renormalized coupling constant is explicitly independent of the source size. However, as the two nucleons come closer together, the $\ln(\Lambda/R)$ term approaches zero, and the dependence of the nuclear force on the unrenormalized coupling constant alone becomes increasingly pronounced. This situation is analogous to the case of the quantum electrodynamics in which two electrons see each other through renormalized charges at the large separations, while, as they approach each other, they gradually feel the unrenormalized (bare) charges.

V. CONCLUSION

This work achieves its main objective in that it proves explicitly the conjecture that the dependence of the nuclear force on the unrenormalized coupling constant alone becomes increasingly pronounced as the two nucleons come closer together. For this purpose the Serber-Pais method turns out to be both powerful and consistent. One can expand the Hamiltonian to any desired order in the coupling constant through a series of unitary transformations, and then one may ask an interesting question of whether our conclusion, valid for the most significant terms, which are of order g_r^2, is also valid to all orders in g_r. The answer to this question requires some tedious calculations in a most systematic way. It is our feeling that our conclusion is also valid to higher order terms in g_r.

ACKNOWLEDGMENT

The author should like to thank Professor Robert Serber for many enlightening discussions, which were indispensable for this work.

PHYSICAL REVIEW VOLUME 122, NUMBER 3 MAY 1, 1961

Construction of Unitary Scattering Amplitudes*

R. BLANKENBECLER
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received December 22, 1960)

A general linear technique is discussed which constructs unitary scattering amplitudes without expanding in partial waves and in the presence of inelastic channels. Two- and three-particle intermediate states are discussed explicitly, but the method can be extended directly to any finite number of particles.

A new approximation technique suggested by this formalism is applied to electroproduction of pions from pions and pion–K-meson scattering. A form of the impulse approximation is derived for both the coupled form factor and the coupled scattering amplitude problems. The nucleon and deuteron form factor system is briefly discussed.

Finally, a model field theory which contains three-particle intermediate states is formulated and solved by the linear technique for purely pedagogical reasons.

I. INTRODUCTION

THE conjecture of Mandelstam[1] concerning the analytical structure of the scattering amplitude has led to considerable insight into the interrelation of the various strong interactions. This representation has been verified in perturbation theory by Eden[2] and Polkinghorne.[3] The work of Chew and Mandelstam[4] on the low-energy pion-pion system utilized this analyticity together with unitarity in the form of an expansion in partial waves to develop a set of dynamical equations for the determination of the scattering amplitude. This program has met with difficulties, not the least of which is the fact that the partial-wave expansion cannot converge along the negative cut, and if the series is terminated beyond S waves, false divergences are introduced into the equations. In order to make progress here, one must evidently learn how to calculate the inelastic contributions.

This work has led to considerable interest in reactions where a final-state pion-pion interaction is important and perhaps observable. The partial-wave approach has been applied to a variety of processes; the most important for our later purposes are the photoproduction of pions from pions[5,6] and pion–K-meson scattering.[7] The Chew-Mandelstam program has been extended to multichannel situations by Bjorken[8] and Nauenberg.[9] This generalization will be of particular interest to us in a discussion of the impulse approximation.

A related but different approach to the problem of the coupling of processes has been developed by Cutkosky.[10] His approach is a graphical calculus which uses the techniques of analysis of the singularities of a Feynman graph developed by Landau[11] and Bjorken.[12] This type of consideration will undoubtedly be an

* This work was supported in part by the U. S. Air Force Office of Scientific Research.

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important computational tool. Such techniques could be used with the results of this paper to develop a nonperturbation scheme for the calculation of the scattering amplitude.

