resonance peak is many times higher than the average of the distribution outside the resonance. This is not always the case, however. One precautionary check that may easily be made is to plot the angular distribution of all the events under the resonance peak with respect to the direction $\mathbf{k}_f$ of the M_2. This distribution must be symmetric in the absence of a final-state interaction, but is likely to be appreciably asymmetric if the M_2-Y or M_2-π_1 interaction is sufficiently important to alter the general distribution of Eq. (2) or (3). One should make sure that there is no large asymmetry in $\mathbf{k}\cdot\mathbf{k}_f$ present in the data.

It has come to the author's attention that the results of this paper have been derived independently by R. Gatto and H. P. Stapp [see Phys. Rev. **121**, 1555 (1961)].

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High-Energy Potential Scattering with Short-Range Forces

B. J. MALENKA
Northeastern University, Boston, Massachusetts

AND

H. S. VALK
University of Nebraska, Lincoln, Nebraska

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An attempt is made to separate out long- and short-range effects for high-energy elastic scattering. Within the context of a high-energy approximation, expressions for the scattering amplitudes are obtained for the cases $kR\gg ka\gg 1$ and $kR\gg 1>ka$, where R and a denote the long and short ranges, respectively. For the latter case, the entire short-range effect is included in a phenomenological S-wave term while the long-range contributions are written explicitly.

I. INTRODUCTION

WITH the increase in energy of scattering experiments, we see more details of the interaction of particles at short ranges. Such experiments are complicated by the presence of the longer range interactions generally employed to interpret experimental results at lower energies. Thus, it is of interest to see to what extent we can separate these two effects, hopefully in such a manner that will enable us to utilize our previous knowledge about the long-range interactions in some relatively simple way.[1] As a first approach, it seems convenient to work within the context of a high-energy small-angle approximation for elastic scattering based on the work of Molière[2] and developed in some detail by Glauber[3] and others.[4–6]

II. HIGH-ENERGY APPROXIMATION

We are interested in the case where the scattering amplitude for a high-energy particle of reduced mass m with an incident momentum propagation vector $\mathbf{k}$ and a final propagation vector $\mathbf{k}'$ is given by[3]

$$f(\mathbf{k}',\mathbf{k})=\frac{k}{2\pi i}\int \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}] \times\{\exp[i\chi(\mathbf{b})]-1\}d^{(2)}b, \quad (1)$$

where

$$\chi(\mathbf{b})=-\frac{m}{\hbar^2 k}\int_{-\infty}^{\infty} V(\mathbf{b}+\boldsymbol{\kappa}z)dz, \quad (2)$$

with

$$\boldsymbol{\kappa}=(\mathbf{k}+\mathbf{k}')/|\mathbf{k}+\mathbf{k}'|. \quad (3)$$

Here $V(\mathbf{r})=V(\mathbf{b}+\boldsymbol{\kappa}z)$ is assumed to be a potential which represents the interaction between the projectile and target particles. For simplicity, we will further assume the potential to be azimuthally symmetric, that is $V(\mathbf{b}+\boldsymbol{\kappa}z)=V(b,z)$. The extension to other interesting cases is fairly direct.[3,4] Equation (1) can then be written

$$f_0(\theta)=\frac{k}{i}\int_0^{\infty} J_0(2kb\sin\tfrac{1}{2}\theta)\{e^{i\chi(b;R)+i\chi(b;a)}-1\}bdb, \quad (4)$$

where $J_0(2kb\sin\frac{1}{2}\theta)$ is the Bessel function of order zero. In writing the above expression, we have now explicitly assumed that the potential can be split into a long-range part characterized by a range R and a short-range part characterized by a range a, where $V(b,z)=V(b,z;R)$

[1] B. J. Malenka and H. S. Valk, Bull. Am. Phys. Soc. **5**, 269 (1960). This abstract contains a preliminary report of some of the work in this paper and reference 7.
[2] G. Molière, Z. Naturforsch. **2A**, 133 (1947).
[3] R. J. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1959), Vol. I, p. 315.
[4] B. J. Malenka, Phys. Rev. **95**, 522 (1956).
[5] I. I. Shapiro, thesis, Harvard University, Cambridge, Massachusetts, 1955 (unpublished).
[6] L. I. Schiff, Phys. Rev. **103**, 443 (1956).

$+V(b,z;a)$, so that in Eq. (4), we define

$$\chi(b;[R \text{ or } a])=-\frac{m}{\hbar^2 k}\int_{-\infty}^{\infty}V(b,z;[R \text{ or } a])dz. \quad (5)$$

III. SPECIAL CASES OF INTEREST

The extent to which the long- and short-range parts of Eq. (4) can be separated in a useful fashion will depend on the ranges and energies involved. For the case where $kR \gg ka \gg 1$, the following rearrangement of Eq. (4) is convenient for further approximation, namely

$$f_0(\theta)=\frac{k}{i}\int_0^{\infty}J_0(2kb\sin\tfrac{1}{2}\theta)\{e^{i\chi(b;R)}-1\}bdb$$
$$+\frac{k}{i}\int_0^{\infty}J_0(2kb\sin\tfrac{1}{2}\theta)e^{i\chi(b;R)}\{e^{i\chi(b;a)}-1\}bdb. \quad (6)$$

