$V-\theta$ Collisions in the Lee Model*

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The methods of dispersion theory are used to obtain an exact expression for the $V-\theta$ elastic scattering amplitude and the amplitude for the production process, $V+\theta\rightarrow N+2\theta$, in the Lee model.

I. INTRODUCTION

THE Lee model[1] was introduced to provide a soluble but nontrivial field theory exhibiting renormalization. As such it has proved valuable not only for insight into this problem[1,2] but also as a testing ground for new methods of calculation. For example, recently Goldberger and Treiman[3] and DeCelles and Feldman[4] have applied the methods of dispersion theory to obtain the classical results of the model. In this paper we show how their methods may be used to obtain results not previously known from the usual approaches to the model.

It will be recalled that the Lee model describes the interaction of two Fermions, V and N, with a Boson, θ. The Fermions are taken to be fixed while the θ with mass μ is assumed to have a relativistic momentum-energy relation. The particular characteristic of the model is the interaction, which allows only the elementary process $V\rightleftarrows N+\theta$. With this choice of selection rules, the N and θ fields do not need to be renormalized. (The theory is nonrelativistic and thus there are no antiparticles), while the V-particle self-energy arises only from $N-\theta$ "bubbles." In considering this aspect of the problem one is led naturally to study together the state vectors for the physical V particle as well as for $N-\theta$ scattering. It is the study of these states which is the primary concern of the "classical" Lee model.

The next most complicated state one can try to study is $V-\theta$ scattering. However, the selection rules for the model couple this state to the three-particle state $N+2\theta$. Thus, although one can write an integral equation for the state vector describing the $V-\theta$ system, attempts to solve it have been unsuccessful.[2] In this paper we apply the techniques of dispersion theory to the problem of $V-\theta$ scattering and the related problem of θ production in $V-\theta$ collisions. By following a method suggested by recent work in the theory of nuclear direct reactions,[5] we are able to obtain directly the exact amplitude for the scattering and the production process without having to calculate the state vectors. In addition to providing another solved aspect of the Lee model, these new results are, we believe, the first examples of the exact expression for a scattering amplitude in which production is possible and of a production amplitude in field theory. As such they have a number of analogies in real processes, some of which are discussed in Sec. V.

In Sec. II the formal structure of the Lee model is reviewed briefly, and the methods of dispersion theory are applied to the problem of $V-\theta$ scattering. One is led to an integral equation for an amplitude closely related to the production amplitude and this equation is solved. From this the $V-\theta$ elastic scattering amplitude is constructed. In Sec. III the results of the previous section are used to obtain the amplitude for the process $V+\theta\rightarrow N+2\theta$. In Sec. IV the unitarity of the elastic scattering amplitude is established, and in Sec. V a brief discussion of the results is presented. Calculation of some integrals is relegated to the appendix.

II. $V-\theta$ SCATTERING

The Lee model describes a world of three particles, V, N, and θ. It has been studied extensively particularly with regard to the $N-\theta$ scattering state and the V-particle state and we shall not review the discussion here. For completeness we state the Hamiltonian.[6]

$$H=mZ\psi_V{}^\dagger\psi_V+m\psi_N{}^\dagger\psi_N+\sum_k\omega_k a_k{}^\dagger a_k +g\psi_N{}^\dagger\psi_V A^\dagger+g\psi_V{}^\dagger\psi_N A+\delta m_V Z\psi_V{}^\dagger\psi_V, \quad (1)$$

where

$$A=\sum_k \frac{u(\omega)}{(2\omega\Omega)^{\frac{1}{2}}}a_k, \quad \omega=(\mu^2+k^2)^{\frac{1}{2}}, \quad (2)$$

$$[a_{k'},a_k{}^\dagger]=\delta_{k,k'}, \quad \{\psi_N,\psi_N{}^\dagger\}=1, \quad \{\psi_V,\psi_V{}^\dagger\}=1/Z,$$
$$[a_{k'},a_k]=\{\psi_N,\psi_N\}=\{\psi_V,\psi_V\}=0. \quad (3)$$

We have quantized in a box of volume Ω, later $\Omega\rightarrow\infty$. g is the renormalized coupling constant. ψ_V is the renormalized V-particle field operator and Z is a renormalization constant. These are chosen so that $\langle 0|\psi_V|V\rangle=1$. δm_V is the V-particle mass renormalization. δm_V and Z are easily solved for in terms of the other quantities appearing in the Hamiltonian. We have assumed that the interaction $V\rightleftarrows N+\theta$ contains a source function, $u(\omega)$, such that all integrals which we encounter exist and such that there are no ghost V-particle states.

* Supported in part by the U. S. Atomic Energy Commission.
[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[2] G. Kallen and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 7 (1955).
[3] M. L. Goldberger and S. B. Treiman, Phys. Rev. **113**, 1663 (1959). This paper will be referred to as GT.
[4] P. Decelles and G. Feldman, Nuclear Phys. **14**, 517 (1959/60).
[5] R. D. Amado and R. Blankenbecler (to be published).
[6] Our notation follows closely that of GT, except that we make the simplifying assumption, unrestrictive in our case, that the V and N have the same mass.

We wish to calculate $V-\theta$ elastic scattering within the framework of this model. The S-matrix element of interest is

$$S=\langle V\theta_{\omega}^{(-)}|V\theta_{\omega'}^{(+)}\rangle, \tag{4}$$

where the plus and minus refer to the usual "in" and "out" states. We use the asymptotic definition of these states to write

$$S=\lim_{t\to\infty}\langle V|a_k(t)|V\theta_{\omega'}^{(+)}\rangle e^{i\omega t},$$

where the Heisenberg operator $a_k(t)$ is defined by

$$a_k(t)=\exp(iHt)a_k\exp(-iHt). \tag{5}$$

We can write

$$S-\delta_{k,k'}=i\int_{-\infty}^{\infty}e^{i\omega t}\left(-i\frac{d}{dt}+\omega\right)\langle V|a_k(t)|V\theta_{\omega'}^{(+)}\rangle dt. \tag{6}$$

Defining the θ current by[3,7]

$$j(t)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\left(-i\frac{d}{dt}+\omega\right)a_k(t)=-g\psi_N^{\dagger}(t)\psi_V(t), \tag{7}$$

we have

$$S-\delta_{k,k'}=i\int_{-\infty}^{\infty}e^{i\omega t}\langle V|j(t)|V\theta_{\omega'}^{(+)}\rangle\frac{u(\omega)}{(2\omega\Omega)^{\frac{1}{2}}}dt. \tag{8}$$