Our purpose here is to discuss a generalization of the Chew-Mandelstam approach which does not involve any illegal partial-wave expansion. Even though this extension is quite trivial for the elastic intermediate states, it does seem to have many formal as well as practical advantages. It yields equations which are highly reminiscent of Heitler damping theory, but the knowledge of the analyticity of the scattering amplitude is incorporated in an explicit manner. It is a simple matter to discuss many-channel problems with this method and it leads naturally to a generalization of the impulse approximation. Finally, anomalous thresholds can be discussed, without expanding in partial waves, in a straightforward fashion by analytic continuation in an appropriate variable.[13,14]

The point of view adopted here suggests a type of approximation in which unitarity is satisfied approximately but crossing symmetry is treated exactly. Such a procedure should be adequate except in the case of an elastic scattering resonance. This approach gives, for example, a more satisfactory derivation of the comparison function method,[15] which has already been applied to photoproduction[16] and electroproduction[17] of pions from nucleons. Application of a generalized version of this technique to the problems of pion–K-meson scattering and photoproduction and electroproduction of pions from pions will be made here. A sketch of the problem of the electromagnetic structure of the deuteron and nucleon will be presented in order to illustrate the ease with which a unified treatment of these problems can be made. A new formulation of the impulse approximation is presented which has no off-energy-shell ambiguities. Finally, a model field theory which contains inelastic reactions is exhibited and solved in the two- and three-particle sectors in order to demonstrate the general method.

II. ANALYTICITY AND UNITARITY

The application of the unitarity condition to the four-point scattering amplitude seems to take its simplest form if one chooses as variables the energy and angle. The singularities of the transition amplitude as a function of the energy at fixed angle have been well discussed on the basis of the Mandelstam representation.[18] The essential result that we need is that in addition to the expected physical branch cut whenever the energy is such that a real physical intermediate state can be created, there are other cuts coming from the possible "crossed" intermediate states. These cuts are, in general, in the complex plane, but in the case of equal mass particles they lie on the real axis below the physical threshold. We will refer to these as crossed cuts.

In order to treat the contributions of the inelastic intermediate states in the unitarity condition, it is necessary to make rather definite statements concerning the analyticity domain of the production amplitudes. It will be assumed that as a function of the square of the center-of-mass energy of any pair of particles, for fixed, physical values of the other variables, this function has a physical cut. Other arbitrary singularities may be present. These are, however, disconnected from the physical cut. In the general case, the physical cut must be extended below the normal threshold when the other variables range over their possible physical values. We will assume that the physical amplitude can be found by analytic continuation in the external masses and energies, in the same manner as found in the anomalous threshold problem.[13,14]

Armed with these reasonable assumptions, we can proceed with the construction of a unitary scattering amplitude. In order to develop an understandable notation to be used in more interesting cases, the scattering of scalar particles of equal mass will be considered in detail.

Let us explicitly deal with the two- and three-particle intermediate states only. The following discussion could be extended to any finite number of particles without any new difficulties arising, at least if our assumptions are correct about the singularities of the production amplitudes. The notation is that the incoming (outgoing) particles have four-momentum k_i (k_i'). Intermediate particles will have momentum p_i. Now introduce the scattering matrices:

$$\begin{aligned} M_{22}&=(4\omega_1'\omega_2')^{\frac{1}{2}}\langle k_1'k_2'|\,j_1^\dagger(0)\,|k_2\rangle(2\omega_2)^{\frac{1}{2}},\\ M_{23}&=(2\omega_1')^{\frac{1}{2}}\langle k_1'|\,j_2'(0)\,|k_1k_2k_3\rangle(8\omega_1\omega_2\omega_3)^{\frac{1}{2}},\\ M_{33}&=(8\omega_1'\omega_2'\omega_3')^{\frac{1}{2}}\langle k_1'k_2'k_3'|\,j_1^\dagger(0)\,|k_2k_3\rangle(4\omega_2\omega_3)^{\frac{1}{2}}, \end{aligned} \tag{2.1}$$

and also M_{32}, which is, of course, M_{23} with the initial and final variables interchanged. It should be stressed at this point that the functions defined here contain disconnected graphs. For example, M_{33} contains a term of the form

$$\delta(k_3'-k_3)\langle k_1'k_2'|\,j_1^\dagger\,|k_2\rangle.$$

The nonanalyticity implied by the delta function is of a trivial nature and can be readily dealt with.

