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Charged-Scalar Strong-Coupling Theory for Two-Nucleon System*

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The Serber-Pais charged-scalar strong-coupling method is extended to the two-nucleon system. It is shown explicitly that whereas the nuclear force depends on the renormalized coupling constant alone at the large internucleon separations, its dependence on the unrenormalized coupling constant alone becomes increasingly pronounced as the two nucleons come closer together.

I. INTRODUCTION

This work is motivated by our desire to study explicitly and in a consistent manner the long-standing conjecture in the charged-scalar strong-coupling theory that whereas the nuclear force depends on the renormalized coupling constant alone at the large internucleon separations, its dependence on the unrenormalized coupling constant alone becomes increasingly pronounced as the two nucleons come closer together. If one considers two extreme cases—namely, the case of very large separations, in which the nuclear force depends on the renormalized coupling constant alone, and the case of an exact overlap between the two nucleons, in which the nuclear force depends on the unrenormalized coupling constant alone (Chapter III), one will inevitably arrive at the above conjecture. To achieve our objective it is necessary to carry out numerous transformations and to examine various problems, new and old, which bear physical significance by their own rights.

In this work we make use of the Serber-Pais method, a new method in the charged-scalar strong-coupling theory, which is powerful in the variational procedure and is particularly suitable for the treatment of the effects of the quantum mechanical field fluctuations.

Chapter II is devoted to series of the transformations which will bring into evidence the isobaric energy spectrum of the system. The fields are then separated into a static part (of the order of the coupling constant g) and a fluctuating part (of the order of g^0) so that we may expand the theory in the ratio of the fluctuating field and the static field in a strong coupling limit. Chapter III is devoted to some customary problems, such as self-energy, nuclear force, and isobaric energy. It is shown there that the nuclear force, which has the exchange characteristics at large separations, becomes isotopic spin-independent at small separations. Finally, emphasis is given to Chapter IV, in which we study the normal modes to evaluate the zero-point energy shift of the meson fields due to the presence of the source functions. This energy shift of the order of g^0 combined with the self-energy terms of the order of g^2 exhibits explicitly that the dependence of the nuclear force on the unrenormalized coupling constant alone becomes increasingly pronounced as the two nucleons come closer together.

II. TRANSFORMATIONS

We write the Hamiltonian for two nucleons, A and B, coupled to the charged scalar field as follows:

$$H=H_{\mathrm{mes}}+H_{\mathrm{int}}, \tag{1}$$

where

$$H_{\mathrm{mes}}=\tfrac{1}{2}\sum_\alpha \int[\pi_\alpha{}^2(\mathbf{r})+\phi_\alpha(\mathbf{r})\omega^2\phi_\alpha(\mathbf{r})]d\mathbf{r}, \qquad \omega^2=\kappa^2-\nabla^2, \tag{2}$$

$$H_{\mathrm{int}}=(2\pi)^{\frac{1}{2}}g\sum_\alpha\int[u_A(\mathbf{r})\tau_{A\alpha}\phi_\alpha(\mathbf{r}) +u_B(\mathbf{r})\tau_{B\alpha}\phi_\alpha(\mathbf{r})]d\mathbf{r},\quad \alpha=1,2. \tag{3}$$

The operator of total charge of the system is given by

$$T=\Sigma+\tfrac{1}{2}(1+\tau_{A3})+\tfrac{1}{2}(1+\tau_{B3}),\qquad \Sigma=\int[\phi_1(\mathbf{r})\pi_2(\mathbf{r})-\phi_2(\mathbf{r})\pi_1(\mathbf{r})]d\mathbf{r}. \tag{4}$$

In the above equations we assume the extended source model. The Hermitian fields $\phi_\alpha(\mathbf{r})$ and their conjugate momenta $\pi_\alpha(\mathbf{r})$ satisfy the usual commutation relation,

$$[\pi_\alpha(\mathbf{r}),\phi_\beta(\mathbf{r}')]=-i\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'). \tag{5}$$

A feature which distinguishes the Serber-Pais method[1] from the customary one lies in the fact that they introduce in the theory an arbitrary scalar c-number function f, called the "distribution function," in such a way that all the transformations are homogeneous in f. Since f can be chosen by any suitable criterion, their method opens a general way for the application of variational methods. The Serber-Pais method consists of two distinct sets of transformations. The first one, which is independent of the value of the coupling constant or of any nonrelativistic approximations, represents series of rigorous transformations which bring into evidence the isobaric energy spectrum. The second one represents the transformations which specifically refer to expansions valid only for the extended source model and for large values of the

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[1] A. Pais and R. Serber, Phys. Rev. **105**, 1636 (1957).

coupling constant. In the case of a single nucleon they start with the separation of ϕ and π into a "bound" part and a "free" part (marked by a prime)[2]:

$$\phi_\alpha=\phi_\alpha'+\frac{f}{F}\int f\phi_\alpha,\quad \pi_\alpha=\pi_\alpha'+\frac{f}{F}\int f\pi_\alpha, \tag{6}$$

in which the normalization factor $F(=\int f^2)$ has been so introduced that the bound parts, characterized by collective coordinates,

$$Q_\alpha=\frac{1}{F^{\frac{1}{2}}}\int f\phi_\alpha,\quad P_\alpha=\frac{1}{F^{\frac{1}{2}}}\int f\pi_\alpha, \tag{7}$$

will satisfy the canonical commutation relation

$$[P_\alpha,Q_\beta]=-i\delta_{\alpha\beta}. \tag{8}$$

The transformation $(\phi,\pi)\rightarrow(\phi',\pi')$ is not exactly canonical, corresponding to a transition from a complete to an incomplete set of free-meson states; the latter should be supplemented by bound states generated by the collective motion:

$$[\pi_\alpha'(\mathbf{r}),\phi_\beta'(\mathbf{r}')]=-i\delta_{\alpha\beta}[\delta(\mathbf{r}-\mathbf{r}')-(1/F)f(\mathbf{r})f(\mathbf{r}')]. \tag{9}$$

Note that the primed variables are orthogonal to f and commute with the P's and Q's:

$$\int f\phi_\alpha'=\int f\pi_\alpha'=0, \tag{10}$$

$$[P_\alpha,\phi_\beta']=[\pi_\alpha',Q_\beta]=0. \tag{11}$$

We now return to the two-nucleon system. In analogy to Eq. (6) we expand ϕ and π into a "bound" part and a "free" part (marked by a prime):

$$\begin{aligned}\phi_\alpha&=\phi_\alpha'+\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}[(f_A-bf_B)Q_{A\alpha}+(f_B-bf_A)Q_{B\alpha}],\\ \pi_\alpha&=\pi_\alpha'+\frac{1}{1+b}\frac{1}{F^{\frac{1}{2}}}[f_AP_{A\alpha}+f_BP_{B\alpha}],\end{aligned} \tag{12}$$

where

$$F=\int f_A^2=\int f_B^2,\quad F_{AB}=\int f_Af_B,\quad b=F_{AB}/F. \tag{13}$$

The Q's and P's are so defined that $(P_{A\alpha},Q_{A\alpha})$, $(P_{B\alpha},Q_{B\alpha})$ are independently canonical pairs:

$$\begin{aligned}Q_{A\alpha}&=\frac{1}{1+b}\frac{1}{F^{\frac{1}{2}}}\int f_A\phi_\alpha,\qquad Q_{B\alpha}=\frac{1}{1+b}\frac{1}{F^{\frac{1}{2}}}\int f_B\phi_\alpha,\\ P_{A\alpha}&=\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}\int[f_A-bf_B]\pi_\alpha,\\ P_{B\alpha}&=\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}\int[f_B-bf_A]\pi_\alpha,\end{aligned} \tag{14}$$

$$[P_{A\alpha},Q_{A\beta}]=[P_{B\alpha},Q_{B\beta}]=-i\delta_{\alpha\beta}. \tag{15}$$

