as a first approximation the radial integration will depend essentially only on the nuclear radius and the thickness of the surface region, as discussed in the previous section. The simplest empirical choice we can make for $R(r)[A(r)]^2$ in (13) which involves only these two parameters is

$$R(r)[A(r)]^2 \propto \exp[-(r-R)^2/\lambda^2]. \tag{14}$$

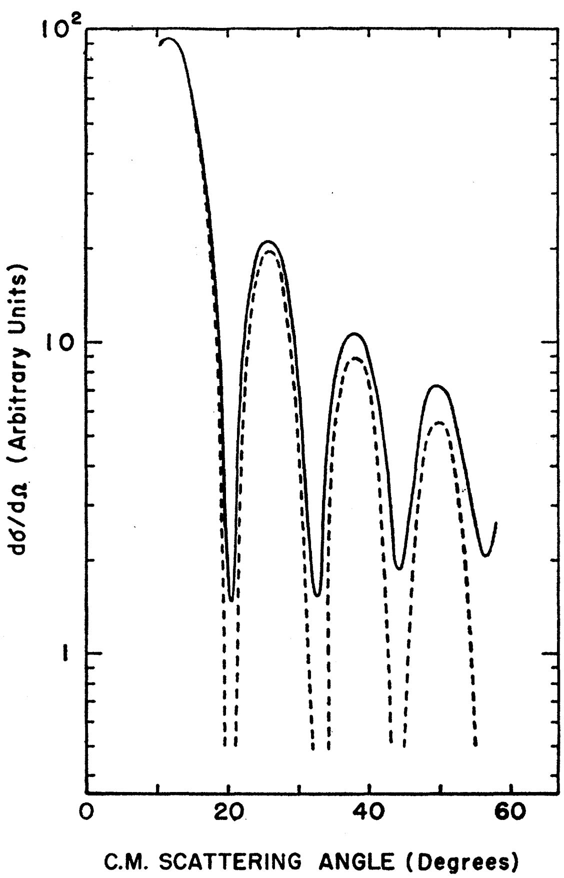

FIG. 5. Effect of anisotropy parameter. The dashed curve represents the Butler theory given by Eq. (16), and the solid curve represents our theory without any radial integration, i.e., Eq. (17).

(An alternative way of regarding this approximation is to consider that for small momentum transfer the radial integral depends on the radial weight function $R(r)[A(r)]^2$ only through its first two moments.) We believe there is no sense in attempting to represent the falloff of $R(r)$ outside the nucleus or of $[A(r)]^2$ inside more accurately than by (14) unless at the same time we make a realistic approximation for $R(r)$ inside the nucleus.

Our final form for the differential cross section for (α,α') scattering is therefore

$$\frac{d\sigma}{d\Omega} \propto \left| \int_0^\infty \exp[-(r-R)^2/\lambda^2] j_L(\xi r) r^2 dr \right|^2, \tag{15}$$

where ξ has been defined in Eqs. (12). In the quasi-elastic approximation (i.e., $k_f \approx k_i$), this can be written in the simpler form

$$\frac{d\sigma}{d\Omega} \propto \left| \int_0^\infty \exp[-(r-R)^2/\lambda^2] \times j_L[2(k_i r+i\gamma)\sin\tfrac{1}{2}\theta] r^2 dr \right|^2. \tag{15$'$}$$

For comparison we quote the analogous results for the Butler theory using a fixed radius, and for our theory also using a fixed radius:

$$\text{(Butler)}\ \frac{d\sigma}{d\Omega} \propto |j_L(2k_i R \sin\tfrac{1}{2}\theta)|^2, \tag{16}$$

$$\text{(Fixed radius)}\ \frac{d\sigma}{d\Omega} \propto |j_L[2(k_i R+i\gamma)\sin\tfrac{1}{2}\theta]|^2. \tag{17}$$

We have discussed qualitatively in the preceding section how the various parameters in (15) or (15′) should affect the cross section. As with the simple Butler theory, the positions of the maxima and minima are controlled by the radius parameter R and the angular momentum transfer L. We find that essentially the same radius parameter should be used to fit the positions of the peaks whichever of (15′), (16), or (17) is used, and therefore it is convenient to obtain R using the simplest formula, namely (16). The anisotropy parameter γ should control the peak-to-valley ratio in the cross section, and also enhance the large-angle cross section somewhat. Lastly, the relative magnitudes of successive peaks is controlled by the thickness parameter λ. Predictions of (16) and (17) are compared in Fig. 5 and typical fits for (α,α') scattering from sulfur and magnesium targets using (15′) are shown in Figs. 6 and 7. For both these calculations the values of the parameters were $\gamma=0.9$, $\lambda=0.88$ f. Similar fits have been obtained with magnesium data at energies down to 28 Mev.