Using the time translation property,

$$j(t)=\exp(iHt)j\exp(-iHt), \tag{9}$$

the time integral can be done, and we obtain

$$S-\delta_{k,k'}=2\pi i\delta(\omega-\omega')\frac{u(\omega)}{(2\omega\Omega)^{\frac{1}{2}}}\langle V|j|V\theta_{\omega'}^{(+)}\rangle. \tag{10}$$

We define the scattering amplitude in the usual way by

$$T(\omega)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\langle V|j|V\theta_{\omega}^{(+)}\rangle, \tag{11}$$

so that

$$S=\delta_{k,k'}+2\pi i\delta(\omega-\omega')\frac{u^2(\omega)}{2\omega\Omega}T(\omega). \tag{12}$$

To make further progress in obtaining T, one must contract another particle. One's success in solving the problem depends on which one he contracts. Following a method developed to deal with nuclear direct reactions,[5] we choose to contract the V particle from the left. We obtain

$$T(\omega)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\lim_{t\to\infty}\langle 0|\{\psi_V(t),j\}|V\theta_{\omega}^{(+)}\rangle e^{imt}. \tag{13}$$

The extra term introduced by the anticommutator is zero. Proceeding as before, we write[8]

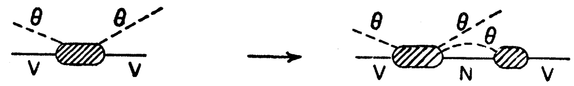

FIG. 1. Dispersion graph for $V-\theta$ scattering corresponding to the division of the amplitude given in Eq. (17).

$$T(\omega)=i\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\int_{-\infty}^{\infty}e^{imt}\left(-i\frac{d}{dt}+m\right)\times\langle 0|\{\psi_V(t),j\}|V\theta_{\omega}^{(+)}\rangle\theta(t)dt. \tag{14}$$

The equal-time anticommutator resulting from differentiation of the theta function gives zero. Thus we may write

$$T(\omega)=i\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\int_{\infty}^{\infty}e^{imt}\theta(t)\langle 0|\{f(t),j\}|V\theta_{\omega}^{(+)}\rangle dt, \tag{15}$$

where

$$f(t)=\left(-i\frac{d}{dt}+m\right)\psi_V(t)=-\delta m_V\psi_V(t)-\frac{g}{Z}\psi_N(t)A(t). \tag{16}$$

If a complete set of intermediate states is inserted in (15), we can do the time integral, and we obtain a Low-type equation[9]:

$$T(\omega)=\sum_S\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\frac{\langle 0|f|S\rangle\langle S|j|V\theta_{\omega}^{(+)}\rangle}{S-m-i\epsilon}. \tag{17}$$

The second term from the anticommutator gives no contribution. The states S in (17) must be states with the same quantum numbers as a V particle. Since $\langle 0|f|V\rangle=0$, only the $N-\theta$ state will contribute. We choose a "plus" state, which choice leads to considerable simplification later on. We obtain for (17)

$$T(\omega)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\sum_{k'}\frac{\langle 0|f|N\theta_{\omega'}^{(+)}\rangle\langle N\theta_{\omega'}^{(+)}|j|V\theta_{\omega}^{(+)}\rangle}{\omega'}. \tag{18}$$

This division of T corresponds to the dispersion graph of Fig. 1. Now we define

$$K(\omega)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\langle 0|f|N\theta_{\omega}^{(+)}\rangle, \tag{19}$$

and

$$F(\omega',\omega)=\frac{(\omega\omega')^{\frac{1}{2}}}{u(\omega)u(\omega')}\langle N\theta_{\omega'}^{(+)}|j|V\theta_{\omega}^{(+)}\rangle, \tag{20}$$

then

$$T(\omega)=\sum_{k'}\frac{K(\omega')u^2(\omega')F(\omega',\omega)}{\omega'^2}. \tag{21}$$

[7] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 205 (1955).

[8] $\theta(t)=0$ for $t<0$; $\theta(t)=1$ for $t>0$.

[9] F. G. Low, Phys. Rev. **97**, 1392 (1955).

The problem of calculation of $T(\omega)$ is then reduced, or better expanded, to computing $K(\omega)$ and $F(\omega',\omega)$. Since the $N\theta$ state is known, K is known. It is obtained for example by GT using dispersion-theoretic methods. Thus we need only calculate $F(\omega',\omega)$.

$F(\omega',\omega)$ is nearly a production amplitude for the process $V+\theta \rightarrow N+2\theta$ but it differs from a normal one in that both the states involved are "plus" or "in" states. As we shall see in Sec. III, the ordinary production amplitude can still be calculated in terms of F, but the presence of the two "plus" states will greatly simplify its calculation. To obtain F, we contract the θ from the left in (20) and obtain

$$F(\omega',\omega)=\frac{(\omega\omega')^{\frac{1}{2}}}{u(\omega)u(\omega')}\lim_{t\rightarrow-\infty}\langle N|[a_{k'}(t),j]|V\theta_\omega^{(+)}\rangle e^{i\omega' t} +\frac{(\omega\omega')^{\frac{1}{2}}}{u(\omega)u(\omega')}\lim_{t\rightarrow-\infty}\langle N|ja_{k'}(t)|V\theta_\omega^{(+)}\rangle e^{i\omega' t}. \quad (22)$$

The second term is introduced to cancel the extra term from the commutator. It is not now zero, but rather

$$\lim_{t\rightarrow-\infty}\langle N|ja_{k'}(t)|V\theta_\omega^{(+)}\rangle e^{i\omega' t}=\delta_{k,k'}\langle N|j|V\rangle,$$

where we have used the definition of the "in" state. Using (7), we also see that

$$\langle N|j|V\rangle=-g. \quad (23)$$

Thus we may write

$$\begin{aligned}F(\omega',\omega)=&-i\frac{(\omega\omega')^{\frac{1}{2}}}{u(\omega)u(\omega')}\int_{-\infty}^{\infty}e^{i\omega' t}\left(-i\frac{d}{dt}+\omega'\right)\\&\times\langle N|[a_{k'}(t),j]|V\theta_\omega^{(+)}\rangle\theta(-t)dt-\frac{\omega g}{u^2(\omega)}\delta_{k,k'}\\=&-i\left(\frac{\omega}{2\Omega}\right)^{\frac{1}{2}}\frac{1}{u(\omega)}\int_{-\infty}^{\infty}\theta(-t)e^{i\omega' t}\\&\times\langle N|[j(t),j]|V\theta_\omega^{(+)}\rangle dt-\frac{\omega g}{u^2(\omega)}\delta_{k,k'},\end{aligned} \quad (24)$$