We require that ϕ_α' and π_α' be orthogonal to the f's:

$$\int f_A\phi_\alpha'=\int f_B\phi_\alpha'=0,\qquad \int f_A\pi_\alpha'=\int f_B\pi_\alpha'=0. \tag{16}$$

The primed variables then commute with the P's and Q's:

$$[P_{A\alpha},\phi_\beta']=[P_{B\alpha},\phi_\beta']=[\pi_\alpha',Q_{A\beta}]=[\pi_\alpha',Q_{B\beta}]=0. \tag{17}$$

Note that the primed variables are not exactly canonical:

$$\begin{aligned}[\pi_\alpha'(\mathbf{r}),\phi_\beta'(\mathbf{r}')]=-i\delta_{\alpha\beta}\Big[\delta(\mathbf{r}-\mathbf{r}')-\frac{1}{1-b^2}\frac{1}{F}\\ \times\{(f_A(\mathbf{r})f_A(\mathbf{r}')+f_B(\mathbf{r})f_B(\mathbf{r}'))\\ -b(f_A(\mathbf{r})f_B(\mathbf{r}')+f_B(\mathbf{r})f_A(\mathbf{r}'))\}\Big].\end{aligned} \tag{18}$$

In terms of these variables we obtain

$$\begin{aligned}H_{\text{mes}}=H_{\text{mes}}'+\tfrac{1}{2}\sum_{\alpha}\Bigg[&\frac{1}{(1+b)^2}(P_{A\alpha}^2+P_{B\alpha}^2+2bP_{A\alpha}P_{B\alpha})\\ &+\frac{1}{1-b}\frac{1}{F}\left\{(Q_{A\alpha}^2+Q_{B\alpha}^2)\left((1+b^2)\int f\omega^2f-2b\int f_A\omega^2f_B\right)+2Q_{A\alpha}Q_{B\alpha}\left((1+b^2)\int f_A\omega^2f_B-2b\int f\omega^2f\right)\right\}\\ &+\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}\left\{Q_{A\alpha}\int\phi_\alpha'\omega^2(f_A-bf_B)+Q_{B\alpha}\int\phi_\alpha'\omega^2(f_B-bf_A)\right\}\Bigg],\end{aligned} \tag{19}$$

$$H_{\text{int}}=H_{\text{int}}'+(2\pi)^{\frac{1}{2}}g\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}\sum_\alpha\int(\tau_{A\alpha}u_A+\tau_{B\alpha}u_B)[(Q_{A\alpha}-bQ_{B\alpha})f_A+(Q_{B\alpha}-bQ_{A\alpha})f_B], \tag{20}$$

$$\Sigma=\Sigma'+(Q_{A1}P_{A2}-Q_{A2}P_{A1})+(Q_{B1}P_{B2}-Q_{B2}P_{B1}). \tag{21}$$

[2] Where variables of integration are omitted they are understood to be the three dimensional volume element $d\mathbf{r}$.

Introduce polar coordinates:

$$\begin{aligned}
Q_{A1}&=Q_A\cos\theta_A,\\
Q_{A2}&=Q_A\sin\theta_A,\\
P_{A1}&=(\cos\theta_A)P_A-(\sin\theta_A)Q_A^{-1}P_{\theta A},\\
P_{A2}&=(\sin\theta_A)P_A+(\cos\theta_A)Q_A^{-1}P_{\theta A},\\
Q_{B1}&=Q_B\cos\theta_B,\\
Q_{B2}&=Q_B\sin\theta_B,\\
P_{B1}&=(\cos\theta_B)P_B-(\sin\theta_B)Q_B^{-1}P_{\theta B},\\
P_{B2}&=(\sin\theta_B)P_B+(\cos\theta_B)Q_B^{-1}P_{\theta B},
\end{aligned}\tag{22}$$

where $(P_{\theta A},\theta_A)$, $(P_{\theta B},\theta_B)$, (P_A,Q_A), (P_B,Q_B) are independently canonical pairs:

$$\begin{aligned}
[P_{\theta A},\theta_A]&=[P_{\theta B},\theta_B]=-i,\\
[P_A,Q_A]&=[P_B,Q_B]=-i.
\end{aligned}\tag{23}$$

Instead of using the transformations (22), it is more convenient to use the new canonical variables:

$$\begin{aligned}
\theta&=\tfrac{1}{2}(\theta_A+\theta_B), & \psi&=\tfrac{1}{2}(\theta_A-\theta_B),\\
P_\theta&=(P_{\theta A}+P_{\theta B}), & P_\psi&=(P_{\theta A}-P_{\theta B}),
\end{aligned}\tag{24}$$

where (P_θ,θ) and (P_ψ,ψ) are independently canonical pairs:

$$[P_\theta,\theta]=[P_\psi,\psi]=-i.\tag{25}$$

Along with the transformations (22) and (24) it is convenient to change the state vector Φ into Φ', given by

$$\Phi'=Q_A^{\frac{1}{2}}Q_B^{\frac{1}{2}}\Phi.\tag{26}$$

The operators H_{mes}, H_{int}, and Σ, considered to act on Φ', then become

$$\begin{aligned}
H_{\text{mes}}=H_{\text{mes}}'&+\frac{1}{2}\frac{1}{(1+b)^2}\{P_A^2+\tfrac{1}{4}Q_A^{-2}[(P_\theta+P_\psi)^2-1]+P_B^2+\tfrac{1}{4}Q_B^{-2}[(P_\theta-P_\psi)^2-1]\}\\
&+\frac{1}{(1+b)^2}b\cos2\psi\{P_AP_B-\tfrac{1}{4}Q_A^{-1}Q_B^{-1}+\tfrac{1}{2}i(Q_B^{-1}P_A+Q_A^{-1}P_B)+\tfrac{1}{4}Q_A^{-1}Q_B^{-1}(P_\theta^2-P_\psi^2)\}\\
&+\frac{1}{2}\frac{1}{(1+b)^2}b\sin2\psi\{(P_A+\tfrac{1}{2}iQ_A^{-1})Q_B^{-1}(P_\theta-P_\psi)-(P_B+\tfrac{1}{2}iQ_B^{-1})Q_A^{-1}(P_\theta+P_\psi)\}\\
&+\frac{1}{2}\frac{1}{(1-b)^2}\frac{1}{F}\left\{(Q_A^2+Q_B^2)\left((1+b^2)\int f\omega^2f-2b\int f_A\omega^2f_B\right)+2\cos2\psi Q_AQ_B\left((1+b^2)\int f_A\omega^2f_B-2b\int f\omega^2f\right)\right\}\\
&+\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}\left\{Q_A\int[\phi_1'\cos(\theta+\psi)+\phi_2'\sin(\theta+\psi)]\omega^2(f_A-bf_B)+Q_B\int[\phi_1'\cos(\theta-\psi)+\phi_2'\sin(\theta-\psi)]\omega^2(f_B-bf_A)\right\},
\end{aligned}\tag{27}$$