There is one feature of Eqs. (17) and (15′) which we would like to clarify. In Sec. 2 we showed that localiza-

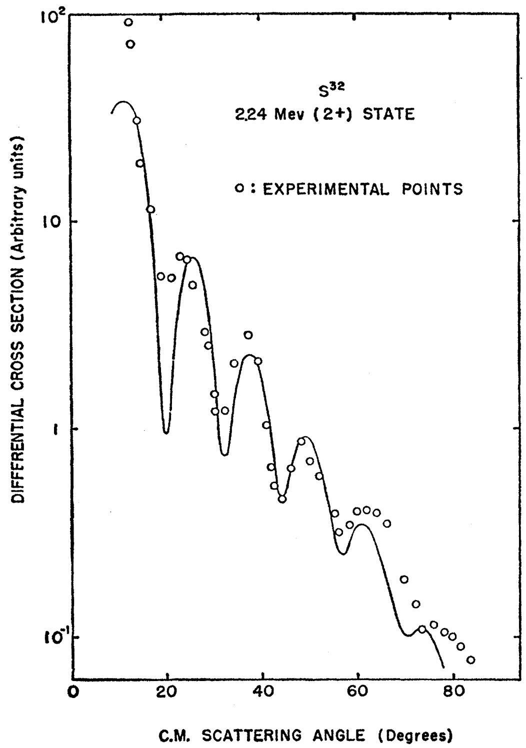

FIG. 6. Typical fit obtained for 41.7-Mev α particles on a sulfur target. The experimental points are taken from P. Robison and G. W. Farwell (private communication).

tion of the interaction to a part only of the surface should partly spoil the resolution for measurement of the angular momentum change of the target nucleus, and we illustrated this by a simple example which made this clear. In that example the simple $j_L(KR)$ of the Butler formula was replaced by a sum of terms $j_{L'}(KR)$ with coefficients whose squared moduli were interpreted as the probability that the angular momentum transfer in the inelastic episode is L' rather than L. On the other hand, Eq. (17) does not seem to have this type of structure, and only one Bessel function appears to be present. However, the argument of the Bessel function in (17) is not simply KR, and the perfect resolution of angular momentum is associated with Bessel functions of this argument, and in particular with the fact that for particular momentum transfers $|j_L(KR)|^2$ is strictly zero. The Bessel function in (17) could, however, be written as a sum of Bessel functions of KR with different L values and with coefficients depending on γ. Explicitly,

$$j_L[2(k_iR+i\gamma)\sin\tfrac{1}{2}\theta] = \sum_{l,L'}(2L'+1)(-1)^{\frac{1}{2}(l+L'-L)}(LL'00|LL'l0)^2 \times j_l(2i\gamma\sin\tfrac{1}{2}\theta)\,j_{L'}(2k_iR\sin\tfrac{1}{2}\theta), \quad (18)$$

as may be seen by comparing the expansion of $\exp[i(a+b)z]$ in a series of Legendre polynomials of z with the product of the expansions of $\exp(iaz)$ and $\exp(ibz)$. The right-hand side of (18) is very similar in structure to (6): in particular, the occurrence of the Clebsch-Gordan coefficient should be noted. The explicit dependence in (18) of the coefficient of $j_{L'}(KR)$ on the scattering angle θ is not an essentially new feature, and its analogy in the simpler model which led to (6) would be a dependence of α on θ. Equation (18) displays clearly the poor resolution for angular momentum transfer, but also spoils the great simplicity of our result. In the form (17) or (15′) the lack of resolution for angular momentum transfer, shows up through the nonzero minima of $d\sigma/d\Omega$.

We shall now discuss the values of the parameters which we have to use to get fits such as those of Figs. 6 and 7. It is notable that the anisotropy parameter γ which is required is close to that which we expect from examination of optical model wave functions. The arguments given earlier show that the maxima and minima should, if anything, be more smoothed out than one would expect solely on the basis of surface anisotropy, and it is satisfactory therefore that the γ value required to fit experimental data is, if anything, larger rather than smaller than the value obtained from optical model wave functions.