In this form, the first term contains only the long-range effects and with a suitably chosen $V(b,z;R)$ should approximately represent the same scattering amplitude as a function of energy and angle as observed at lower energies. In the second term, we note that the long-range phase factor $\chi(b;R)$ will by definition vary slowly over the interval in which $b\leqslant a$. On the other hand, the expression in curly brackets essentially vanishes outside of this interval. These observations suggest that in the second integral, a reasonable simplifying approximation would be to replace $\exp[i\chi(b;R)]$ by $\exp[i\chi(\bar{b};R)]$ where $\bar{b}$ represents some value of b in the interval $0\leqslant b\leqslant a$. Considering the delta-function-like behavior of the rest of the integrand, an appropriate value of $\bar{b}$ could correspond to the value where the rest of the integrand obtains its principal maximum. We could then write Eq. (6) in the approximate form:

$$f_0(\theta)\approx f_R(\theta)+\exp[i\chi(\bar{b};R)]f_a(\theta), \quad (7)$$

where $f_R(\theta)$ and $f_a(\theta)$ are the respective long- and short-range scattering amplitudes.

It is worth pointing out that even in its simplest form, Eq. (7) represents an improvement over the corresponding Born approximation scattering amplitude $f^{(\mathrm{Born})}(\theta)=f_R^{(\mathrm{Born})}(\theta)+f_a^{(\mathrm{Born})}(\theta)$ by virtue of the explicit appearance of the phase factor $\exp[i\chi(\bar{b};R)]$. Possible applications of Eq. (7) will be postponed for future investigations.

A particularly interesting special case occurs when $kR\gg 1>ka$. Here the usual incident angular momentum and range arguments lead us to expect that the short-range interaction would be felt predominantly in the S wave and could be neglected in partial waves of higher orbital momentum. Such consideration suggests that a useful way to separate out these effects would be to write

$$f(\theta)=[1-P^{(0)}(\theta)]f_0(\theta)+k^{-1}e^{i\delta}\sin\delta, \quad (8)$$

where the operator $P^{(0)}(\theta)$ projects out the S-wave part of $f_0(\theta)$ as written in Eq. (4). The second term in Eq. (8) represents the actual S-wave contribution which includes both long- and short-range effects expressed, as usual, in terms of the phase shift δ. Since the high-energy approximations leading to $f_0(\theta)$ are poorest for small l states,[3] Eq. (8) should represent an improvement in $f_0(\theta)$. If as mentioned above, we now further neglect the short-range contribution to the non-S-wave part, we obtain

$$f(\theta)\approx\frac{k}{i}\int_0^{\infty}\left[J_0(2kb\sin\tfrac{1}{2}\theta)-\frac{1}{kb}J_1(2kb)\right]\{e^{i\chi(b;R)}-1\}bdb+k^{-1}\Delta, \quad (9)$$

where

$$\Delta=e^{i\delta}\sin\delta. \quad (10)$$

An application of this approximation is discussed in the following paper.[7]

The extent to which Eq. (9) differs from Eq. (4), particularly in neglecting the short-range effects in the non-S-wave part, can be seen in somewhat more quantitative detail if we first rearrange Eq. (4) in the form

$$f_0(\theta)=f_{01}+f_{02}+f_{03}, \quad (11)$$

where

$$f_{01}=\frac{k}{i}\int_0^{\infty}\left[J_0(2kb\sin\tfrac{1}{2}\theta)-\frac{1}{kb}J_1(2kb)\right]\{e^{i\chi(b;R)}-1\}bdb, \quad (12)$$

$$f_{02}=\frac{k}{i}\int_0^{\infty}\left[J_0(2kb\sin\tfrac{1}{2}\theta)-\frac{1}{kb}J_1(2kb)\right]e^{i\chi(b;R)}\{e^{i\chi(b;a)}-1\}bdb, \quad (13)$$

$$f_{03}=\frac{k}{i}\int_0^{\infty}\frac{1}{kb}J_1(2kb)\{e^{i\chi(b;R)+i\chi(b;a)}-1\}bdb. \quad (14)$$

Comparing Eq. (11) with Eq. (9), we see that f_{01} is just the first term in Eq. (9), while the S-wave part of Eq. (11), f_{03}, has been replaced by an empirical contribution $k^{-1}\Delta$. Thus the expression f_{02} approximately represents the error made in neglecting the short-range contribution to the non-S-wave part of Eq. (9). Under the conditions $kR\gg 1>ka$, the contribution of f_{02} to the scattering amplitude should be relatively small. This may be seen by noting that in the integrand of Eq. (13), for the range of integration in which $b>a$, the expression in the curly bracket essentially vanishes, while for $b<a$, the integrand has a leading term dependence of $(kb)^3$ which is small to the extent that the condition $ka<1$ is satisfied.