We would like to construct a solution of the unitarity relations below the four-particle threshold:

$$\begin{aligned}[][M_{22}^+(t)-M_{22}^-(t)]/2\pi i\\ &=\Sigma M_{22}^-(t)M_{22}^+(t)+\Sigma M_{23}^-(t)M_{32}^+(t),\end{aligned}$$

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[18] M. Cini, S. Fubini, and A. Stanghellini, Phys. Rev. **114**, 1633 (1959).

$$[M_{23}^+(t)-M_{23}^-(t)]/2\pi i$$
$$=\Sigma M_{22}^-(t)M_{23}^+(t)+\Sigma M_{23}^-(t)M_{33}^+(t), \quad (2.2)$$
$$[M_{33}^+(t)-M_{33}^-(t)]/2\pi i$$
$$=\Sigma M_{32}^-(t)M_{23}^+(t)+\Sigma M_{33}^-(t)M_{33}^+(t),$$

where

$$M_{ij}^\pm(t)=M_{ij}(t\pm i\epsilon),$$
$$t=-[\textstyle\sum_i k_i]^2=-[\sum_i k_i']^2=-P^2,$$

and

$$\Sigma=\prod_{i=1}^{N}\left[\int d^4p_i\,\theta(p_i^0)\delta(p_i^2+M^2)/(2\pi)^3\right](2\pi)^3\delta(\sum_1^N p_i-P).$$

With the M^- instruction under the Σ's on the right is an implied complex conjugation of the suppressed intermediate variables. Also, N is the number of particles in the particular intermediate state in question and P is the total available center-of-mass momentum. The integrations involved in Σ are over the suppressed intermediate variables.

Consider the integral equations

$$\Sigma M_{22}(t)\frac{1}{\rho_2}D_{22}(t)+\Sigma M_{23}(t)\frac{1}{\rho_3}D_{32}(t)=N_{22}(t),$$
$$\Sigma M_{22}(t)\frac{1}{\rho_2}D_{23}(t)+\Sigma M_{23}(t)\frac{1}{\rho_3}D_{33}(t)=N_{23}(t),$$
$$\Sigma M_{32}(t)\frac{1}{\rho_2}D_{22}(t)+\Sigma M_{33}(t)\frac{1}{\rho_3}D_{32}(t)=N_{32}(t),$$
$$\Sigma M_{32}(t)\frac{1}{\rho_2}D_{23}(t)+\Sigma M_{33}(t)\frac{1}{\rho_3}D_{33}(t)=N_{33}(t), \quad (2.3)$$

where t is the square of the center-of-mass energy of the initial and hence final particles in the various M_{ij}. The other independent variables have been suppressed. The $N_{ij}(t)$ will be chosen to have crossed cuts in t corresponding to those in $M_{ij}(t)$, and, in addition, the physical cuts arising from intermediate states containing four or more particles. $N_{ij}(t)$ might in some cases have kinematic cuts coming from the three-particle phase-space integral of a trivial nature. These will be discussed in more detail later. In order that N have these properties, D_{ij} is defined as

$$D_{ij}(t)=\delta_{ij}-\int^{16M2} dt'\,\rho_i(t')N_{ij}(t')(t'-t)^{-1}, \quad (2.4)$$

where the lower limit is either $4M^2$ or $9M^2$ depending on whether i is two or three. The delta function, δ_{ij}, if formal and is actually a Dirac delta function in all the suppressed intermediate variables. Then the condition

$$\Sigma M_{ij}(t)\frac{1}{\rho_j}\delta_{jj}=M_{ij}(t) \quad (2.5)$$

determines the ρ_i as functions of the intermediate momenta. The ρ's are, of course, real along the physical cut. This condition insures that the iteration of these equations will yield a normal perturbation series. For example, if the Born approximation is inserted for the N_{ij}, then we require that to lowest order, $M_{ij}=N_{ij}$. This is assured by the condition (2.5).

Explicitly, the two-particle phase-space factor is

$$\Sigma_2(t)=\frac{1}{8(2\pi)^3}\int d\Omega_Q\left(\frac{t-4M^2}{t}\right)^{\frac{1}{2}}\Theta(t-4M^2), \quad (2.6)$$

where Q is the relative coordinate between particles one and two. The three-particle phase-space factor is expressible as

$$\Sigma_3(t)=\frac{1}{(2\pi)^3}\int d^4p_3\,\Theta(p_3^0)\delta(p_3^2+M^2)$$
$$\times\Sigma_2(t+M^2-2p_3^0t^{\frac{1}{2}}). \quad (2.7)$$