As is usual with simple direct-interaction calculations, the radius parameter R is very large, close to 2.2 $A^{\frac{1}{3}}$ f for incident α particles of 40-Mev energy. Also the width parameter λ which gives the best fit is considerable less

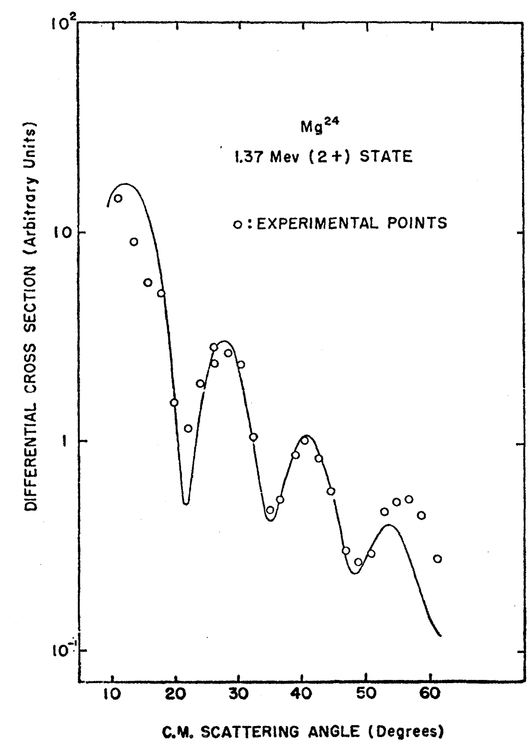

FIG. 7. Typical fit obtained for 41-Mev α particles on a magnesium target. The experimental points are from Blair, Farwell, and McDaniels.[3]

than one would expect from the amount of penetration of the α-particle wave functions into the nucleus predicted by the optical model. However, the extremely rough nature of the approximation for the radial overlap functions in (13) should warn us against too literal an interpretation of the parameters R and λ. A more realistic approximation must take account of the radial dependence of the phase and amplitude of the distorted wave functions as well as the angle dependence, and must also make a more realistic approximation for the radial dependence of the nuclear overlap function. Work in these directions is in progress.

As has already been indicated, we expect the inadequacy of our approximations for the radial integration in the matrix element to lead to failure of our predictions for large-angle scattering. There is evidence of this in Figs. 6 and 7, and attempts to fit data with magnesium[15] and argon targets[16] fail seriously at angles greater than about 100°. We hope to obtain an improvement with a more realistic radial integration. There is also an indication that our predictions fail for forward scattering, with scattering angles less than 10°–15°. From the different directions of the failure for sulfur and magnesium targets, one might suspect interference effects. There are at least two contributions to the forward scattering which we have omitted from our present calculations, and which might cause interferences. These are Coulomb excitation, and the contribution from the overlap of the focus for one channel with the surface term for the other. It is easily seen that this last effect should have a maximum for forward scattering, and also the surface-surface overlap term should then be a minimum.

4. FOCUS EFFECTS

In this section we shall discuss the focus term in our approximation (8) for the distorted wave functions in the neighborhood of the nuclear surface. We have already shown that our approximation for the amplitude of this term is reasonable. Reference to the work of McCarthy[9] shows that reasonable values for the size parameter are in the region of 1 to 2 f. For nucleons of medium energies, the B parameter giving the intensity at the focus should be chosen so that $|\psi|^2$ at the most intense point is about 20–30 times $|\psi|^2$ at the brightest part of the front surface. This parameter, however, is sensitive to optical potential parameters, and should therefore be fitted empirically. For strongly absorbed particles such as α particles, $|\psi|^2$ at the focus should be around 1–2 times $|\psi|^2$ on the front surface. The remaining parameter in the amplitude distortion is a, the distance of the center of the focus from the center of the nucleus. At zero energy, a is zero (since only the s wave contributes to the scattering), i.e., the focus is at the center of the nucleus. However by the time a nucleon energy of order of 10 Mev is reached the focus has moved out considerably towards the surface, and from 20 Mev onwards a appears to be a reasonably linear function of energy.