The relations of the S-wave phase shift δ to the potentials producing the scattering will usually depend

[7] B. J. Malenka and H. S. Valk, following paper [Phys. Rev **122**, 934 (1961)].

on the specific details of a problem. Except in very special cases, the expressions usually obtained are based on various approximations such as the Born approximation and effective-range theory which are not particularly applicable in our case. Here, as a first step, we would like to separate out the specifically short-range contribution to δ. This can be done by observing that the conditions employed[3] for obtaining Eq. (1) are that the ratio of potential to total energy, $V/E \ll 1$ and that $kR \gg 1$. These are just the usual "quasi-classical" conditions[8] for the validity of the WKB expressions for partial waves. Thus, in the absence of a short-range interaction, r times the radial part of the S-wave wave function $u_0(r)$ would be approximately

$$u_0(r) \approx \sin\left(\int_0^r [k^2 - (2m/\hbar^2)V(r';R)]^{\frac{1}{2}} dr' \right). \quad (15)$$

If the short-range interaction $V(r;a)$ of range $a \ll R$ is then introduced, we have in the region for which $r > a$

$$u_0(r) \approx \sin\left(\int_0^r [k^2 - (2m/\hbar^2)V(r';R)]^{\frac{1}{2}} dr' + \delta_a \right), \quad (16)$$

where δ_a represents the phase shift produced by $V(r;a)$. Finally, in the region where $r \gg R \gg a$, Eq. (16) then takes the usual asymptotic form

$$u_0(r) = \sin(kr + \delta), \quad (17)$$

where the total S-wave phase shift[9] is

$$\delta = \delta_R + \delta_a, \quad (18)$$

and

$$\delta_R \approx -\frac{m}{\hbar^2 k} \int_0^\infty V(r;R) dr. \quad (19)$$

Referring back to Eq. (10), we see that this result enables us to write

$$\delta_a \approx -\tfrac{1}{2} i \ln(2i\Delta + 1) + \frac{m}{\hbar^2 k} \int_0^\infty V(r;R) dr, \quad (20)$$

thus expressing the short-range part of the phase shift in terms of the experimentally determined Δ and presumably known long-range potential $V(r;R)$.

Of course, it remains to express δ_a in terms of $V(r;a)$. However, this will depend on the value obtained for δ_a from Eq. (20) and the form assumed for $V(r;a)$ which generally will be based on considerations relating to the specific scattering problem being investigated.

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APPENDIX

The separation of the S-wave phase shift into long- and short-range parts to obtain Eq. (18) depended on assuming $V/E \ll 1$ and $kR \gg 1$ for the long-range interaction. It is therefore of some interest to briefly show that similar results are obtained by the exact solution of a simple example.

Consider a two-step well such that

$$\begin{aligned} &V(r) = V_a, && k_2^2 = (2m/\hbar^2)(E - V_a), && 0 \leqslant r \leqslant a; \\ &V(r) = V_R, && k_1^2 = (2m/\hbar^2)(E - V_R), && a < r \leqslant R; \\ &V(r) = 0, && k^2 = (2m/\hbar^2)E, && R < r; \end{aligned} \quad (A.1)$$

where k_2, k_1, k are the magnitudes of the respective propagation vectors. By the usual conditions of continuity at a and R, it is easily shown that the S-wave phase shift can be written as

$$\tan\delta = \frac{(k/k_1)[\tan k_1 R + \tan\delta_a] - [1 - \tan k_1 R \tan\delta_a] \tan kR}{1 - \tan k_1 R \tan\delta_a + (k/k_1)[\tan k_1 R + \tan\delta_a] \tan kR}, \quad (A.2)$$

where δ_a is determined from the equation

$$k_2 \cot k_2 a = k_1 \cot(k_1 a + \delta_a), \quad (A.3)$$

and may be considered the phase shift produced by V_a in the region $a < r \leqslant R$.

We now introduce δ_R which represents the phase shift in the absence of the short-range step at a, that is, when V_a is replaced by V_R in the interval $0 \leqslant r \leqslant a$. At $r = R$, δ_R satisfies the usual equation

$$k_1 \cot k_1 R = k \cot(kR + \delta_R). \quad (A.4)$$

Using this equation in Eq. (A.2), we find

$$\tan\delta = \frac{\tan\delta_R + \tan\delta_a[1+\phi]}{1 - \tan\delta_R \tan\delta_a[1+\psi]}, \quad (A.5)$$

where

$$\phi = \left(\frac{k_1 - k}{k}\right)\left[\frac{1 - \tan k_1 R \tan kR}{(k_1/k) + \tan k_1 R \tan kR}\right], \quad (A.6)$$

$$\psi = \left(\frac{1}{kR}\right)\left[\frac{\tan k_1 R + \tan kR}{(k_1 - k)^{-1}[\tan k_1 R - \tan kR] - (kR)^{-1} \tan kR}\right]. \quad (A.7)$$

We see from Eqs. (A.6) and (A.7) that for $V_R/E \ll 1$ and $kR \gg 1$ both ϕ and ψ are small. From Eq. (A.5), it immediately follows that $\delta \approx \delta_R + \delta_a$ under the same conditions as shown above.

[8] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1958), p. 399 and p. 158.

[9] See Appendix.