If we choose as independent variables the set Ω_Q, Ω_3, P_3^0, then

$$\Sigma_3(t)=\frac{1}{16(2\pi)^6}\int d\Omega_Q d\Omega_3 dp_3^0\,\Theta(p_3^0-M)$$
$$\times\Theta(t-3M^2-2p_3^0t^{\frac{1}{2}})\{[(p_3^0)^2-M^2]$$
$$\times[t-3M^2-2p_3^0t^{\frac{1}{2}}][t+M^2-2p_3^0t^{\frac{1}{2}}]^{-1}\}^{\frac{1}{2}}.$$

Instead of p_3^0, it may prove convenient in some cases to choose the center-of-mass energy of the (1–2) pair as a variable. It should be pointed out that in both the Σ_2 and the Σ_3 terms, there is one trivial azimuthal variable. For reasons of symmetry, it is convenient to keep this redundancy in both integrals.

Using this set of variables, we find

$$\delta_{22}=\delta(\Omega_Q'-\Omega_Q),\quad \delta_{33}=\delta(\Omega_Q'-\Omega_Q)\delta(\Omega_3'-\Omega_3)\delta(p_3^{0\prime}-p_3^0),$$

and

$$\rho_2=\frac{1}{8(2\pi)^3}\left(\frac{t-4M^2}{t}\right)^{\frac{1}{2}}\Theta(t-4M^2), \quad (2.8)$$

$$\rho_3=\frac{1}{16(2\pi)^6}\{[(p_3^0)^2-M^2][t-3M^2-2p_3^0t^{\frac{1}{2}}]$$
$$\times[t+M^2-2p_3^0t^{\frac{1}{2}}]^{-1}\}^{\frac{1}{2}}\Theta(p_3^0-M)$$
$$\times\Theta(t-3M^2-2p_3^0t^{\frac{1}{2}}). \quad (2.9)$$

Therefore, the three-particle contribution to the M_{22} equation can be written explicitly as

$$\Sigma M_{23}\frac{1}{\rho_3}D_{32}(t)=\int d\Omega_Q d\Omega_3\int_M^\infty dp_3^0$$
$$\times M_{23}(t;\Omega_f;\Omega_Q,\Omega_3,p_3^0\pm i\epsilon)\int_{9M^2}^\infty dt'(t'-t)^{-1}$$
$$\times\rho_3(t',p_3^0)N_{32}(t';\Omega_Q,\Omega_3,p_3^0\mp i\epsilon;\Omega_i).$$

The $(P_3{}^0 \pm i\epsilon)$ instruction is to be particularly noted, as well as the fact that the lower limit on the t' integration is actually

$$[p_3{}^0 + (3M^2 + (p_3{}^0)^2)^{\frac{1}{2}}]^2,$$

instead of $9M^2$, due to the theta function present in ρ_3. This particular choice for ρ_3 has the advantage that possible kinematic cuts coming from the three-particle intermediate state do not occur. These would occur, for example, if ρ_3 was chosen to be given by Eq. (2.9) without the theta functions. Then the limits of integration of $P_3{}^0$ are functions of t with square-root type singularities. Below the three-particle threshold, one is also forced into an analytic continuation in these limits.

By taking the discontinuity of (2.3) across the physical cut and by using (2.2), it is seen that any solution of (2.3) will satisfy unitarity. Therefore, this is a generalization of the procedure used by Chew and Mandelstam to discuss the elastic unitarity condition for the partial-wave amplitude.[4] The Castillejo, Dalitz, and Dyson[19] ambiguity due to the zeroes of M_{ij} is present here, as it must be in any such formulation. This approach does clarify the effect of a CDD zero in a coupled situation.

A solvable field theory is discussed in the Appendix to clarify the structure of these equations and their physical content.