So far we have not discussed the phase distortion in the focus term, which certainly is not negligible. However, since the focus is a relatively small region of the nucleus, we expect the gradient of the phase to be reasonably constant throughout the focal region. From the axial symmetry of the problem, the direction of this gradient must be the direction of $\mathbf{k}_i$. Hence we expect the phase at the focus to be represented fairly well by

$$S = c\mathbf{k}_i \cdot \mathbf{r} + \delta, \tag{19}$$

where $c<1$. Reference to flux calculations does indeed support the belief that the direction of ∇S within the focus is just the direction of $\mathbf{k}_i$. The approximation (19) can be shown to be exact if the wave function at the focus is a superposition of functions each with a wave vector of the same magnitude as $\mathbf{k}_i$ but twisted through a fixed angle ω (see Appendix 3). In this case $c=\cos\omega$. Since the amplitude of the wave function falls off in the interior of the nucleus because of absorption, most of the particles which reach the focus must have come through the lower density surface region, and this suggests that the "fixed twist" approximation for the phase distortion might be good enough to help us to estimate c. Reference to flux pictures shows that the particles do indeed enter the focus at quite a large angle to the direction of the incident beam, and values of ω as large as 60° or more are not unreasonable. This would imply $c \lesssim \frac{1}{2}$.

In order to determine the additional constant δ in (19), it is necessary to know the relative phase of the wave function at the focus and at the front surface. We do not see any convincing way of relating this to the other parameters of our theory, and therefore δ must be retained as a free parameter.

It is unfortunate that the treatment of the focus requires the introduction of five new parameters (at a fixed energy), namely B, σ, a, c, and δ. Obviously it will not be possible to determine all of these from scattering data, and some at least will have to be obtained from optical model calculations. We do not have available at present calculations which would justify any more precise statements about the values of the parameters than we have already given.

Nevertheless, it is possible to discuss qualitatively the effect the focus will have on differential cross sections. The overlap function (4) will have contributions from the overlap of the two surface terms, from the overlap of the two focus terms, and from the overlap of one surface and one focus term. Let us call these SS, FF, and SF terms, respectively. The SS terms have been discussed in the preceding section. Before going further, let us ask for which angles each type of term will dominate. Obviously the SS and FF overlaps will be greatest for large-angle scattering, and the reason for the usual falloff of the differential cross sections at large angles is the

[15] G. B. Shook, Phys. Rev. **114**, 310 (1959).
[16] L. Seidlitz, E. Bleuler, and D. J. Tendam, Phys. Rev. **110**, 682 (1958).

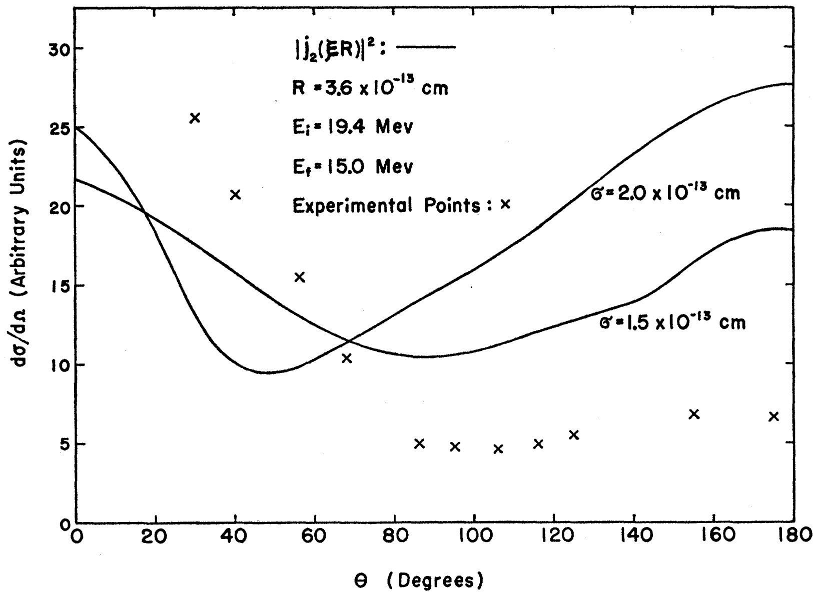

FIG. 8. Comparison of predicted cross section using a single SF term and no radial integration with experimental cross section for (p,p') scattering on a C^{12} target. The experimental points are taken from R. W. Peele [Phys. Rev. **105**, 1311 (1957)]. σ is the parameter controlling the size of the focus.

absence of large momentum-transfer components in the nucleus. The SF terms in the overlap, however, will be largest for forward scattering, least for back scattering. Consequently, provided the intensity at the focus is great enough to compensate for its comparatively small size, we might expect the forward scattering to be dominated by the SF terms.