$$\begin{aligned}
H_{\text{int}}=H_{\text{int}}'&+\frac{1}{1-b}\left(\frac{2\pi}{F}\right)^{\frac{1}{2}}g\Big\{Q_A[\tau_{A1}\cos(\theta+\psi)+\tau_{A2}\sin(\theta+\psi)]\int u_A(f_A-bf_B)\\
&+Q_B[\tau_{A1}\cos(\theta-\psi)+\tau_{A2}\sin(\theta-\psi)]\int u_A(f_B-bf_A)+Q_B[\tau_{B1}\cos(\theta-\psi)+\tau_{B2}\sin(\theta-\psi)]\int u_B(f_B-bf_A)\\
&+Q_A[\tau_{B1}\cos(\theta+\psi)+\tau_{B2}\sin(\theta+\psi)]\int u_B(f_A-bf_B)\Big\},
\end{aligned}\tag{28}$$

$$\Sigma=\Sigma'+P_\theta.\tag{29}$$

The variable θ can be eliminated from the Hamiltonian by applying the unitary transformation

$$S_1=\exp[i\theta(\tfrac{1}{2}\tau_{A3}+\tfrac{1}{2}\tau_{B3}-\Sigma')].\tag{30}$$

Furthermore, the terms proportional to Q_A^2, Q_B^2, Q_AQ_B, Q_A, Q_B, P_A^2, P_B^2, P_AP_B, can be eliminated from the Hamiltonian through the following transformations:

$$\begin{aligned}
\phi_1''&=\phi_1'+\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}[(f_A-bf_B)Q_A+(f_B-bf_A)Q_B]\cos\psi, & \phi_2''&=\phi_2'+\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}[(f_A-bf_B)Q_A-(f_B-bf_A)Q_B]\sin\psi,\\
\pi_1''&=\pi_1'+\frac{1}{1+b}\frac{1}{F^{\frac{1}{2}}}(f_AP_A+f_BP_B)\cos\psi, & \pi_2''&=\pi_2'+\frac{1}{1+b}\frac{1}{F^{\frac{1}{2}}}(f_AP_A-f_BP_B)\sin\psi,
\end{aligned}\tag{31}$$

which give the following anomalous commutation relations:

$$[\pi_1''(\mathbf{r}),\phi_1''(\mathbf{r}')]=-i\left[\delta(\mathbf{r}-\mathbf{r}')-\frac{1}{1-b^2}\frac{\sin^2\psi}{F}\{[f_A(\mathbf{r})f_A(\mathbf{r}')+f_B(\mathbf{r})f_B(\mathbf{r}')]-b[f_A(\mathbf{r})f_B(\mathbf{r}')+f_B(\mathbf{r})f_A(\mathbf{r}')]\}\right],$$

$$[\pi_2''(\mathbf{r}),\phi_2''(\mathbf{r}')]=-i\left[\delta(\mathbf{r}-\mathbf{r}')-\frac{1}{1-b^2}\frac{\cos^2\psi}{F}\{[f_A(\mathbf{r})f_A(\mathbf{r}')+f_B(\mathbf{r})f_B(\mathbf{r}')]-b[f_A(\mathbf{r})f_B(\mathbf{r}')+f_B(\mathbf{r})f_A(\mathbf{r}')]\}\right], \quad (32)$$

$$[\pi_1''(\mathbf{r}),\phi_2''(\mathbf{r}')]=[\pi_2''(\mathbf{r}),\phi_1''(\mathbf{r}')]=-i\frac{\cos\psi\,\sin\psi}{(1-b^2)F}\{[f_A(\mathbf{r})f_A(\mathbf{r})-f_B(\mathbf{r})f_B(\mathbf{r}')]-b[f_A(\mathbf{r})f_B(\mathbf{r}')-f_B(\mathbf{r})f_A(\mathbf{r}')]\}.$$

With these new variables, we obtain

$$\begin{aligned}
H_{\text{mes}}=H_{\text{mes}}''&+\frac{1}{8}\frac{1}{(1+b)^2}Q_A^{-2}[(P_\theta+P_\psi-\tfrac{1}{2}\tau_{A3}-\tfrac{1}{2}\tau_{B3}-\Sigma'')^2-1]\\
&+\frac{1}{8}\frac{1}{(1+b)^2}Q_B^{-2}[(P_\theta-P_\psi-\tfrac{1}{2}\tau_{A3}-\tfrac{1}{2}\tau_{B3}-\Sigma'')^2-1]\\
&+\frac{1}{4}\frac{1}{(1+b)^2}b\cos2\psi\{-Q_A^{-1}Q_B^{-1}+2i(Q_B^{-1}P_A+Q_A^{-1}P_B)+Q_A^{-1}Q_B^{-1}[(P_\theta-\tfrac{1}{2}\tau_{A3}-\tfrac{1}{2}\tau_{B3}-\Sigma'')^2-P_\psi^2]\}\\
&+\frac{1}{2}\frac{1}{(1+b)^2}b\sin2\psi\{Q_B^{-1}(P_A+\tfrac{1}{2}iQ_A^{-1})(P_\theta-P_\psi-\tfrac{1}{2}\tau_{A3}-\tfrac{1}{2}\tau_{B3}-\Sigma'')\\
&\qquad -Q_A^{-1}(P_B+\tfrac{1}{2}iQ_B^{-1})(P_\theta+P_\psi-\tfrac{1}{2}\tau_{A3}-\tfrac{1}{2}\tau_{B3}-\Sigma'')\}, \quad (33)
\end{aligned}$$

$$H_{\text{int}}=H_{\text{int}}'', \quad (34)$$

$$\Sigma=P_\theta-\tfrac{1}{2}\tau_{A3}-\tfrac{1}{2}\tau_{B3}, \quad (35)$$

where

$$\begin{aligned}
Q_A&=\frac{1}{1+b}\frac{1}{F^{\frac{1}{2}}}\int f_A(\phi_1''\cos\psi+\phi_2''\sin\psi),\quad Q_B=\frac{1}{1+b}\frac{1}{F^{\frac{1}{2}}}\int f_B(\phi_1''\cos\psi-\phi_2''\sin\psi),\\
P_A&=\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}\left[\left(\cos\psi\int f_A\pi_1''+\sin\psi\int f_A\pi_2''\right)-b\left(\cos\psi\int f_B\pi_1''+\sin\psi\int f_B\pi_2''\right)\right], \quad (36)\\
P_B&=\frac{1}{1-b}\frac{1}{F^{\frac{1}{2}}}\left[\left(\cos\psi\int f_B\pi_1''-\sin\psi\int f_B\pi_2''\right)-b\left(\cos\psi\int f_A\pi_1''-\sin\psi\int f_A\pi_2''\right)\right].
\end{aligned}$$