Let us now rewrite the linear unitarity relations in a more concise notation which can be readily generalized to multichannel situations. The matrices M, N, D, and the diagonal matrix ρ are introduced[8,9] in an obvious fashion. The unitarity relation then takes the form

$$M^+(t) - M^-(t) = 2\pi i M^-(t) M^+(t), \tag{2.10}$$

where the Σ on the right-hand side of this equation has been suppressed, as it is to be implied by the matrix multiplication. The integral equation for the scattering matrix takes the familiar looking form

$$M\rho^{-1}D = N, \tag{2.11}$$

where

$$D = 1 - \int^{16M^2} dt'(t'-t)^{-1}\rho(t')N(t'). \tag{2.12}$$

Along the crossed cuts and above the four-particle threshold, define the discontinuity function A as

$$M^+(t) - M^-(t) = 2\pi i A(t).$$

Then

$$N(t) = \left[\int_{-\infty} + \int_{16M^2}^{\infty}\right] dt'(t'-t)^{-1}A(t')\rho^{-1}D(t'). \tag{2.13}$$

Let us prove that if A is a symmetric matrix in all its variables, including the suppressed ones, then the solution to (2.11) will satisfy time-reversal invariance, that is, M will also be symmetric. Now, if M is symmetric, then the matrix (T means transpose in all variables)

$$D^T\rho^{-1}M\rho^{-1}D$$

will certainly have the same property. This, in turn, requires that

$$D^T\rho^{-1}N(t) = N^T(t)\rho^{-1}D. \tag{2.14}$$

Since Eq. (2.14) must hold between two analytic functions of t, it is sufficient in this case to prove that the equality is satisfied at the singularities of each. Using the definition of D, Eq. (2.14) is readily seen to be satisfied along the two- and three-particle physical cuts. If A is symmetric, then it is satisfied along the remaining cuts as well. Thus M is unitary and satisfies time reversal invariance if A is chosen symmetric.

The relation between this formulation of the unitarity condition and the partial-wave procedure of Chew and Mandelstam is easily made apparent. If the inelastic contributions to unitarity are neglected, and the first equation of (2.3) is expanded in partial waves, one finds the standard N/D result. However, if one imagines that the functions A_{ij} are given, then (2.3) allows a concise statement of unitarity including the inelastic contributions for all partial waves. Equations (2.3) bear a very close relationship to the Heitler integral equation, which is essentially a linear unitarity requirement.

The Mandelstam representation for M_{22} allows one to express the negative energy part of the function A in terms of the absorptive amplitudes for the crossed reactions. This then allows one to develop a dynamical set of equations since the same type of unitarity equations as (2.11) can be written down for each of the variables describing the energies of the three possible reactions implicit in M_{22}. At present, our knowledge of the structure of the five- and six-point functions does not allow such definite statements to be made, so that we do not yet have a complete dynamical scheme. It is clear that this state of ignorance will not last.[20] However, the possibility of using (2.11) as the basis for approximations on the three-particle contributions is very appealing.

One might entertain the possibility in practical calculations of replacing the inelastic N's by simple functions with a few adjustable parameters in the spirit of an effective range approach. This procedure will be illustrated in a later section, where the nucleon and deuteron form factors are discussed with emphasis upon the effect of the three-pion state.

The form factors corresponding to the external current J are readily obtainable in terms of the scattering solutions. Introduce the row matrix F, where the elements are labeled by the channel to which they refer, i.e.,

$$F_i = \langle 0|J|i\rangle. \tag{2.15}$$

[19] Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. **101**, 453 (1956); referred to hereafter as CDD.

[20] See, for example, the work on the five-point function by L. Cook and J. Tarski, Phys. Rev. Letters **5**, 585 (1960), and by Y. S. Kim (to be published).

Also, introduce the row matrix g, which has constant (or polynomial) elements for the two-particle channels and perhaps crossed cuts in the three-particle channels. As we shall see in an example, in a certain approximation these are determined by an analytic continuation of the Mandelstam representation for the four-point function. Then a solution for F which satisfies unitarity is

$$F\rho^{-1}D(t)=g(t). \tag{2.16}$$

This result will be discussed further in Sec. IV B. It can be interpreted as a type of impulse approximation for the coupled-form-factor problem.

Let us now turn to a brief discussion of the relation of this method to perturbation theory.

III. PERTURBATION THEORY

In order to clarify the physical content of this generalized matrix approach to unitarity, let us attempt to reconstruct the perturbation series for M_{22} from the set of equations (2.3). Much of the following discussion was first given by Mandelstam.[1] It is reproduced here for completeness.