In order to get a feel for the effect of a large SF term, we may study the cross section predicted using only this part of the matrix element. The angular integrals can again be performed explicitly, and the differential cross section is proportional to the squared modulus of a complex Bessel function and a Legendre polynomial. The latter does not vary with angle as rapidly as the former, so that the structure of the cross section is dominated by the Bessel function. Furthermore, the qualitative behavior of the Bessel function is not altered significantly if we put $c=1$ in (19), although of course the detailed behavior is changed.

Figures 8 and 9 show the differential cross section obtained using a single SF term (instead of two interfering SF terms), and using an interaction confined to the nuclear surface, for 19.4-Mev protons on C^{12} and 40-Mev protons on Fe. The rise in the cross section at large angles is due to the omission of the radial integration. The SF terms lead to large scattering cross sections for forward scattering, in qualitative agreement with experimental results. Of course, one should not

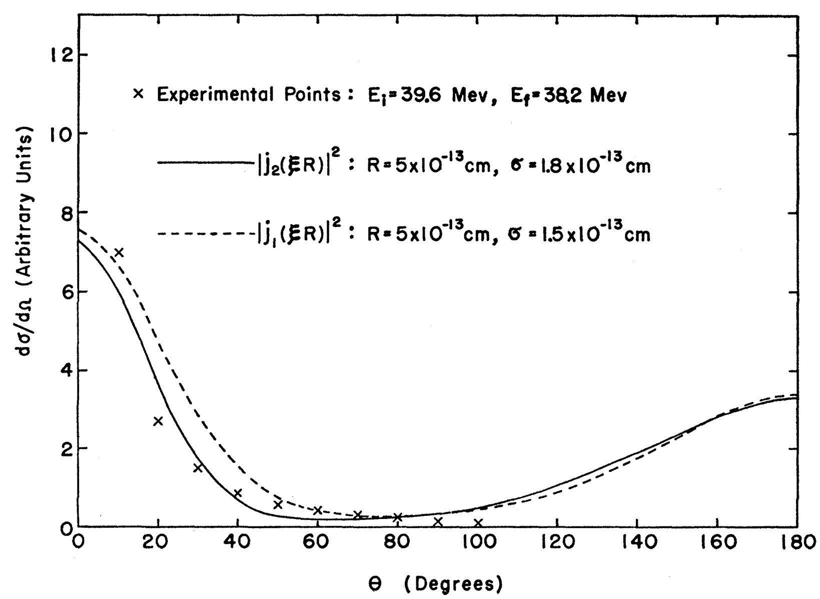

FIG. 9. Comparison of predicted cross section using a single SF term for (p,p') scattering on an iron target at 40 Mev. The experimental points are taken from M. K. Brussel and J. H. Williams [University of Minnesota Linear Accelerator Laboratory Annual Progress Report, March, 1958 (unpublished), p. 41; see also M. K. Brussel and J. H. Williams, Phys. Rev. **114**, 525 (1959)].

expect more than rough qualitative agreement with experiment with the very crude approximations used in Figs. 8 and 9.

The large forward scattering cross section can be understood in terms of the uncertainty principle arguments of Sec. 2. The SF terms are localized to such a small region of the nuclear surface that very little resolution for angular momentum transfer is retained. Consequently there can be a very large contribution to the cross section from $j_0(KR)$ even though in fact the angular momentum transfer L is 2. Butler, Austern, and Pearson[17] have given what appears to be an alternative explanation of the anomalous forward scattering typical of (p,p') experiments, namely that the bending of the incoming beam introduces an extra transverse component of momentum which permits a scattering to take place which one might naively expect to be forbidden by angular momentum conservation. This is really not a different interpretation, however. In Appendix 3 it is shown that the transverse momentum components introduced in this way contribute to the amplitude distortion rather than the phase distortion, so that the existence of the focus is possible only because of the effect considered by Butler, Austern, and Pearson. When the process is considered from a wave-mechanical rather than a quasi-classical viewpoint, however, the required transverse momentum components arise because of the localization in the focus and the uncertainty principle.