where we have used the fact that the equal-time commutator, $[a_{k'},j]$, is zero. Inserting intermediate states and doing the time integrals, we obtain

$$\begin{aligned}F(\omega',\omega)=&\left(\frac{\omega}{2\Omega}\right)^{\frac{1}{2}}\frac{1}{u(\omega)}\sum_S\langle N|j|S\rangle\langle S|j|V\theta_\omega^{(+)}\rangle\\&\times\left(\frac{1}{S-\omega'-m+i\epsilon}+\frac{1}{\omega'+S-m-\omega-i\epsilon}\right)\\&-g\frac{\omega}{u^2(\omega)}\delta_{k,k'}.\end{aligned} \quad (25)$$

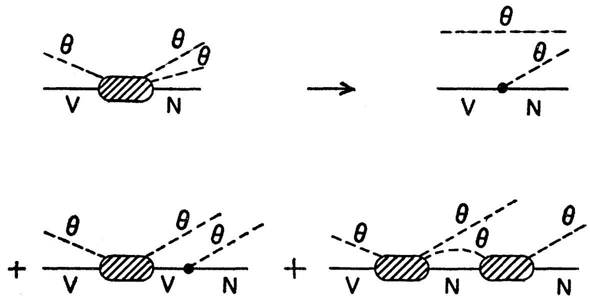

FIG. 2. Dispersion graph for the amplitude $F(\omega',\omega)$ as divided in Eq. (26). To each of the last two graphs there corresponds another with the roles of the outgoing θ particles interchanged.

The states S can be either the V-particle state or the $N-\theta$ state and once more we choose the "plus" state. We then get

$$\begin{aligned}F(\omega',\omega)=&-g\frac{\omega}{u^2(\omega)}\delta_{k,k'}\\&+\left(\frac{\omega}{2\Omega}\right)^{\frac{1}{2}}\frac{1}{u(\omega)}\langle N|j|V\rangle\langle V|j|V\theta_\omega^{(+)}\rangle\\&\times\left(\frac{1}{\omega'-\omega-i\epsilon}-\frac{1}{\omega'}\right)+\left(\frac{\omega}{2\Omega}\right)^{\frac{1}{2}}\frac{1}{u(\omega)}\\&\times\sum_{k_1}\langle N|j|N\theta_{\omega_1}^{(+)}\rangle\langle N\theta_{\omega_1}^{(+)}|j|V\theta_\omega^{(+)}\rangle\\&\times\left(\frac{1}{\omega'+\omega_1-\omega-i\epsilon}+\frac{1}{\omega_1-\omega'+i\epsilon}\right).\end{aligned} \quad (26)$$

The graphic contribution of these terms to F is represented in Fig. 2.

If we define the $N-\theta$ scattering amplitude corresponding to T of (11) as

$$\mathfrak{M}(\omega)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\langle N|j|N\theta_\omega^{(+)}\rangle, \quad (27)$$

then using (11), (20), (23), and (27) we can write for (26)

$$\begin{aligned}F(\omega',\omega)=&-g\frac{\omega}{u^2(\omega)}\delta_{k,k'}\\&+\frac{g}{2\Omega}T(\omega)\left(\frac{1}{\omega'}-\frac{1}{\omega'-\omega-i\epsilon}\right)\\&+\frac{1}{2\Omega}\sum_{k_1}\frac{u^2(\omega_1)}{\omega_1}\mathfrak{M}(\omega_1)F(\omega_1,\omega)\\&\times\left(\frac{1}{\omega_1+\omega'-\omega-i\epsilon}+\frac{1}{\omega_1-\omega'+i\epsilon}\right).\end{aligned} \quad (28)$$

This is an integral equation for F in which the kernel is known in terms of the known $N-\theta$ scattering ampli-

tude. One might worry that the inhomogeneous term contains $T(\omega)$, which is certainly unknown at this stage, but if we consider the integral equation as an equation in the variable ω', for fixed ω, then $T(\omega)$ appears as an unknown constant in the equation. Since the sum in (28) is over k', the appearance of this unknown parameter in the solution for $F(\omega',\omega)$ will not affect our ability to do the sum and finally we will obtain a linear *algebraic* equation for $T(\omega)$ which we shall certainly be able to solve.

The analytic properties of $F(\omega',\omega)$ as a function of ω' for fixed ω may be read off from Eq. (28). Apart from the delta-function term, $F(\omega',\omega)$ has poles at $\omega'=0$ and at $\omega'=\omega+i\epsilon$ and branch cuts along the real axis in the interval $\mu\leq\omega'\leq\infty$ and in the interval $\omega-\mu\geq\omega'\geq-\infty$. So long as $\omega<2\mu$, these branch cuts will not overlap. This condition on ω is, of course, just the condition that ω be below the production threshold, that is below the threshold for the process $V+\theta\rightarrow N+2\theta$. In constructing a solution for $F(\omega',\omega)$ we shall impose the condition that $\omega<2\mu$, but once the solution is obtained explicitly it may be taken as valid for all ω. $F(\omega',\omega)$ has no other singularities in the finite part of the ω' plane. The singularities of $F(\omega',\omega)$ are completely analogous to the singularities encountered in the study of the full relativistic production amplitude, or five-point function. In particular, the overlapping of the cuts for $\omega>2\mu$ is a characteristic feature of physical production amplitudes and its interpretation is clear in the simple model presented here.

In order to solve Eq. (28) we shall need to know the behavior of $F(\omega',\omega)$ for very large ω', ω fixed. From the unitarity of the S-matrix (it will be recalled that ghost states are explicitly ruled out), it follows that $T(\omega)$ is bounded and therefore that the sum in Eq. (24) must exist. $K(\omega')$ becomes constant for very large ω',[3] and hence $u^2(\omega')F(\omega',\omega)/\omega'^2$ must go to zero sufficiently rapidly in ω' for the sum to exist. For very large ω_1, $\mathfrak{M}(\omega_1)$ goes to C/ω_1 (C a constant). Thus for ω_1 very high, the summand in Eq. (28) may be written

$$C\frac{u^2(\omega_1)}{\omega_1^2}F(\omega_1,\omega)\left(\frac{1}{\omega_1+\omega'-\omega-i\epsilon}+\frac{1}{\omega_1-\omega'+i\epsilon}\right).$$

The contribution of this to the sum for very large ω_1 may be made arbitrarily small, even if ω' is also very large, since $u^2(\omega_1)F(\omega_1,\omega)/\omega_1^2$ decreases sufficiently rapidly. Hence $F(\omega',\omega)$ goes to zero for very large ω' and fixed finite ω.