Through sixth order, which means that the four-particle states are neglected, we find

$$M_{22}^{(2)}=N_{22}^{(2)}, \tag{3.1}$$

$$M_{22}^{(4)}=N_{22}^{(4)}-\Sigma M_{22}^{(2)}\frac{1}{\rho_2}D_{22}^{(2)}, \tag{3.2}$$

$$M_{22}^{(6)}=N_{22}^{(6)}-\Sigma M_{22}^{(4)}\frac{1}{\rho_2}D_{22}^{(2)}-\Sigma M_{22}^{(2)}\frac{1}{\rho_2}D_{22}^{(4)} -\Sigma M_{23}^{(3)}\frac{1}{\rho_3}D_{32}^{(3)}, \tag{3.3}$$

$$M_{23}^{(3)}=N_{23}^{(3)}=N_{32}^{(3)}, \tag{3.4}$$

where the superscripts refer to the order in perturbation theory of the function involved.

The difficulty in this procedure is that we must imagine that only $N_{22}^{(2)}$ (or equivalently $A_{22}^{(2)}$) is given as input information, since it corresponds to the Born approximation. In terms of the two appropriate variables, energy and angle, Eq. (3.2) can be written explicitly as

$$M_{22}^{(4)}(s,z_{13})=N_{22}^{(4)}(s,z_{13})-\int d\Omega_2\, N_{22}^{(2)}(s,z_{12}) \times\int_{4M^2} ds'(s'-s)^{-1}\rho_2 N_{22}^{(2)}(s',z_{23}). \tag{3.5}$$

In order to determine $N_{22}^{(4)}$, one uses the fact that a dispersion relation in s for fixed t holds if t is sufficiently small. This will determine $N_{22}^{(4)}$ except for terms with denominators of the form $(t'-t)$ and/or $(u'-u)$. In order to determine these contributions, one must repeat the procedure using the unitarity condition in the t or u reaction. Now imagine that this has been carried out to fourth order. Then in sixth order in the s reaction, one must determine two functions, $N_{22}^{(6)}$ and $N_{23}^{(3)}$, by the same requirement of a fixed t dispersion relation. This procedure is almost identical with that of Mandelstam[1] in his iterative construction of the scattering amplitude.

The new point here is the possibility of using these equations for a nonperturbative construction of the scattering amplitude. There seems to be no obvious difficulty in doing this except for the technical problem of solving these nonsingular integral equations. The scheme presented here would, in principle at least, augment the partial-wave approach with a procedure that does not inherently contain false or unphysical divergences. In practice, it seems very difficult to use this approach as a dynamical tool in most physically interesting problems even if one neglects the inelastic contributions. It is quite useful in relating one process to another, and we will now turn to this particular application.

IV. APPLICATIONS AND APPROXIMATIONS

A

As the first application of this method, let us consider electroproduction of pions from pions. In this discussion we will follow closely the notation of Gourdin and Martin.[5] The squares of the center-of-mass energies in the three reactions described by the Green's function of interest are called s, t, and u. They are connected by the relation

$$s+t+u=3+\lambda^2, \tag{4.1}$$

where the pion mass has been set equal to unity and λ is the mass of the virtual photon. It is convenient to introduce a symmetry point by

$$s_0=t_0=u_0=1+\lambda^2/3. \tag{4.2}$$

The matrix element has the form

$$M=\frac{1}{2i}\epsilon_{\lambda\mu\nu\rho}\frac{\epsilon_\lambda p_{1\mu}p_{2\nu}p_{3\rho}}{[16p_1{}^0p_2{}^0p_3{}^0p_4{}^0]^{\frac{1}{2}}}\epsilon_{\alpha\beta\gamma}F(s,t,u), \tag{4.3}$$

where α, β, and γ are the isotopic labels of the pions and F is a completely symmetric function of its arguments. It is easily seen that the final pions are in a $T=1$ isotopic state and hence their relative angular momentum is odd. The unitarity condition then forces $F(s,t,u)$ to have the phase of the $T=J=1$ state in pion-pion scattering if higher angular momentum states are neglected.