Following the customary procedure in the strong-coupling theory, we are going to diagonalize the interaction Hamiltonian. Before doing this, however, it is necessary to discuss more about the distribution function, f, whose significance was mentioned earlier. We may greatly narrow down the choices of f if we try to develop a self-consistent procedure, starting with that part of the Hamiltonian which is expected to give the leading contributions for a given region of coupling, and for that part making a best choice for f. At this point the choice of f is not unique. For instance, the choice of $f=u$, the source function, will reduce the theory into the customary one. In this case the interaction Hamiltonian (34) will be diagonalized in a representation in which τ_{A1} and τ_{B1} are diagonal if we write $S_1=\exp\{i[\frac{1}{2}\tau_{A3}(\theta+\psi)+\frac{1}{2}\tau_{B3}(\theta-\psi)-\theta\Sigma']\}$ in Eq. (30). However, the effects of the quantum-mechanical field fluctuations in this case give the perturbation series with an expansion parameter $1/g^2\kappa a$, a being the source size. Since the theory is not more singular than the self-energy singularity ($\sim 1/a$), the perturbation series will be valid only if the condition $g^2\ll 1/\kappa a$ in addition to the usual condition $g^2\gg 1$, is satisfied. The conventional procedure thus faces serious divergence difficulties in the limit of vanishing source size.[3–5] On the other hand, it is shown that the choice of $f=v$, the self-field, gives results in which the perturbation series are given in terms of the expansion parameter $(1/g^2)\ln(1/\kappa a)$. Furthermore, it is shown that the renormalized coupling constant contains terms which have logarithmic dependence on the source size, and if the

[3] H. Nickle and R. Serber, Phys. Rev. **119**, 449 (1960).
[4] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940).
[5] G. Wentzel, Helv. Phys. Acta **14**, 633 (1941).

expansion is expressed in terms of the renormalized coupling constant, then, aside from the self-energy terms, the theory becomes independent of the source size.[3] Here lies the physical significance of the choice of $f=v$. Hereafter we will adhere to this choice.

The interaction Hamiltonian can be diagonalized by rotating the secular components of the nuclear isotopic spins τ_{A1} and τ_{B1} to the direction of the vectors $\int u_A\phi''$ and $\int u_B\phi''$, respectively. This is accomplished by a unitary transformation

$$S_2=\exp\left\{\frac{i}{2}\left[\tau_{A3}\tan^{-1}\left(\int u_A\phi_2''\Big/\int u_A\phi_1''\right)+\tau_{B3}\tan^{-1}\left(\int u_B\phi_2''\Big/\int u_B\phi_1''\right)\right]\right\}. \quad (37)$$

The interaction energy is then diagonalized in a representation in which τ_{A1} and τ_{B1} are diagonal:

$$H_{\text{int}}=(2\pi)^{\frac{1}{2}}g(\tau_{A1}W_A+\tau_{B1}W_B), \quad (38)$$

where

$$W_A=\left[\left(\int u_A\phi_1''\right)^2+\left(\int u_A\phi_2''\right)^2\right]^{\frac{1}{2}},$$
$$W_B=\left[\left(\int u_B\phi_1''\right)^2+\left(\int u_B\phi_2''\right)^2\right]^{\frac{1}{2}}. \quad (39)$$

Since we are considering the lowest energy states of the system, we consider only the lower state ($\tau_{A1}=-1$, $\tau_{B1}=-1$). The upper states ($\tau_{A1}=+1, \tau_{B1}=+1$; $\tau_{A1}=\pm1, \tau_{B1}=\mp1$), which are separated from the lower state by the energy gap of the order of g^2/a, the magnitude of the self-energy, have completely different characteristics from those of the lower state. In this work we will consider neither the upper states nor the interference between lower and upper states.

We also obtain

$$H_{\text{mes}}=H_{\text{mes}}''+\frac{1}{8}\left(\frac{N}{W_A^2}+\frac{N}{W_B^2}\right)-\frac{1}{8}\frac{q^2}{FW_A^4}\left\{\int u_A(\phi_1''\cos\psi+\phi_2''\sin\psi)\right\}^2-\frac{1}{8}\frac{q^2}{FW_B^4}\left\{\int u_B(\phi_1''\cos\psi-\phi_2''\sin\psi)\right\}^2$$
$$+\frac{1}{8}\frac{1}{(1+b)^2}Q_A^{-2}\{(P_\theta+P_\psi-\Sigma'')^2-1\}+\frac{1}{8}\frac{1}{(1-b^2)^2}Q_B^{-2}\{(P_\theta-P_\psi-\Sigma'')^2-1\}$$
$$+\frac{1}{2}\frac{1}{(1+b)^2}b\cos2\psi\cdot[-\tfrac{1}{2}Q_A^{-1}Q_B^{-1}+(Q_A^{-1}P_B+Q_B^{-1}P_A)+\tfrac{1}{2}Q_A^{-1}Q_B^{-1}\{(P_\theta-\Sigma'')^2-P_\psi^2\}]$$
$$+\frac{1}{2}\frac{1}{(1+b)^2}b\sin2\psi\cdot[Q_B^{-1}(P_A+\tfrac{1}{2}iQ_A^{-1})(P_\theta-P_\psi-\Sigma'')-Q_A^{-1}(P_B+\tfrac{1}{2}iQ_B^{-1})(P_\theta+P_\psi-\Sigma'')], \quad (40)$$

$$\Sigma=P_\theta-\tfrac{1}{2}\tau_{A3}-\tfrac{1}{2}\tau_{B3}, \quad (41)$$

where

$$N=\int u_A^2=\int u_B^2, \quad q=\int u_Af_A=\int u_Bf_B.$$

In Eq. (40) the off-diagonal terms (linear in τ_{A3} and τ_{B3}) coupling widely separated states are neglected.

Our next step is to split the field into a quasi-classically determined self-field ($\sim g$) plus a field of free mesons and fluctuations ($\sim g^0$) so that we can expand the theory in the ratio of free field and self-field:

$$\phi_1''=\phi_1'''+(v_A+v_B)\cos\psi,$$
$$\phi_2''=\phi_2'''+(v_A-v_B)\sin\psi, \quad (42)$$

where ϕ_1''' and ϕ_2''' are the fluctuating fields, v_A and v_B, the self-fields of nucleon A and B, respectively. The above transformations correspond to a contact transformation

$$S_3=\exp\left\{-i\int[\pi''(v_A+v_B)\cos\psi+\pi''(v_A-v_B)\sin\psi]\right\}. \quad (43)$$