A useful crossing symmetry of $F(\omega',\omega)$ also follows from (28). We see that

$$F(\omega',\omega)+\frac{g\omega}{u^2(\omega)}\delta_{k,k'}=F(\omega-\omega',\,\omega)+\frac{g\omega}{u^2(\omega)}\delta_{k,k-k'}. \quad (29)$$

This symmetry corresponds to an interchange of the two outgoing θ particles in Fig. 2.

To solve the integral equation for $F(\omega',\omega)$ it is convenient to make the transition to ∞ volume and change the sums to integrals. We obtain for (28)

$$F(\omega',\omega)=-g\frac{\omega}{u^2(\omega)}\delta_{k,k'}+\frac{g}{2\Omega}T(\omega)\left(\frac{1}{\omega'}-\frac{1}{\omega'-\omega-i\epsilon}\right)$$
$$+\frac{1}{\pi}\int_\mu^\infty e^{i\delta_1}\sin\delta_1 F(\omega_1,\omega)$$
$$\times\left(\frac{1}{\omega_1+\omega'-\omega-i\epsilon}+\frac{1}{\omega_1-\omega'+i\epsilon}\right)d\omega_1, \quad (30)$$

where we have expressed $\mathfrak{M}(\omega_1)$ in terms of the known phase shifts for $N-\theta$ scattering by

$$\frac{1}{4\pi}(\omega_1^2-\mu^2)^{\frac{1}{2}}u^2(\omega_1)\mathfrak{M}(\omega_1)=e^{i\delta_1}\sin\delta_1, \quad (31)$$

using the notation $\delta(\omega_1)=\delta_1$; $\delta(\omega')=\delta'$, etc. The integral equation (30) is of a general structure well-known in dispersion theory.[10] In particular, equations of just this type have been studied by Blankenbecler and Gartenhaus,[11] so long as $\omega<2\mu$. In fact the equation they study has a real inhomogeneous term and a kernel under the integral given in terms of the imaginary part of the unknown function, which is assumed to have a known phase. It is easily verified that their method of solution is still valid in our case in which the inhomogeneous terms are not real and in which $\exp(i\delta_1)\sin\delta_1 F(\omega_1,\omega)\neq\operatorname{Im}F(\omega_1,\omega)$. Thus applying their method, we obtain the solution

$$F(\omega',\omega)=-g\frac{\omega}{u^2(\omega)}\delta_{k,k'}$$
$$+\frac{g}{2\Omega}T(\omega)\left(\frac{1}{\omega'}-\frac{1}{\omega'-\omega-i\epsilon}\right)$$
$$+\frac{e^{\rho(\omega')-i\delta'}}{\pi}\int_\mu^\infty e^{-\rho(\omega_1)}\sin\delta_1 d\omega_1$$
$$\times\left(\frac{1}{\omega_1-\omega'+i\epsilon}+\frac{h}{\omega_1+\omega'-\omega-i\epsilon}\right)$$
$$\times\left[\frac{g}{2\Omega}T(\omega)\left(\frac{1}{\omega_1}-\frac{1}{\omega_1-\omega-i\epsilon}\right)-\frac{g\omega}{u^2(\omega)}\delta_{k,k_1}\right]$$
$$+\mathcal{P}(\omega')e^{\rho(\omega')-i\delta'}, \quad (32)$$

where

$$\rho(\omega')=\frac{P}{\pi}\int_\mu^\infty d\omega_1\delta_1\left(\frac{1}{\omega_1-\omega'}+\frac{1}{\omega_1+\omega'-\omega}\right). \quad (33)$$

[10] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff, N. V. Groningen, Holland, 1953); R. Omnes, Nuovo cimento **8**, 316 (1958).

[11] R. Blankenbecler and S. Gartenhaus, Phys. Rev. **116**, 1297 (1959).

h is an arbitrary function. $\mathcal{P}(\omega)$ is an arbitrary polynomial representing the possibility of adding arbitrary amounts of solutions of the homogeneous equation to the solution of the inhomogeneous equation. We may now use the fact that both $\rho(\omega')$ and $F(\omega',\omega)$ go to zero for large ω'. This can only be if $\mathcal{P}=0$. Further the crossing symmetry (29) requires that $h=1$.

The problem now is solved. All that remains is to do the integrals and assemble the results. The integral containing $\delta_{k,k'}$ is done immediately, giving

$$\begin{aligned}F(\omega',\omega)&\\ =&-\frac{g\omega}{u^2(\omega)}\delta_{k,k'}+\frac{g}{2\Omega}\left(\frac{1}{\omega'}-\frac{1}{\omega'-\omega-i\epsilon}\right)\\ &\times\left(T(\omega)-\frac{4\pi\sin\delta e^{\rho(\omega')-\rho(\omega)-i\delta'}}{(\omega^2-\mu^2)^{\frac{1}{2}}u^2(\omega)}\right)+\frac{ge^{\rho(\omega')-i\delta'}T(\omega)}{2\Omega\pi}\\ &\times\int_\mu^\infty e^{-\rho(\omega_1)}\sin\delta_1\left(\frac{1}{\omega_1}-\frac{1}{\omega_1-\omega-i\epsilon}\right)\\ &\times\left(\frac{1}{\omega_1-\omega'+i\epsilon}+\frac{1}{\omega_1+\omega'-\omega}\right)d\omega_1.\end{aligned}\tag{34}$$

In order to calculate $\rho(\omega)$ we define the function $\beta(\omega)$ by[3]

$$\mathfrak{M}(\omega)=\frac{-g^2}{\omega}\frac{1}{1-\beta(\omega)},\tag{35}$$

where

$$\beta(\omega)=\frac{-g^2\omega}{4\pi^2}\int_\mu^\infty\frac{d\omega_1(\omega_1^2-\mu^2)^{\frac{1}{2}}u^2(\omega_1)}{\omega_1^2(\omega_1-\omega-i\epsilon)},\tag{36}$$

and

$$\operatorname{Im}\beta(\omega)=\frac{-g^2u^2(\omega)(\omega^2-\mu^2)^{\frac{1}{2}}}{4\pi\omega},\quad \omega>\mu.\tag{37}$$