We do not expect the SF terms to dominate the cross section at all angles. Rather, we expect SF terms to dominate the forward scattering, but SS terms to be important at medium angles, and FF terms at angles greater than about 150° (provided the foci are sufficiently intense). This agrees with the observed fact that most (p,p') scattering results show a nonzero forward cross section, even though $L\neq 0$, and show a definite structure at medium angles with peaks whose position can be fitted fairly well with the Butler theory. It is also notable that large back-scattering peaks have been observed in (α,p) and (d,p) cross sections.[18] We also expect the effect of the SF terms to be dominant primarily when the focus falls in the region of the nuclear surface. If the focus is further in, then the SF overlap will be considerably reduced, and if it is further out, then the nuclear density will be too low for there to be an appreciable contribution to the matrix element. With the parameters we have suggested, the proton focus should be approaching the surface for energies near 40 Mev, but should still be largely inside at 20 Mev. This agrees qualitatively with the experimental data for Fe, which show structure corresponding to the Butler theory with $L=2$ (although with some "anomalous" forward scattering) at 20 Mev, but show only the forward peak at 40 Mev. We expect that at greater energies the structure will reappear in the cross section.

The arguments of this section are necessarily lacking in detail, but we feel that our wave functions lead us to expect results qualitatively sufficiently close to the observed cross sections for it to be worth attempting more detailed calculations. Such calculations are in progress.

5. CONCLUSIONS

The results of this paper fall into four categories. In the first place we have given a method of predicting qualitatively the effect on direct-interaction differential cross sections of distortion of the wave functions of the scattered particle. Secondly, we have given an empirical approximation for this distortion in the region of the nuclear surface, and have discussed the justification of the approximation on the basis of the optical model. Thirdly, we have shown that our approximation works extremely well for (α,α') scattering in the energy range 20–40 Mev and for angles up to 90°, and have demonstrated that our method of physical interpretation correctly describes the effect of modifying our parameters. Lastly, we have used our method of interpretation to understand qualitatively some of the more complicated features of direct interactions, especially interactions involving nucleons. We have not so far included spin effects in our calculations, nor have we yet detailed calculations of the effect of the focus, but such calculations are in progress.

Perhaps the most important of our detailed results is to learn how to approximate the angle dependence of the amplitude of distorted wave functions, and the simple expression for the differential cross section for inelastic scattering which follows in a natural way. The close agreement of the γ parameter chosen to fit the scattering data and its value predicted from optical model wave functions convinces us that our interpretation of this parameter is substantially correct. On the other hand, the two parameters R and λ, which relate to the radial dependence of the integrand in the matrix element, do not have any very direct physical interpretation.

The use of general arguments to predict qualitatively the effects of localization of the direct interaction is helpful whenever it is useful to treat the distorted wave functions as a whole rather than as a sum of partial waves. These arguments were of great help to us in developing and understanding physically the results of the more detailed calculations presented in Secs. 3 and 4. However, they are also relevant to other direct interaction processes, wherever there is reason to believe the interaction is largely confined to just a small region of the nucleus.

The use of simple approximations to the distorted wave functions in the theory of direct interactions is limited by the difficulty of finding a simple approximation involving only a few parameters which yet reason-

[17] S. T. Butler, N. Austern, and C. Pearson, Phys. Rev. **112**, 1227 (1958).

[18] D. H. Wilkinson, *Proceedings of the International Conference on Nuclear Structure, Kingston*, edited by D. A. Bromley and E. W. Vogt (University of Toronto Press, Toronto, Canada, 1960).

ably accurately reproduces the results of direct optical model calculations. The value lies in the fact that the complete wave function has certain simple features which may easily be parametrized, but which are hard to discover from an examination of the partial waves separately. Approximations such as we use therefore give a more direct insight into the physical reasons behind those aspects of direct-interaction processes which are due to simple properties of the complete wave function than could be obtained using an analysis in terms of partial waves. Of course, a partial wave analysis should, if correctly performed, lead to the same results as the use of an exact expression for the complete wave functions, and for some purposes this may frequently be the best procedure. We believe that the most important further development of our use of approximations to the complete optical model wave function near the nucleus will be the investigation of effects due to the focus.