Using the matrix formulation for two channels, and neglecting higher order electromagnetic effects, we have

$$\int d\Omega_2\, F(s,z_{12})D_{\pi\pi}(s,z_{23})=N(s,z_{13}), \tag{4.4}$$

for the reaction described by the energy s. It is consistent with the neglect of rescattering in the higher

angular momentum states to approximate $D_{\pi\pi}$ by

$$D_{\pi\pi}(s,z_{23})=\delta(\Omega_2'-\Omega_3)e^{-\Delta(s)}, \tag{4.5}$$

where

$$\pi\Delta(s)=\int_4^\infty ds'\,\delta(s')[(s'-s-i\epsilon)^{-1}-P(s'-s_0)^{-1}],$$

and $\delta(s)$ is the $J=T=1$ phase shift and P means principal value. Actually, any function of angle and s with the required analyticity can be used in place of the delta function. Then, restoring the original variables, F achieves the form

$$F(s,t,u)=f(s,t,u)e^{\Delta(s)}.$$

Since F must be symmetric in all variables, the simplest function satisfying this crossing condition is

$$F(s,t,u)=g_3\exp[\Delta(s)+\Delta(t)+\Delta(u)]. \tag{4.6}$$

We will assume that g_3 has a weak dependence on λ^2 [see Eq. (4.14)].

A comparison with the results of the partial-wave method for $\lambda^2=0$ can easily be made by expanding about the point $s=4$. The result, using the same phase shift as in reference 5 but with the new Frazer-Fulco parameters, is

$$F=(\text{constant})[1-A(s-4)/4]\exp[\Delta(s)],$$

where $A=0.38$. The numerical evaluation by Gourdin and Martin[5] of their partial-wave equation yields $A=0.31$. The result of Wong's[6] one-pole approximation is $A=0.41$. Thus, the agreement is quite good. If $\lambda^2\neq 0$, one should replace the coefficient of $A/4$ by $(s-4-\lambda^2)$.

The three-dimensional solution (4.6) can be used in the description of electroproduction of pions from nucleons and the process virtual photon $\rightarrow 3\pi$, which arises in the problem of the isotopic scalar nucleon form factor. It would seem quite difficult to use the partial-wave approach in the present stage of approximation to the latter process. In order to apply any solution for F to this three-particle production process, the photon mass λ^2 must be continued analytically above 9. Thus all three variables, s, t, u, approach their cuts from the same half-plane, which insures that the three-particle state will satisfy the final-state phase theorem. This point will be discussed further in part B of this section.

The comparison function method[15] can also be derived in a very simple manner. Consider, for example, a two-channel problem in which the final state interactions are important in only one of the channels. The transition matrix M_{12} satisfies the unitarity relation,

$$\sum_{\rho_2}^{1} M_{12}D_{22}=N_{12}.$$

Now assume that rescattering in only one angular momentum state, l, is important. Then approximate D_{22} by

$$D_{22}=\delta_{22}-[1-\exp(\Delta_l(s))]\Lambda_l,$$

where Λ_l is a projection operator for the lth state. The transition amplitude becomes

$$M_{12}=N_{12}+M_{12}{}^l[1-\exp(-\Delta_l(s))].$$

Applying Λ_l to this equation yields a relation between M_{12} and N_{12} which leads to the result

$$M_{12}=N_{12}+N_{12}{}^l[e^{\Delta(s)}-1]. \tag{4.7}$$

If N_{12} is chosen so as to have the cuts and poles coming from the Born approximation and the crossed rescattering process, one achieves the type of solution given explicitly in reference 15, but with the assurance that no spurious singularities occur.

Let us now consider a slightly different type of example, that of pion-kaon scattering. To reiterate, our philosophy here is to treat crossing symmetry exactly but to take into account approximately, if necessary, the lowest possible partial waves in each of the three reactions. The notation of Lee[7] will be followed, for ease in reading.