In terms of this function, we show in the Appendix that

$$e^{\rho(\omega')+i\delta'}=\frac{Z^2}{[1-\beta(\omega')][1-\beta(\omega-\omega')]},\tag{38}$$

from which it follows that

$$e^{-\rho(\omega')}\sin\delta'=-\frac{[1-\beta(\omega-\omega')]\operatorname{Im}[1-\beta(\omega')]}{Z^2},\tag{39}$$

since for $\omega<2\mu$, $\beta(\omega-\omega')$ is real for $\mu<\omega'<\infty$. Thus using (37), (38), and (39) in (34), we may write

$$\begin{aligned}F(\omega',\omega)=&-\frac{g\omega}{u^2(\omega)}\delta_{k,k'}-\frac{g}{2\Omega}T(\omega)\frac{\omega}{\omega'(\omega'-\omega-i\epsilon)}\\ &-\frac{g^3}{2\Omega}\frac{1}{\omega'(\omega'-\omega-i\epsilon)[1-\beta(\omega-\omega')][1-\beta^*(\omega')]}\\ &+\frac{g}{2\Omega}\frac{T(\omega)I}{[1-\beta(\omega-\omega'')][1-\beta^*(\omega')]},\end{aligned}\tag{40}$$

where

$$I=\frac{\omega}{\pi}\int_\mu^\infty\frac{\operatorname{Im}[1-\beta(\omega_1)][1-\beta(\omega-\omega_1)](2\omega_1-\omega)d\omega_1}{\omega_1(\omega_1-\omega-i\epsilon)(\omega_1-\omega'+i\epsilon)(\omega_1+\omega'-\omega)},\tag{41}$$

which we show in the Appendix is

$$I=\frac{\omega}{\omega'(\omega'-\omega-i\epsilon)}\times\{[1-\beta^*(\omega')][1-\beta(\omega-\omega')]+\beta(\omega)-1\}.\tag{42}$$

Thus finally we obtain for $F(\omega',\omega)$ the expression

$$\begin{aligned}F(\omega',\omega)=&-g\frac{\omega}{u^2(\omega)}\delta_{k,k'}-\frac{g^3}{2\Omega}\frac{1}{\omega'(\omega'-\omega-i\epsilon)[1-\beta(\omega-\omega')][1-\beta^*(\omega')]}\\ &-\frac{g\omega T(\omega)}{2\Omega}\frac{[1-\beta(\omega)]}{[1-\beta(\omega-\omega')][1-\beta^*(\omega')](\omega'-\omega-i\epsilon)\omega'}.\end{aligned}\tag{43}$$

To get $T(\omega)$, we insert (43) into (21) along with[3]

$$K(\omega')=\frac{-g}{1-\beta(\omega')},\tag{44}$$

Each of the three terms of (43) will give a contribution to $T(\omega)$ which we call T_1, T_2, and T_3 in turn. The first is trivially evaluated to give

$$T_1=\frac{g^2}{\omega[1-\beta(\omega)]}.\tag{45}$$

Changing the sum to an integral and using (37), we may write for the others

$$\begin{aligned}T_2=&\frac{g^2}{\pi}\int_\mu^\infty d\omega'\frac{1}{\omega'(\omega'-\omega-i\epsilon)}\times\frac{\operatorname{Im}[1-\beta(\omega')]}{|1-\beta(\omega')|^2}\\ &\times\frac{1}{1-\beta(\omega-\omega')},\\ =&g^2A,\end{aligned}\tag{46}$$

and

$$T_3=\omega[1-\beta(\omega)]T(\omega)A,\tag{47}$$

where the integral A is

$$A=\frac{1}{\pi}\int_\mu^\infty d\omega'\frac{\mathrm{Im}[1-\beta(\omega')]}{\omega'(\omega'-\omega-i\epsilon)|1-\beta(\omega')|^2}\times\frac{1}{1-\beta(\omega-\omega')}$$
$$=\frac{1}{\pi}\int_\mu^\infty d\omega'\frac{\mathrm{Im}[1-\beta(\omega')]}{\omega'(\omega'-\omega-i\epsilon)|1-\beta(\omega')|^2}\times\left(1+\frac{\beta(\omega-\omega')}{1-\beta(\omega-\omega')}\right)$$
$$=B+C. \quad (48)$$

In the Appendix we show that

$$B=\frac{1}{\pi}\int_\mu^\infty d\omega'\frac{\mathrm{Im}[1-\beta(\omega')]}{\omega'(\omega'-\omega-i\epsilon)|1-\beta(\omega')|^2}=\frac{-\beta(\omega)}{\omega[1-\beta(\omega)]}, \quad (49)$$

which leaves only C:

$$C=\frac{1}{\pi}\int_\mu^\infty d\omega'\frac{\mathrm{Im}[1-\beta(\omega')]}{\omega'(\omega'-\omega-i\epsilon)|1-\beta(\omega')|^2}\times\frac{\beta(\omega-\omega')}{1-\beta(\omega-\omega')}$$
$$=\frac{1}{\pi}\int_\mu^\infty d\omega'\frac{\mathrm{Im}[1-\beta(\omega')]}{\omega'(\omega'-\omega)|1-\beta(\omega')|^2}\times\frac{\beta(\omega-\omega')}{1-\beta(\omega-\omega')}, \quad (50)$$

the second form following from the fact that $\beta(0)=0$. Thus for $\omega<2\mu$, C is real. It clearly represents the effect of production on the scattering amplitude. Combining (45), (46), (47), (48), and (49), we have

$$T(\omega)=\frac{g^2}{\omega[1-\beta(\omega)]}-\frac{g^2\beta(\omega)}{\omega[1-\beta(\omega)]}+g^2C-T(\omega)\beta(\omega)+\omega T(\omega)[1-\beta(\omega)]C, \quad (51)$$
$$=\frac{(g^2/\omega)+g^2C}{1+\beta(\omega)-\omega[1-\beta(\omega)]C}$$
$$=\frac{g^2/\omega}{(1-\omega C)/(1+\omega C)+\beta(\omega)}. \quad (52)$$

Since the integral C is in principle known, this completes the solution.[12]

III. PRODUCTION

Having calculated the amplitude $F(\omega',\omega)$ of the previous section, we are in a position to calculate the amplitude for the production process $V+\theta\rightarrow N+2\theta$. We begin with the S-matrix element,

$$S=\langle N\theta_{\omega''}\theta_{\omega'}{}^{(-)}|V\theta_\omega{}^{(+)}\rangle. \quad (53)$$

[12] If we neglect production ($C=0$) we get $T(\omega)=g^2/\{\omega[1+\beta(\omega)]\}$. Essentially this result has been obtained previously in this approximation by D. A. Geffen (private communication).