ACKNOWLEDGMENTS

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APPENDIX 1

In this Appendix, we demonstrate that if a probe described by $\rho\exp(iS)$, where ρ is a known function of position, is interpreted as measuring momentum, then the probe will have maximum resolution for detecting a momentum $\mathbf{k}\hbar$, consistent with the uncertainty principle and the known localization ρ of the probe, if $S=\mathbf{k}\cdot\mathbf{r}$. By this interpretation of the probe, we mean that $\int f(\mathbf{r})\rho\exp(iS)d^3r$ is to be regarded as an approximation to the Fourier transform $\int f(\mathbf{r})\exp(i\mathbf{k}\cdot\mathbf{r})d^3r$ of f corresponding to wave number vector $\mathbf{k}$. Clearly the resolution of the probe is better, the better $\rho\exp(iS)$ approximates $\exp(i\mathbf{k}\cdot\mathbf{r})$. The criterion we adopt that $\rho\exp(iS)$ approximates $\exp(i\mathbf{k}\cdot\mathbf{r})$ in some (perhaps poor) sense is that, in the Fourier decomposition of $\rho\exp(iS)$, the mean wave number vector is $\mathbf{k}$, and the criterion for maximum resolution for $\mathbf{k}$ is that the mean square deviation of wave number vectors about $\mathbf{k}$ is a minimum.

For convenience, we suppose ρ normalized so that $\int\rho^2d^3r=1$. Then the conditions for the interpretation we wish for $\rho\exp(iS)$ are

$$\int\rho e^{-iS}(-i\nabla-\mathbf{k})\rho e^{iS}d^3r=0, \tag{A1.1}$$

$$\int\rho e^{-iS}(-i\nabla-\mathbf{k})^2\rho e^{iS}d^3r=\text{minimum}. \tag{A1.2}$$

Simplification, and introduction of Lagrange multipliers, lead to the equivalent condition

$$\int\rho^2\{-\rho^{-1}\nabla^2\rho-2i(\nabla\ln\rho)\cdot\boldsymbol{\sigma}-i\nabla\cdot\boldsymbol{\sigma}+\sigma^2+k^2 +\boldsymbol{\lambda}\cdot[\boldsymbol{\sigma}-i\nabla\ln\rho-\mathbf{k}]\}d^3r=\text{minimum}, \tag{A1.3}$$

where $\boldsymbol{\sigma}=\nabla S$. We now treat this as a variational problem, varying $\boldsymbol{\sigma}$, and obtain

$$-2i\rho^2(\nabla\ln\rho)+i\nabla\rho^2+\boldsymbol{\lambda}\rho^2+2\rho^2\boldsymbol{\sigma}=0, \tag{A1.4}$$

or

$$\boldsymbol{\sigma}=-\boldsymbol{\lambda}=\text{constant}, \tag{A1.5}$$

and also

$$\int\rho^2[\boldsymbol{\sigma}-i\nabla\ln\rho-\mathbf{k}]d^3r=0, \tag{A1.6}$$

or if $\rho\rightarrow 0$ sufficiently rapidly for large r, then

$$0=\int\rho^2(\boldsymbol{\sigma}-\mathbf{k})d^3r=-\int\rho^2(\boldsymbol{\lambda}+\mathbf{k})d^3r=-(\boldsymbol{\lambda}+\mathbf{k}). \tag{A1.7}$$

Hence $\boldsymbol{\lambda}=-\mathbf{k}$, and

$$\nabla S=\boldsymbol{\sigma}=\mathbf{k}.$$

Hence

$$S=\mathbf{k}\cdot\mathbf{r}+\text{constant},$$

which is what we wished to prove.

APPENDIX 2

In this Appendix we justify the assertion made in the body of the paper that $\cos\Theta\approx 1$, or equivalently $|\sin\Theta|\ll 1$, where Θ was defined following Eq. (12). The approximation is obviously exact for 0° and 180° scattering or if $k_f=k_i$. We now consider the more general case.

It is easy to show that

$$\boldsymbol{\xi}\times\boldsymbol{\xi}'^*=i\gamma(1/r+1/r')(k_i-k_f)\hat{k}_i\times\hat{k}_f, \tag{A2.1}$$

and

$$\boldsymbol{\xi}\cdot\boldsymbol{\xi}'^*=(k_i-k_f)^2+(1-\cos\theta)[2k_ik_f+2\gamma^2/rr' +i\gamma(k_i+k_f)(1/r-1/r')], \tag{A2.2}$$

where $\cos\theta=\hat{k}_i\cdot\hat{k}_f$, i.e., θ is the scattering angle.