The triple-primed variables obey the same commutation relations as those obeyed by the double-primed variables (32). With these variables, to the order of

g^{-2}, we obtain

$$H=(2\pi)^{\frac{1}{2}}g(\tau_{A1}W_A+\tau_{B1}W_B)+\frac{1}{2}\int v_A\omega^2v_A+\frac{1}{2}\int v_B\omega^2v_B+\cos 2\psi\int v_A\omega^2v_B$$
$$+\int(\phi_1\cos\psi+\phi_2\sin\psi)\omega^2v_A+\int(\phi_1\cos\psi-\phi_2\sin\psi)\omega^2v_B+\frac{1}{8}\left(\frac{N}{W_A{}^2}+\frac{N}{W_B{}^2}\right)+H_{\text{mes}}$$
$$+\frac{1}{8}\frac{1}{(1+b)^2}Q_A{}^{-2}[(P_\theta+P_\psi-\Sigma)^2-1]+\frac{1}{8}\frac{1}{(1+b)^2}Q_B{}^{-2}[(P_\theta-P_\psi-\Sigma)^2-1]$$
$$+\frac{1}{2}\frac{1}{(1+b)^2}b\cos 2\psi[-\tfrac{1}{2}Q_A{}^{-1}Q_B{}^{-1}+(Q_A{}^{-1}P_B+Q_B{}^{-1}P_A)+\tfrac{1}{2}Q_A{}^{-1}Q_B{}^{-1}\{(P_\theta-\Sigma)^2-P_\psi{}^2\}]$$
$$+\frac{1}{2}\frac{1}{(1+b)^2}b\sin 2\psi[Q_B{}^{-1}(P_A+\tfrac{1}{2}iQ_A{}^{-1})(P_\theta-P_\psi-\Sigma)-Q_A{}^{-1}(P_B+\tfrac{1}{2}iQ_B{}^{-1})(P_\theta+P_\psi-\Sigma)],\tag{44}$$

where

$$W_A{}^2=\left\{\int u_A[\phi_1+(v_A+v_B)\cos\psi]\right\}^2+\left\{\int u_A[\phi_2+(v_A-v_B)\sin\psi]\right\}^2,$$
$$W_B{}^2=\left\{\int u_B[\phi_1+(v_A+v_B)\cos\psi]\right\}^2+\left\{\int u_B[\phi_2+(v_A-v_B)\sin\psi]\right\}^2,$$
$$Q_A=\frac{1}{1+b}\frac{1}{V^{\frac{1}{2}}}\left[\int v_A(v_A+v_B\cos 2\psi)+\int v_A(\phi_1\cos\psi+\phi_2\sin\psi)\right],$$
$$Q_B=\frac{1}{1+b}\frac{1}{V^{\frac{1}{2}}}\left[\int v_B(v_B+v_A\cos 2\psi)+\int v_B(\phi_1\cos\psi-\phi_2\sin\psi)\right],$$
$$V=\int v_A{}^2=\int v_B{}^2,\quad V_{AB}=\int v_Av_B.$$

In thė above equations we omit all the triple primes.

III. SOME OLD PROBLEMS

We are now ready to examine some old problems.[4–6]

1. Self-Energy

In the strong-coupling limit, the leading contributions to the Hamiltonian come from the terms quadratic in the coupling constant:

$$H_l=-(2\pi)^{\frac{1}{2}}g(W_A+W_B)+\frac{1}{2}\int v_A\omega^2v_A+\frac{1}{2}\int v_B\omega^2v_B+\cos 2\psi\cdot\int v_A\omega^2v_B$$
$$=-g^2(I^2+J^2+2IJ\cos 2\psi)^{\frac{1}{2}}+\tfrac{1}{2}g^2(I+J\cos 2\psi),\tag{45}$$

where

$$I=4\pi\int u_A\omega^{-2}u_A=4\pi\int u_B\omega^{-2}u_B=1/a,$$
$$J=4\pi\int u_A\omega^{-2}u_B=4\pi\int u_B\omega^{-2}u_A\approx\frac{e^{-\kappa R}}{R}\quad(\text{for}\quad R\gg a).\tag{46}$$

[6] S. M. Dancoff and R. Serber, Phys. Rev. **63**, 143 (1943).

H_l represents the static self-energy and the nuclear force of the system, whose dependence on the separation R between the two nucleons comes through J. In the above equations v_A and v_B are determined from the condition that v_A and v_B be chosen so as to minimize the static part of the Hamiltonian. To the first approximation we obtain $\omega^2 v_A=(2\pi)^{\frac{1}{2}} g u_A$ and $\omega^2 v_B=(2\pi)^{\frac{1}{2}} g u_B$ in the limit that $(P_{\theta A}/g)\ll 1$ and $(P_{\theta B}/g)\ll 1$.

For large g, ψ will adjust itself so as to minimize H_l:

$$\cos 2\psi=1, \quad \psi=0 \quad \text{or} \quad \pi, \tag{47}$$

and

$$H_l \xrightarrow[(\cos 2\psi\to 1)]{} -\tfrac{1}{2}g^2(I+J). \tag{48}$$

When the two nucleons overlap each other exactly, i.e., $R\to 0$, $J\to 1$, the self-energy becomes

$$H_l \xrightarrow[(R\to 0)]{} -g^2 I=-g^2/a, \tag{49}$$

which is four times as large as in the case of a single nucleon. Physically this comes about due to the fact that the coupling constant in this case is effectively twice as large as in the case of a single nucleon.

The condition $\cos 2\psi=1$ corresponds to the "freezing of the isotopic spin" at small separations. In this case the nuclear force appearing in (48) through J is isotopic spin-independent. A question naturally arises at what separation the "freezing of the isotopic spin" sets in. This question can be answered in a semiquantitative way by considering the situation from the other side of the extreme case of $R=\infty$, when the variable ψ is completely unspecified. As the two nucleons approach each other, H in Eq. (45) can be expanded as $H_l=-(g^2/2)(I+J\cos 2\psi)$, the term $-(g^2/2)J\cos 2\psi$ thus appearing as correction to g^2 term; a correction term also appears in the isobaric energy, which, however, is small down to separations of the order of κ^{-1}. Thus, one may roughly estimate the critical separation from the relation $\kappa/g^2=g^2J/2$, or $2/g^4=e^{-\kappa R}/\kappa R$.

2. Nuclear Forces at Large Separations

We next consider the case of separations larger than the critical one discussed above. In this case the Hamiltonian can be approximated as a sum of the isobaric terms of the individual nucleon and H_l:

$$H\cong -\frac{g^2}{2}(I+J\cos 2\psi)+\frac{1}{4V}(P_\theta{}^2+P_\psi{}^2). \tag{50}$$

It is found that the treatment of $-(g^2/2)J\cos 2\psi$ as a perturbation to $(1/4V)P_\psi{}^2$ leads to the exchange forces. By making use of the unperturbed stationary states consisting of the symmetric ($\cos n\psi$) and the antisymmetric ($\sin n\psi$) isotopic spin states, where n is the charge differences between the two nucleons, we obtain the following characteristics of the nuclear forces: For $n=0$ (p-p, or n-n) the perturbation vanishes in first order. For $n=1$ (n-p), in the case of deuteron, the perturbation energies are given as $V_T(\text{triplet})=g^2J/4$ and $V_S(\text{singlet})=-g^2J/4$. The factor 1/4 appearing in the perturbation energies has significant physical meaning in a sense that $g^2/4\to g_r{}^2$ (g_r: renormalized coupling constant) to the extent that we neglect here the logarithmic dependence of g_r on the source size. Thus we may write

$$V_T=g_r{}^2J, \quad V_S=-g_r{}^2J. \tag{51}$$

The two nucleons thus feel each other through the renormalized coupling constant at large separations. The nuclear forces discussed thus far cannot escape from the usual criticisms that (1) they are independent of the isotopic spin inside the radius of the critical separation, and (2) the singlet state is lower than the triplet state at large separations.