Contracting in a θ from the left, we have

$$S=2\pi i\delta(\omega''+\omega'-\omega)\times\frac{u(\omega'')}{(2\omega''\Omega)^{\frac{1}{2}}}\frac{1}{\sqrt{2}}\langle N\theta_{\omega'}{}^{(-)}|j|V\theta_\omega{}^{(+)}\rangle, \quad (54)$$

the extra factor of $1/\sqrt{2}$ comes from the identity of the θ particles. The S-matrix element is thus expressed in terms of the amplitude $P(\omega',\omega)$ defined as

$$P(\omega',\omega)=\frac{(\omega\omega')^{\frac{1}{2}}}{u(\omega)u(\omega')}\langle N\theta_{\omega'}{}^{(-)}|j|V\theta_\omega{}^{(+)}\rangle. \quad (55)$$

The problem is to relate $P(\omega',\omega)$ to $F(\omega',\omega)$. Inserting a complete set of states in (55), we can write

$$P(\omega',\omega)=\frac{(\omega\omega')^{\frac{1}{2}}}{u(\omega)u(\omega')}\sum_S\langle N\theta_{\omega'}{}^{(-)}|S\rangle\langle S|j|V\theta_\omega{}^{(+)}\rangle. \quad (56)$$

We take "plus" states for the states S. The selection rules of the model restrict S to the V-particle state or the $N-\theta$ scattering state, but $\langle N\theta^{(-)}|V\rangle=0$. Thus we have

$$P(\omega',\omega)=\frac{(\omega\omega')^{\frac{1}{2}}}{u(\omega)u(\omega')}\sum_{k''}\langle N\theta_{\omega'}{}^{(-)}|N\theta_{\omega''}{}^{(+)}\rangle\times\langle N\theta_{\omega''}{}^{(+)}|j|V\theta_\omega{}^{(+)}\rangle, \quad (57)$$

which expresses the result in terms of the $N-\theta$ scattering matrix. Using the energy conservation imposed by this element, the sum may be done, and we obtain

$$P(\omega',\omega)=e^{2i\delta'}F(\omega',\omega), \quad (58)$$

where δ' is the $N-\theta$ scattering phase shift for energy ω'. Thus finally Eq. (54) may be written

$$S=2\pi i\delta(\omega''+\omega'-\omega)\frac{u(\omega)u(\omega')u(\omega'')}{(2\Omega\omega\omega'\omega'')^{\frac{1}{2}}}\frac{e^{2i\delta'}}{\sqrt{2}}F(\omega',\omega). \quad (59)$$

IV. FULL UNITARITY

It is instructive as well as amusing to verify that our solution (52) satisfies unitarity. To see this we calculate the imaginary part of $T(\omega)$ from (11). We write

$$T(\omega)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\lim_{t\rightarrow-\infty}\langle V|[j,a_k{}^\dagger(t)]|V\rangle e^{-i\omega t}; \quad (60)$$

the extra ordering introduced by the commutator gives zero. Using the fact that the equal-time commutator $[j,a_k{}^\dagger]$ vanishes, we have

$$T=i\int_{-\infty}^\infty e^{-i\omega t}\theta(-t)\langle V|[j,j^\dagger(t)]|V\rangle dt. \quad (61)$$

$\mathrm{Im}T(\omega)$ is obtained from the first term $\theta(-t)=\frac{1}{2}$

$+\frac{1}{2}\epsilon(-t)$. Hence

$$\mathrm{Im}T(\omega)=\frac{1}{2}\int_{-\infty}^{\infty}e^{-i\omega t}\langle V|[j,j^{\dagger}(t)]|V\rangle dt. \quad (62)$$

Inserting a complete set of states and noting that only the first ordering of the commutator contributes, we get

$$\mathrm{Im}T(\omega)=\pi\sum_{S}|\langle V|j|S\rangle|^{2}\delta(m+\omega-E_S)$$
$$=\pi\sum_{k'}|\langle V|j|V\theta_{\omega'}{}^{(-)}\rangle|^{2}\delta(\omega-\omega')$$
$$+\pi\sum_{k',k''}|\langle V|j|V\theta_{\omega'}\theta_{\omega''}{}^{(-)}|^{2}\delta(\omega'+\omega''-\omega). \quad (63)$$

With the convention for Boson states implicit in the factor $1/\sqrt{2}$ in (54) the sum in the second term may run over all k' and k''. The first term of (63) relates ImT to $|T|^2$ and the second term represents the contribution from production. To express it simply, we notice that the production S-matrix element (53) may be written

$$S=\lim_{t\to-\infty}\langle N\theta_{\omega'}\theta_{\omega''}{}^{(-)}|a_k(t)|V\rangle e^{-i\omega t}$$
$$=2\pi i\delta(\omega'+\omega''-\omega)\frac{u(\omega)}{(2\omega\Omega)^{\frac{1}{2}}}\langle N\theta_{\omega'}\theta_{\omega''}{}^{(-)}|j|V\rangle. \quad (64)$$

Thus on the energy shell, which is all that concerns us here, we have from (59)

$$\delta(\omega'+\omega''-\omega)\langle N\theta_{\omega'}\theta_{\omega''}{}^{(-)}|j|V\rangle$$
$$=\delta(\omega'+\omega''-\omega)\frac{u(\omega')u(\omega'')}{(2\omega'\omega'')^{\frac{1}{2}}}e^{2i\delta'}F(\omega',\omega). \quad (65)$$

Inserting this into (63) using (11) and (65) and doing the first sum, we have

$$\mathrm{Im}T(\omega)$$
$$=\frac{u^2(\omega)}{4\pi}(\omega^2-\mu^2)^{\frac{1}{2}}|T(\omega)|^2$$
$$+\left(\frac{2\Omega}{4\pi}\right)^2\frac{\pi}{2}\int d\omega'd\omega''\, u^2(\omega')u^2(\omega'')$$
$$\times(\omega'^2-\mu^2)^{\frac{1}{2}}(\omega''^2-\mu^2)^{\frac{1}{2}}|F(\omega',\omega)|^2\delta(\omega'+\omega''-\omega), \quad (66)$$

or using (37),

$$\mathrm{Im}T(\omega)=\frac{\omega}{g^2}\,\mathrm{Im}[1-\beta(\omega)]|T(\omega)|^2$$
$$+\frac{(2\Omega)^2}{2\pi g^4}\int d\omega'd\omega''\,\omega'\omega''\,\mathrm{Im}[1-\beta(\omega')]$$
$$\times\mathrm{Im}[1-\beta(\omega'')]|F(\omega',\omega)|^2\delta(\omega'+\omega''-\omega). \quad (67)$$