Since $(\boldsymbol{\xi}\times\boldsymbol{\xi}'^*)^2=\xi^2\xi'^{*2}-(\boldsymbol{\xi}\cdot\boldsymbol{\xi}'^*)^2$, it is clear that

$$|\sin\Theta|^2\equiv|(\boldsymbol{\xi}\times\boldsymbol{\xi}'^*)^2/\xi^2\xi'^{*2}|\ll 1,$$

provided

$$|(\boldsymbol{\xi}\times\boldsymbol{\xi}'^*)/(\boldsymbol{\xi}\cdot\boldsymbol{\xi}'^*)|\ll 1.$$

Now

$$|\boldsymbol{\xi}\cdot\boldsymbol{\xi}'^*|\geqslant(k_i-k_f)^2+2(1-\cos\theta)(k_ik_f+\gamma^2/rr'), \tag{A2.3}$$

and therefore

$$|(\boldsymbol{\xi}\times\boldsymbol{\xi}'^*)/(\boldsymbol{\xi}\cdot\boldsymbol{\xi}'^*)| \leqslant\frac{\gamma(k_i-k_f)(1/r+1/r')|\sin\theta|}{(k_i-k_f)^2+2(1-\cos\theta)(k_ik_f+\gamma^2/rr')}. \tag{A2.4}$$

We maximize this expression with respect to the scattering angle θ, and obtain

$$|(\xi\times\xi'^*)/(\xi\cdot\xi'^*)|\leqslant\gamma(1/r+1/r')\times[(k_i+k_f)^2+4\gamma^2/rr']^{-\frac{1}{2}}. \quad \text{(A2.5)}$$

From this we see that the approximation $\cos\Theta\approx 1$ will be good if $k_i+k_f\gg\gamma(1/r+1/r')$, and still will not be too bad for most scattering angles even if $k_i+k_f\approx\gamma(1/r+1/r')$. However, even for 10-Mev incident nucleons, this limit will not be approached until r or r' or both are smaller than or of the order of 1 f, and the contribution to the radial integrals from such values of r should be unimportant. Hence we conclude that even in this rather extreme case, $\cos\Theta\approx 1$ should be an adequate approximation. Of course the approximation becomes still better at higher energies, or for heavier projectiles such as α particles at the same energy.

APPENDIX 3

In this Appendix we consider the nature of the phase distortion in the neighborhood of the focus if in this region the α-particle wave function is a superposition of waves with wave number vectors each of magnitude k but twisted out of the direction of $\mathbf{k}$ through a definite angle ω. Thus we write

$$\psi\sim\rho(\mathbf{r})\int f(\boldsymbol{\eta})\exp(ik\boldsymbol{\eta}\cdot\mathbf{r})d\Omega_\eta, \quad \text{(A3.1)}$$

where $\eta^2=1$, $\boldsymbol{\eta}\cdot\mathbf{k}=k\cos\omega$, and we assume that $f(\boldsymbol{\eta})$ depends only on the angle between $\boldsymbol{\eta}$ and $\mathbf{k}$, i.e., on ω, which is fixed. (A3.1) can then be rewritten

$$\begin{aligned}\psi\sim\rho(\mathbf{r})\int_0^{2\pi}&\exp\{ikr[\cos\omega\cos\theta\\&+\sin\omega\sin\theta\cos(\varphi_\eta-\varphi)]\}d\varphi_\eta,\\=\rho(\mathbf{r})&\exp(i\mathbf{k}\cdot\mathbf{r}\cos\omega)\\&\times\int_0^{2\pi}\exp[ikr\sin\omega\sin\theta\cos(\varphi_\eta-\varphi)]d\varphi_\eta. \quad \text{(A3.2)}\end{aligned}$$

Now since the integral in (A3.2) is over a complete cycle of φ_η, it does not matter at which point in the cycle the integration is begun. Hence

$$\begin{aligned}&\int_0^{2\pi}\exp[ikr\sin\omega\sin\theta\cos(\varphi_\eta-\varphi)]d\varphi_\eta\\&=\int_\pi^{3\pi}\exp[ikr\sin\omega\sin\theta\cos(\varphi_\eta-\varphi)]d\varphi_\eta,\\&=\int_0^{2\pi}\exp[-ikr\sin\omega\sin\theta\cos(\varphi_\eta-\varphi)]d\varphi_\eta,\\&=\left(\int_0^{2\pi}\exp[ikr\sin\omega\sin\theta\cos(\varphi_\eta-\varphi)]d\varphi_\eta\right)^*. \quad \text{(A3.3)}\end{aligned}$$

Hence (A3.2) is of the form

$$\psi\sim\rho'(\mathbf{r})\exp(i\mathbf{k}\cdot\mathbf{r}\cos\omega), \quad \text{(A3.4)}$$

which is what was to be proved.