3. Isobaric Energy

Next we consider the region in which the separation R is of such a magnitude that the Hamiltonian is minimized, to first order, by minimizing the terms proportional to g^2. In this region ψ is in the neighborhood of 0 or π, so that $\cos 2\psi$ and $\sin 2\psi$ may be expanded, taking only the leading terms in the expansion in the Hamiltonian. Replacing Q_A and Q_B by their equilibrium value Q_0,

$$Q_0=\frac{1}{1+b}\frac{1}{V^{\frac{1}{2}}}(V+V_{AB})=V^{\frac{1}{2}}, \tag{52}$$

we may write the Hamiltonian to the order of g^{-2} as a sum of the self-energy and the nuclear force H_S, normal modes H_0, and isobaric energy H_I:

$$H=H_S+H_0+H_I, \tag{53}$$

where

$$\begin{aligned} H_S&=-\frac{g^2}{2}(I+J)+\frac{2}{g^2}\frac{I}{1+(J/I)^2},\\ H_0&=H_{\text{mes}}-\frac{2\pi}{I+J}\left\{\left(\int u_A\phi_2\right)^2+\left(\int u_B\phi_2\right)^2\right\},\\ H_I&=\frac{(P_\theta-\Sigma)^2-1}{4V(1+b)}+\frac{1-b}{4V(1+b)^2}P_\psi{}^2+g^2J\psi^2.\end{aligned} \tag{54}$$

The terms linear in the coupling constant, $\int(\phi_1\cos\psi+\phi_2\sin\psi)\omega^2 v_A$ and $\int(\phi_1\cos\psi-\phi_2\sin\psi)\omega^2 v_B$, in Eq. (44) cancel out exactly by the expansion series generated by the interaction energy, $-(2\pi)^{\frac{1}{2}}g(W_A+W_B)$.

In the above equations $(2/g^2)I/[1+(J/I)^2]$ is a g^{-2} correction to the self-energy and the nuclear force. The normal modes, H_0, which will be discussed in detail in the following chapter, diagonalize the free-field terms of the order of g^0 in the lower states. The terms,

$$\frac{1-b}{4V(1+b)^2}P_\psi{}^2+g^2J\psi^2,$$

in H_I represent the linear harmonic oscillations of ψ about the "frozen" position. The amplitude of the oscillation is

$$\psi_m=\frac{1}{g}\left[\frac{2\pi(1-b)}{Y(1+b)^2J}\right]^{\frac{1}{4}},\quad Y=\frac{8\pi}{g^2}V=\frac{2\pi}{\kappa}. \tag{55}$$

It should be pointed out that our results in this chapter are consistent with those by Serber and Dancoff.[6] However, if we choose $f=u$, then ψ_m will have $(1/a)^{\frac{1}{4}}$ dependence on the source size. Thus in this case ψ_m will face physically unreasonable divergence difficulties in the limit of vanishing source size.

IV. NORMAL MODES

In this chapter we will study the effects of quantum-mechanical field fluctuations by examining the normal modes which are chosen to diagonalize the free-field terms of the order of g^0 in the Hamiltonian.

The renormalized coupling constant g_r is obtained from the relation $g_r/g=\langle N|\tau_-|P\rangle$, the matrix element of τ_- taken between physical proton and neutron states.[7,8] To the order of g^0, $g_r{}^2$ is given by[3]

$$g_r{}^2=\tfrac{1}{4}[g^2+(2/\pi)\ln\kappa\Lambda],\quad a=2\Lambda(1+\kappa\Lambda)^2. \tag{56}$$

Thus the study of the nuclear force to the order of $g_r{}^2$ requires a thorough knowledge of the terms of the order of g^0.

We will restrict our attention here to the region in which the isotopic spin is completely frozen, i.e., $\cos2\psi=1$. Naturally this restriction greatly simplifies our calculation. In this region the normal modes are represented by

$$H_0=H_{\text{mes}}-\frac{2\pi}{I+J}\left[\left(\int u_A\phi_2\right)^2+\left(\int u_B\phi_2\right)^2\right]. \tag{57}$$

The field equations are given by

$$\dot{\phi}_\alpha(\mathbf{r},t)=i[H_0,\phi_\alpha(\mathbf{r},t)],\quad \dot{\pi}_\alpha(\mathbf{r},t)=i[H_0,\pi_\alpha(\mathbf{r},t)]. \tag{58}$$

Using the anomalous commutation relations (32), we obtain the equations of motion[9]:

$$\begin{aligned}&\dot{\phi}_1(\mathbf{r},t)=\pi_1(\mathbf{r},t),\quad \dot{\phi}_2=\pi_2(\mathbf{r},t),\\ &\dot{\pi}_1(\mathbf{r},t)=-\omega_{\text{op}}{}^2\phi_1(\mathbf{r},t),\\ &\dot{\pi}_2(\mathbf{r},t)=-\omega_{\text{op}}{}^2\phi_2(\mathbf{r},t)\\ &\qquad+\frac{4\pi}{I+J}\left[u_A\int u_A\phi_2+u_B\int u_B\phi_2\right].\end{aligned} \tag{59}$$

Eliminating π_1 and π_2, we obtain the second-order field equations:

$$\begin{aligned}&\ddot{\phi}_1(\mathbf{r},t)+\omega_{\text{op}}{}^2\phi_1(\mathbf{r},t)=0,\\ &\ddot{\phi}_2(\mathbf{r},t)+\omega_{\text{op}}{}^2\phi_2(\mathbf{r},t)\\ &\qquad-\frac{4\pi}{I+J}\left[u_A\int u_A\phi_2+u_B\int u_B\phi_2\right]=0.\end{aligned} \tag{60}$$

Thus in this case the 1-field solutions simply represent the free meson states.

Consider special solutions for the 2-field equations of the form

$$\phi_2(\mathbf{r},t)=\phi_{\mathbf{k}}{}^{(2)}(\mathbf{r})e^{-i\omega t}, \tag{61}$$

where $\phi_{\mathbf{k}}{}^{(2)}(\mathbf{r})$ are normalized such that

$$\int\phi_{\mathbf{k}'}{}^{(2)}(\mathbf{r})\phi_{\mathbf{k}}{}^{(2)}(\mathbf{r})d\mathbf{r}=\delta(\mathbf{k}-\mathbf{k}'). \tag{62}$$

If we write

$$\phi_{\mathbf{k}}{}^{(2)}(\mathbf{r})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\mathbf{k}'\,\phi_{\mathbf{k}\mathbf{k}'}e^{i\mathbf{k}'\cdot\mathbf{r}}, \tag{63}$$

and let $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ denote the Fourier amplitude of $u(r)$, $v(r)$:

$$\begin{aligned}u_{\mathbf{k}}&=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\mathbf{r}\,u(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}},\\ v_{\mathbf{k}}&=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\mathbf{r}\,v(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}},\end{aligned} \tag{64}$$

Eq. (60) then becomes

$$(\omega_{\text{op}}{}^2-\omega^2)\phi_{\mathbf{k}\mathbf{k}'}=\frac{4\pi}{I+J}\left[u_{A\mathbf{k}'}\int u_A\phi_2+u_{B\mathbf{k}'}\int u_B\phi_2\right], \tag{65}$$

[7] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[8] H. Jahn, Fortschr. Physik **7**, 451 (1959).