The term containing $\delta_{k,k'}$ in $F(\omega',\omega)$ gives no contribution in (67) since the delta function can never be satisfied on the energy shell. Thus from (43) we have

$$\mathrm{Im}T(\omega)$$
$$=\frac{\omega}{g^2}\,\mathrm{Im}[1-\beta(\omega)]|T(\omega)|^2$$
$$-\frac{|g^2+T(\omega)\omega[1-\beta(\omega)]|^2}{2g^2}$$
$$\times\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega'\,\mathrm{Im}[1-\beta(\omega')]\,\mathrm{Im}[1-\beta(\omega-\omega')]}{\omega'(\omega'-\omega)|1-\beta(\omega')|^2|1-\beta(\omega-\omega')|^2}. \quad (68)$$

The last integral may be related to C of (50) since

$$\mathrm{Im}C=\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega'\,\mathrm{Im}[1-\beta(\omega')]}{|1-\beta(\omega')|^2\omega'(\omega'-\omega)}\,\mathrm{Im}\left(\frac{\beta(\omega-\omega')}{1-\beta(\omega-\omega')}\right)$$
$$=-\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega'\,\mathrm{Im}[1-\beta(\omega')]\,\mathrm{Im}[1-\beta(\omega-\omega')]}{\omega'(\omega'-\omega)|1-\beta(\omega')|^2|1-\beta(\omega-\omega')|^2}. \quad (69)$$

Thus the condition for unitarity is

$$\mathrm{Im}T(\omega)=\frac{\omega}{g^2}\,\mathrm{Im}[1-\beta(\omega)]|T(\omega)|^2$$
$$+\frac{|g^2+\omega T(\omega)[1-\beta(\omega)]|^2}{2g^2}\,\mathrm{Im}C, \quad (70)$$

but also from (52)

$$\mathrm{Im}T(\omega)=\frac{\omega}{g^2}|T|^2\left(\frac{2\omega\,\mathrm{Im}C}{|1+\omega C|^2}+\mathrm{Im}[1-\beta(\omega)]\right). \quad (71)$$

Below the production threshold, C is purely real and unitarity is obviously satisfied. When $\omega>2\mu$ and ImC is not zero, the unitarity condition reduces to

$$2\omega^2|T(\omega)|^2/|1+\omega C|^2=|g^2+\omega T(\omega)[1-\beta(\omega)]|^2/2, \quad (72)$$

or

$$4\omega^2|T(\omega)|^2=|1+\omega C|^2|g^2+\omega T(\omega)[1-\beta(\omega)]|^2,$$

which is easily verified for our solution (52).

V. DISCUSSION

We have seen that using a novel form of contraction it is possible to obtain a soluble integral equation for the $V-\theta$ scattering amplitude and for the production amplitude for the process $V+\theta\to N+2\theta$ in the Lee model. These results are of interest both as a filling

out of the solved part of the Lee model, and also, perhaps more interestingly, as an example of an exactly soluble field theory with production. Probably some insight into the structure of scattering amplitudes with production and of production amplitudes in relativistic field theory can be gained from those presented here.

There are many extensions and applications of these results that come immediately to mind. Within the framework of the model and using the methods introduced here, one should be able to calculate the amplitude for $N+2\theta \to N+2\theta$. This would complete the discussion of amplitudes in the $V-\theta$ sector of the Lee model and hence should allow a determination of the $V-\theta$ and $N-\theta-\theta$ state vectors. Further, one can study the $V-\theta$ amplitude in more detail looking for the cusp that should occur at the production threshold[13] or studying the structure of the perturbation series. These aspects are presently under investigation.

One can also think of interesting directions for extending the model. In a sense, $V-\theta$ scattering is the analog of pion-nucleon scattering in the $T=\frac{3}{2}$ state and $N-\theta$ scattering in the $T=\frac{1}{2}$ state. This analogy is most apparent from the Born terms. It is seen from the structure of (35) and (52) that no resonance occurs at low energies for $N-\theta$ scattering, whereas a resonance is possible at low energies in $V-\theta$ scattering, since from (35) we see that $\mathrm{Re}\beta<0$ for small ω. The analogy can be made closer if the scalar coupling in the Lee model is changed to pseudovector. We are investigating this possibility. An even more amusing change is to introduce a $\theta-\theta$ interaction into the model in analogy with the pion-pion interaction. $N-\theta$ scattering is unchanged but $V-\theta$ scattering and the production amplitude are altered. A more difficult extension would be to try to relax the static assumption and finally to try these methods on the field theories of the real world. In particular, work is under way to apply these techniques to nuclear direct reactions in order to include the effect of initial- and final-state interactions in the field-theoretic approach to this problem.[5]

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APPENDIX

For Eq. (33) we must evaluate

$$\rho(\omega')+i\delta'=\frac{1}{\pi}\int_\mu^\infty dx\delta(x)\left(\frac{1}{x-\omega'-i\epsilon}+\frac{1}{x+\omega'-\omega}\right).$$

[13] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).

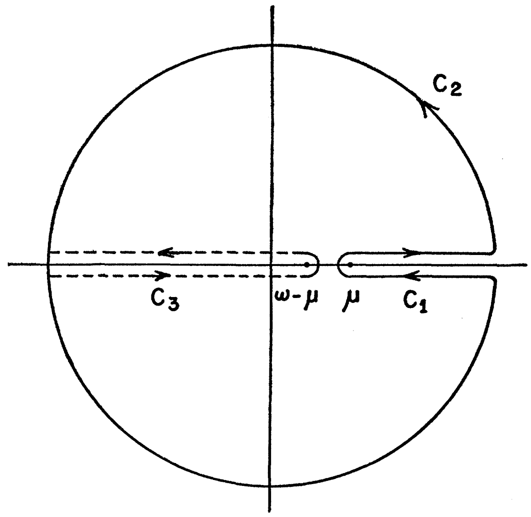

FIG. 3. Contour in the complex ω plane.

