PHYSICAL REVIEW VOLUME 122, NUMBER 3 MAY 1, 1961

Note on the Modified Gravitational Equations $g^2 R_{ik}=0$

DANIEL M. LIPKIN
6 Sealey Avenue, Hempstead, New York

(Received August 29, 1960; revised manuscript received January 9, 1961)

The single example of a gravitational manifold which Einstein and Rosen presented in support of their modified field equations $g^2R_{ik}=0$ is shown to uniformly satisfy the unmodified field equations $R_{ik}=0$ and hence to provide insufficient logical support for their proposed modification. The demonstration is made in the original coordinates used by Einstein and Rosen, by direct calculation of the curvature tensor of the manifold.

I. INTRODUCTION

THE gravitational manifold discussed by Einstein and Rosen in connection with their suggested modification[1] of the field equations was the following transformed version of Schwarzschild's solution:

$$dl^2=\frac{u^2}{A+u^2}dT^2-4(A+u^2)du^2 -(A+u^2)^2(d\theta^2+\sin^2\theta d\phi^2), \quad (1)$$

where A represents the positive constant $2m$ utilized by Einstein and Rosen. There is no indication that Einstein and Rosen regarded their manifold as containing regions not included in the range of real values of the four coordinates T, u, θ, ϕ, and this interpretation of the term "Einstein-Rosen manifold" will accordingly be adhered to here. The fact[2] that this manifold is extensible is not relevant to the purpose of the present paper, which is to criticize the introduction of the modified field equations $g^2R_{ik}=0$ on the basis of a simple formal consideration.

As pointed out by Einstein and Rosen, the metric tensor of the manifold (1) has no singularities, but is degenerate on the locus $u=0$ in the sense that the metric determinant g vanishes on that locus. (It may be interesting to note that without this degeneracy, the manifold would not be extensible.) On such a locus of degeneracy the field equations $R_{ik}=0$ may fail to be satisfied, because the curvature tensor and its contraction, the Ricci tensor R_{ik}, take indeterminate forms and may be discontinuous or have other singularities. Einstein and Rosen noted that this difficulty could be avoided by using the modified field equations $g^2R_{ik}=0$, which, being free of g as a denominator, would necessarily always remain satisfied across loci of degeneracy $g=0$. Whether this modification of the field equations may be desirable in a large class of solutions having physical interest is not known at present; however, this modification will be shown to be unnecessary in the case of the manifold (1) which Einstein and Rosen used as a principal example in support of the modification, essentially because the indeterminate form assumed by the curvature tensor is resolvable without discontinuity or other singularity on the locus of degeneracy $u=0$.

II. THE CURVATURE FIELD OF THE EINSTEIN-ROSEN MANIFOLD

The regularity of the curvature field of the Einstein-Rosen manifold will be studied in the original coordinates T, u, θ, ϕ used by Einstein and Rosen. At points not on the locus of degeneracy of the manifold (1), the nonvanishing components $R^i{}_{klm}$ of its Riemann-Christoffel curvature tensor can be evaluated by direct calculation, and are summarized as follows:

$$\begin{aligned}
R^1{}_{001}&=-R^1{}_{010}=-2R^2{}_{002}=2R^2{}_{020}=-2R^3{}_{003}\\
&=2R^3{}_{030}=-Au^2/(A+u^2)^4,\\
R^0{}_{110}&=-R^0{}_{101}=-2R^2{}_{112}=2R^2{}_{121}=-2R^3{}_{113}\\
&=2R^3{}_{131}=4A/(A+u^2)^2,\\
R^3{}_{223}&=-R^3{}_{232}=-2R^0{}_{220}=2R^0{}_{202}=-2R^1{}_{221}\\
&=2R^1{}_{212}=A/(A+u^2),\\
R^2{}_{332}&=-R^2{}_{323}=-2R^0{}_{330}=2R^0{}_{303}=-2R^1{}_{331}\\
&=2R^1{}_{313}=[A/(A+u^2)]\sin^2\theta,
\end{aligned} \quad (2)$$

with the assignment $(x^0,x^1,x^2,x^3)\equiv(T,u,\theta,\phi)$. By considering the values of the curvature components at locations arbitrarily close to $u=0$, it can be seen that these components are finite, single-valued, and continuous across this locus, and in fact differentiable to all orders there. Inspection of (2) now shows that the curvature tensor satisfies the unmodified field equations $R_{ik}=0$ unambiguously, everywhere in the Einstein-Rosen manifold.

III. CONCLUSION

Einstein and Rosen's suggestion to modify the field equations so that they read $g^2R_{ik}=0$ instead of $R_{ik}=0$ has been shown to be logically unnecessary in the single example of a curved gravitational manifold which those authors considered. In the absence of any new evidence in its favor, this suggested modification can be tentatively rejected on the grounds that it represents an unnecessary weakening of the field equations.

[1] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).
[2] D. Finkelstein, Phys. Rev. **110**, 965 (1958); C. Fronsdal, Phys. Rev. **116**, 778 (1959); M. D. Kruskal, Phys. Rev. **119**, 1743 (1960).