[9] $\omega_{\text{op}}{}^2=\kappa^2-\nabla^2$, $\omega^2=\chi^2+k^2$.

whose solution is readily obtained as

$$\phi_{\mathbf{kk}'}=\cos\delta\left[\delta(\mathbf{k}-\mathbf{k}')+\frac{4\pi}{I+J}\frac{\left(1-\frac{4\pi}{I+J}\gamma\right)(u_{A\mathbf{k}'}u_{A-\mathbf{k}}+u_{B\mathbf{k}'}u_{B-\mathbf{k}})+\frac{4\pi}{I+J}\gamma_{AB}(u_{A\mathbf{k}'}u_{B-\mathbf{k}}+u_{B\mathbf{k}'}u_{A-\mathbf{k}})}{\left(1-\frac{4\pi}{I+J}\gamma\right)^2-\left(\frac{4\pi}{I+J}\gamma_{AB}\right)^2}P\frac{1}{\omega'^2-\omega^2}\right]$$

$$=\cos\delta\Bigg[\delta(\mathbf{k}-\mathbf{k}')+\frac{4\pi}{I+J}u_{\mathbf{k}'}u_{-\mathbf{k}}$$

$$\times\frac{\left(1-\frac{4\pi}{I+J}\gamma\right)(e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}/2}+e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}/2})+\frac{4\pi}{I+J}\gamma_{AB}(e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}/2}+e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{R}/2})}{\left(1-\frac{4\pi}{I+J}\gamma\right)^2-\left(\frac{4\pi}{I+J}\gamma_{AB}\right)^2}P\frac{1}{\omega'^2-\omega^2}\Bigg],\tag{66}$$

where

$$\gamma=\frac{1}{(2\pi)^{\frac{1}{2}}g}\left[\frac{g}{2(2\pi)^{\frac{1}{2}}}I+\omega^2P\int\frac{u_{A\mathbf{k}'}v_{A-\mathbf{k}'}}{\omega'^2-\omega^2}d\mathbf{k}'\right],\quad \gamma_{AB}=\frac{1}{(2\pi)^{\frac{1}{2}}g}\left[\frac{g}{2(2\pi)^{\frac{1}{2}}}J+\omega^2P\int\frac{u_{A\mathbf{k}'}v_{B-\mathbf{k}'}}{\omega'^2-\omega^2}d\mathbf{k}'\right],\tag{67}$$

and

$$u_{A\mathbf{k}}=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\mathbf{r}\,u_A(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}=u_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{R}/2},\quad u_{B\mathbf{k}}=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\mathbf{r}\,u_B(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}=u_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{R}/2}.$$

If we expand the fields according to

$$\phi_\alpha=\int\frac{d\mathbf{k}}{(2\omega)^{\frac{1}{2}}}[a_\alpha(\mathbf{k})+a_\alpha^\dagger(\mathbf{k})]\phi_{\mathbf{k}}^{(\alpha)}(\mathbf{r}),\quad \pi_\alpha=-i\int\frac{d\mathbf{k}}{(2/\omega)^{\frac{1}{2}}}[a_\alpha(\mathbf{k})-a_\alpha^\dagger(\mathbf{k})]\phi_{\mathbf{k}}^{(\alpha)}(\mathbf{r}),\tag{68}$$

where $a_\alpha(\mathbf{k})$, $a_\alpha^\dagger(\mathbf{k})$ are the time-dependent annihilation and creation operators satisfying the usual commutation relations, $[a_\alpha(\mathbf{k}),a_\beta^\dagger(\mathbf{k}')]=\delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{k}')$. H_0 is then written as

$$H_0=\int d\mathbf{k}[a_1^\dagger(\mathbf{k})a_1(\mathbf{k})+a_2^\dagger(\mathbf{k})a_2(\mathbf{k})]\omega+\Delta E,\tag{69}$$

where ΔE represents the change in the zero-point energy of the 2-field due to the presence of the source functions. This energy shift is obtained by quantization in a spherical box[3]:

$$\Delta E=-\frac{1}{2\pi}\int_0^\infty\delta\frac{k}{\omega}dk-\tfrac{1}{2}\kappa=-\frac{1}{2\pi}\int_0^\infty dk\frac{\omega}{1+k^2}\frac{\partial h}{\partial k},\quad \delta=\tan^{-1}[-h(k)],\tag{70}$$

where δ is the phase shift of the $\phi_{\mathbf{k}}^{(2)}(\mathbf{r})$ solutions.

We are ready now to evaluate ΔE for few cases of special interest.

Case 1. $R=0$

This is the case when the two nucleons overlap each other exactly. The solution is a spherically symmetric one and we need consider S-wave phase shift only. Since $\gamma_{AB}=\gamma$ and $J=1$ for $R=0$, Eq. (66) becomes

$$\phi_{\mathbf{kk}'}=\cos\delta\left[\delta(\mathbf{k}-\mathbf{k}')+\frac{4\pi}{I}u_{\mathbf{k}'}u_{-\mathbf{k}}\frac{1}{[1-(4\pi/I)\gamma]}P\frac{1}{\omega'^2-\omega^2}\right],\tag{71}$$

from which we obtain the phase shifts:

$$\delta_A=\delta_B=\tan^{-1}_{\mathbf{k}'\to\mathbf{k}}\left[(2\pi^2k)\frac{4\pi}{I}u_{\mathbf{k}'}u_{-\mathbf{k}}\frac{1}{[1-(4\pi/I)\gamma]}\right].\tag{72}$$

Assuming the Yukawa source function

$$u(\mathbf{r})=\frac{1}{4\pi\Lambda^2}\frac{1}{r}e^{-r/\Lambda},\quad u_{\mathbf{k}}=\frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{[1+(\Lambda k)^2]},\quad a=2\Lambda(1+\kappa\Lambda)^2, \tag{73}$$

we obtain

$$\gamma=\frac{1}{4\pi}\left[I+\frac{1}{2}\frac{\kappa(2+\kappa\Lambda)-\Lambda(3+2\kappa\Lambda+\kappa^2\Lambda^2)k^2-\Lambda^3k^4}{(1+\kappa\Lambda)^2(1+\Lambda^2k^2)^2}\right]. \tag{74}$$

The energy shift ΔE then becomes

$$\Delta E=2\left[-\frac{\ln 2}{\pi\Lambda}-\frac{\kappa}{2\pi}\left(\ln\frac{4}{\kappa\Lambda}+1\right)\right]. \tag{75}$$

Note that ΔE in this case is exactly twice as large as the energy shift corresponding to a single nucleon. This is physically correct since this is g^0 correction.