Our methods follow those of GT. We write

$$\delta(\omega)=-\frac{1}{2i}\{\ln[1-\beta(\omega)]-\ln[1-\beta^*(\omega)]\}, \quad (A.1)$$

and since

$$\beta^*(\omega+i\epsilon)=\beta(\omega-i\epsilon), \quad (A.2)$$

$$\rho(\omega')+i\delta'=-\frac{1}{2\pi i}\int_{C_1}\ln[1-\beta(x)]\times\left(\frac{1}{x-\omega'}+\frac{1}{x+\omega'-\omega}\right)dx, \quad (A.3)$$

where the contour C_1 goes from ∞ to μ just below the real axis and returns to ∞ just above, as shown in Fig. 3. Calling C_2 the contour of the infinite circle, we may write

$$\int_{C_1}=\oint-\int_{C_2},$$

using the fact[3] that $1-\beta(\omega)\to Z$ as $|\omega|\to\infty$ we have

$$\begin{aligned}\rho(\omega')+i\delta' &= -\{\ln[1-\beta(\omega')]+\ln[1-\beta(\omega-\omega')]-2\ln Z\}\\ &=\ln\frac{Z^2}{[1-\beta(\omega')][1-\beta(\omega-\omega')]},\end{aligned} \quad (A.4)$$

from which (38) follows.

Next we consider I defined by (41):

$$I=\frac{\omega}{\pi}\int_\mu^\infty\frac{\mathrm{Im}[1-\beta(z)][1-\beta(\omega-z)](2z-\omega)dz}{z(z-\omega-i\epsilon)(z-\omega'+i\epsilon)(z+\omega'-\omega)}=I_a+I_b, \quad (A.5)$$

where

$$I_a=\frac{\omega}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}[1-\beta(z)](2z-\omega)dz}{z(z-\omega-i\epsilon)(z-\omega'+i\epsilon)(z+\omega'-\omega)},\quad (A.6)$$

using

$$\mathrm{Im}[1-\beta(z)]=-\mathrm{Im}\beta(z)=-\frac{1}{2i}[\beta(z)-\beta(z^*)],$$

we write

$$I_a=-\frac{\omega}{2\pi i}\int_{C1}\frac{(2z-\omega)\beta(z)dz}{z(z-\omega-i\epsilon)(z-\omega'+i\epsilon)(z+\omega'-\omega)}.\quad (A.7)$$

In this case the integral around the infinite contour gives nothing and we may write

$$\int_{C1}\to\oint,$$

in which case (A.7) can be evaluated directly and we get

$$I_a=\frac{\omega}{\omega'(\omega'-\omega-i\epsilon)}[\beta(\omega)-\beta^*(\omega')-\beta(\omega-\omega')],\quad (A.8)$$

where we have used the fact that $\beta(0)=0$. We now must calculate I_b,

$$I_b=-\frac{\omega}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}[1-\beta(z)]\beta(\omega-z)(2z-\omega)dz}{z(z-\omega-i\epsilon)(z-\omega'+i\epsilon)(z+\omega'-\omega)}.\quad (A.9)$$

Proceeding as above, we write this as a contour integral:

$$I_b=\frac{\omega}{2\pi i}\oint\frac{\beta(z)\beta(\omega-z)(2z-\omega)dz}{z(z-\omega-i\epsilon)(z-\omega'+i\epsilon)(z+\omega'-\omega)}.\quad (A.10)$$

Now the integrand in addition to poles within the contour has a cut due to $\beta(\omega-z)$ in the interval $-\infty<z\leq\omega-\mu$. Thus we can write

$$\oint=\oint_{\text{poles}}+\int_{C_3},$$

where the contour C_3 is the integral around the negative cut indicated by the dashed contour in Fig. 3. Thus

$$\begin{aligned}I_b&=\frac{2\beta^*(\omega')\beta(\omega-\omega')\omega}{\omega'(\omega'-\omega-i\epsilon)}\\&\quad+\frac{\omega}{2\pi i}\int_{C_3}\frac{\beta(z)\beta(\omega-z)(2z-\omega)dz}{z(z-\omega-i\epsilon)(z-\omega'+i\epsilon)(z+\omega'-\omega)}\\&=\frac{2\beta^*(\omega')\beta(\omega-\omega')\omega}{\omega'(\omega'-\omega-i\epsilon)}\\&\quad-\frac{\omega}{\pi}\int_{-\mu}^{-\infty}\frac{\beta(z)\,\mathrm{Im}[\beta(\omega-z)](2z-\omega)dz}{z(z-\omega-i\epsilon)(z-\omega'+i\epsilon)(z+\omega'-\omega)},\end{aligned}\quad (A.11)$$

making the variable change $z=\omega-y$, the integral in (A.11) is seen to be just $-I_b$; hence

$$I_b=\frac{\beta^*(\omega')\beta(\omega-\omega')}{\omega'(\omega'-\omega-i\epsilon)},\quad (A.12)$$

thus

$$I=\frac{\omega}{\omega'(\omega'-\omega-i\epsilon)}\times[\beta(\omega)-\beta^*(\omega')-\beta(\omega-\omega')+\beta^*(\omega')\beta(\omega-\omega')],\quad (A.13)$$

which gives (42).

Finally we must evaluate B of Eq. (49), defined by

$$B=\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega'\,\mathrm{Im}[1-\beta(\omega')]}{\omega'(\omega'-\omega-i\epsilon)}\left|\frac{1}{1-\beta(\omega')}\right|^2,$$

using

$$\left|\frac{1}{1-\beta(\omega')}\right|^2=-\frac{1}{\mathrm{Im}[1-\beta(\omega')]}\,\mathrm{Im}\left(\frac{1}{1-\beta(\omega')}\right),\quad (A.14)$$

we have

$$\begin{aligned}B&=-\frac{1}{\pi}\int_{\mu}^{\infty}\frac{d\omega'}{\omega'(\omega'-\omega-i\epsilon)}\,\mathrm{Im}\left(\frac{1}{1-\beta(\omega')}\right)\\&=-\frac{1}{2\pi i}\oint\frac{d\omega'}{\omega'(\omega'-\omega-i\epsilon)}\left(\frac{1}{1-\beta(\omega')}\right)\\&=-\frac{\beta(\omega)}{\omega[1-\beta(\omega)]},\end{aligned}\quad (A.15)$$

as stated in (49).