Case 2. $R\to\infty$

This is the case when the two nucleons are infinitely far apart from each other, so that we can treat each nucleon independently. Obviously ΔE is then just the sum of the energy shifts of the individual nucleons.

Case 3. $R\neq 0$, $\kappa\Lambda<\kappa R<1$

In this case we are considering the region in which R is well within the nuclear force range, while it is greater than the source size a. Since the solution is not spherically symmetric, we should consider higher partial waves in addition to the S wave.

For the S wave the phase shifts are given by[10]

$$(\delta_A)_S=(\delta_B)_S=\tan^{-1}\left\{(2\pi^2k)\frac{4\pi}{I+J}u_{\mathbf{k}}^2\frac{\left(1-\frac{4\pi}{I+J}\gamma\right)+\frac{4\pi}{I+J}\gamma_{AB}\frac{\sin kR}{kR}}{\left(1-\frac{4\pi}{I+J}\gamma\right)^2-\left(\frac{4\pi}{I+J}\gamma_{AB}\right)^2}\right\}, \tag{76}$$

where, using the Yukawa source function (73), γ_{AB} is readily obtained as

$$\gamma_{AB}=\frac{1}{4\pi}\left\{J+\frac{1}{R}\left[\frac{\cos kR}{(1+\Lambda^2k^2)^2}-\frac{e^{-\kappa R}}{(1-\kappa^2\Lambda^2)^2}+e^{-R/\Lambda}\left[\frac{1}{(1-\kappa^2\Lambda^2)^2}-\frac{1}{(1+\Lambda^2k^2)^2}+\frac{R}{2\Lambda}\left(\frac{1}{1-\kappa^2\Lambda^2}-\frac{1}{1-\Lambda^2k^2}\right)\right]\right]\right\}. \tag{77}$$

The energy shift ΔE is obtained by inserting (76) into (71). The calculations are straightforward, but they are extremely tedious. Neglecting the finite and nonessential terms, we finally obtain

$$(\Delta E)_S\cong-(J/2)[(2/\pi)(\ln\kappa\Lambda-\ln\kappa R)]. \tag{78}$$

We pointed out previously that only the S wave is involved in this problem for two extreme cases (i.e., $R=0$ and $R=\infty$). Consequently, any contributions to the energy shift ΔE coming from the higher partial waves should vanish as $R\to 0$ and $R\to\infty$, a situation which prompts one to assume that the higher partial waves play only a minor role in this problem. A careful study indeed shows that their contribution to the energy shift is finite and nonessential, so that we may neglect them in this problem.[11]

We may then write the Hamiltonian to the order of g^0 as

$$\begin{aligned}H&=-\frac{g^2}{2}I-\frac{J}{2}\left(4g_r^2-\frac{2}{\pi}\ln\kappa R\right)\\&=-\frac{g^2}{2}I-\frac{J}{2}\left(g^2+\frac{2}{\pi}\ln\frac{\Lambda}{R}\right).\end{aligned} \tag{79}$$

The implication of the above equation is obvious and significant. The nuclear force expressed in terms of the

[10] K. A. Brueckner, Phys. Rev. **89**, 834 (1953).

[11] The energy shift due to the higher partial waves is represented by the series of integral equations of the form

$$\frac{1}{R}\int_0^\infty\frac{dx(x^2+\kappa^2R^2)^{\frac{1}{2}}x^{n-1}h_0}{x^{2n}+h_0^2[a_n\sin x/x+b_n\cos x]^2}\times\left[-\{(n+1)a_n+b_nx^2\}\frac{\sin x}{x}+\{a_n-nb_n\}\cos x\right],$$

where a_n and b_n are finite constants ($a_n\sin x/x+b_n\cos x\neq 0$), n is a positive integer ($n\neq 0$); h_0 has very weak dependence on k and R, varying slowly from a positive definite at $k=0$ to 0 at $k=\infty$. Obviously the above integral equation is convergent, yielding a finite and nonessential result for this problem.

renormalized coupling constant is explicitly independent of the source size. However, as the two nucleons come closer together, the $\ln(\Lambda/R)$ term approaches zero, and the dependence of the nuclear force on the unrenormalized coupling constant alone becomes increasingly pronounced. This situation is analogous to the case of the quantum electrodynamics in which two electrons see each other through renormalized charges at the large separations, while, as they approach each other, they gradually feel the unrenormalized (bare) charges.

V. CONCLUSION

This work achieves its main objective in that it proves explicitly the conjecture that the dependence of the nuclear force on the unrenormalized coupling constant alone becomes increasingly pronounced as the two nucleons come closer together. For this purpose the Serber-Pais method turns out to be both powerful and consistent. One can expand the Hamiltonian to any desired order in the coupling constant through a series of unitary transformations, and then one may ask an interesting question of whether our conclusion, valid for the most significant terms, which are of order g_r^2, is also valid to all orders in g_r. The answer to this question requires some tedious calculations in a most systematic way. It is our feeling that our conclusion is also valid to higher order terms in g_r.

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Construction of Unitary Scattering Amplitudes*

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A general linear technique is discussed which constructs unitary scattering amplitudes without expanding in partial waves and in the presence of inelastic channels. Two- and three-particle intermediate states are discussed explicitly, but the method can be extended directly to any finite number of particles.

A new approximation technique suggested by this formalism is applied to electroproduction of pions from pions and pion–K-meson scattering. A form of the impulse approximation is derived for both the coupled form factor and the coupled scattering amplitude problems. The nucleon and deuteron form factor system is briefly discussed.

Finally, a model field theory which contains three-particle intermediate states is formulated and solved by the linear technique for purely pedagogical reasons.

I. INTRODUCTION

THE conjecture of Mandelstam[1] concerning the analytical structure of the scattering amplitude has led to considerable insight into the interrelation of the various strong interactions. This representation has been verified in perturbation theory by Eden[2] and Polkinghorne.[3] The work of Chew and Mandelstam[4] on the low-energy pion-pion system utilized this analyticity together with unitarity in the form of an expansion in partial waves to develop a set of dynamical equations for the determination of the scattering amplitude. This program has met with difficulties, not the least of which is the fact that the partial-wave expansion cannot converge along the negative cut, and if the series is terminated beyond S waves, false divergences are introduced into the equations. In order to make progress here, one must evidently learn how to calculate the inelastic contributions.

This work has led to considerable interest in reactions where a final-state pion-pion interaction is important and perhaps observable. The partial-wave approach has been applied to a variety of processes; the most important for our later purposes are the photoproduction of pions from pions[5,6] and pion–K-meson scattering.[7] The Chew-Mandelstam program has been extended to multichannel situations by Bjorken[8] and Nauenberg.[9] This generalization will be of particular interest to us in a discussion of the impulse approximation.

A related but different approach to the problem of the coupling of processes has been developed by Cutkosky.[10] His approach is a graphical calculus which uses the techniques of analysis of the singularities of a Feynman graph developed by Landau[11] and Bjorken.[12] This type of consideration will undoubtedly be an

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