The square of the energies for the three possible reactions are again called s, t, and u, where t is the energy squared for the annihilation process $\pi+\pi\rightarrow k+\bar{k}$. Following Lee, we will assume that as long as baryon loops are neglected, there is no splitting of the two isotopic spin states. Thus, the invariant amplitude is assumed to be of the form,

$$M=\delta_{\alpha\beta}A(s,t,u), \tag{4.8}$$

where α and β are the pion isotopic labels and

$$A(s,t,u)=A(u,t,s).$$

The unitarity relation for low energies in the t reaction is

$$\int d\Omega_2 A(t,z_{12})D_{\pi\pi}(t,z_{23})=N(t,z_{13}),$$

where $D_{\pi\pi}$ involves only the $T=0$ states of the pion system. Making the same type of approximation as before, we find that

$$A(s,t,u)=a(s,t,u)e^{\Delta(t)}, \tag{4.9}$$

where the relevant $\pi-\pi$ phase shift is that in the $T=J=0$ state. We are now in a position to satisfy the unitarity condition in the s and u reactions for the $J=0$ partial waves by the approximate solution

$$A(s,t,u)=A_0e^{\Delta(t)}\left\{1-\frac{A_0}{\pi}\int_{(M+1)^2}^\infty\frac{dx\,\rho(x)f(x)}{(x-s_0)}\times\left[\frac{s-s_0}{x-s}+\frac{u-u_0}{x-u}\right]\right\}^{-1}, \tag{4.10}$$

where

$$\begin{aligned} f(x)&=\tfrac{1}{2}\int_{-1}^1 dz\,\exp[\Delta(t(x,z))],\\ t(x,z)&=-2q^2(1-z), \qquad (4.11)\\ q^2&=[x-(M+1)^2][x-(M-1)^2]/4x,\\ s_0&=u_0,\end{aligned}$$

and

$$\rho(x)=q(x)/4\pi x^{\frac{1}{2}}.$$

Unitarity is approximately satisfied since near threshold the angular dependence of the denominator is weak.

The subtraction points are arbitrary, but in order to improve the agreement with unitarity near the threshold for s or u, it is convenient to choose

$$s_0=u_0=(M-1)^2, \quad t_0=4M.$$

B

In order to illustrate the application of the matrix approach, let us consider the form factors of the deuteron and the isoscalar nucleon. This coupled system is assumed to consist of four channels: 1—deuteron pair, 2—nucleon pair, 3—three-pion state, 4—nucleon pair plus pion. If Eq. (2.15) is written out in component form, the result is

$$\Sigma F_1D_{11}/\rho_1+\Sigma F_2D_{21}/\rho_2+\Sigma F_3D_{31}/\rho_3 +\Sigma F_4D_{41}/\rho_4=g_1(t), \quad (4.12)$$

$$\Sigma F_2D_{22}/\rho_2+\Sigma F_3D_{32}/\rho_3=g_2(t), \quad (4.13)$$

$$\Sigma F_3D_{33}/\rho_3=g_3(t), \quad (4.14)$$

$$\Sigma F_2D_{24}/\rho_2+\Sigma F_4D_{44}/\rho_4=g_4(t), \quad (4.15)$$

where the effect of the deuteron and nucleon pair states have been neglected in the less massive channels. It should be noted that the g's are polynomials, except for g_3 and g_4 which may have crossed cuts.

An approximate solution to (4.14) for F_3 was given in Eq. (4.6). It corresponds roughly to using the fact that D_{33} contains disconnected parts which allows the interactions to be between only two pions at a time. The explicit three-pion interaction terms which do not come from iterated two-particle interactions were neglected. This is the physical statement implicit in our final-state theorem.

The form factor F_4 is an analytic continuation of electroproduction of pions on nucleons.[17] It was found in a certain approximation to involve the nucleon form factor $F_2(t)$ as simple factor. This contribution corrects the zero-range wave function character of the term D_{21} for effects due to the finite range of the binding potential, as has been discussed in references 10 and 14.

The anomalous thresholds present in F_1 are in the terms $D_{21}{}^J$ and $D_{41}{}^J$. Their contribution is found by standard continuation methods.[13,14]

In order to calculate $F_1(F_2)$, it is convenient and obviously sufficient to approximate $D_{11}(D_{22})$ by $\delta_{11}(\delta_{22})$. It is now necessary to evaluate D_{31} in the particular angular momentum state J which is projected out by the integrations. In order to evaluate the matrix D^J, one must supply N^J. This, in turn, is best expressed in terms of the matrix A^J [see Eq. (2.13)].

Thus, the problem of calculating F_1, F_2, and F_3 is to choose a suitable symmetric matrix A^J, solve for the matrix D^J and then to evaluate (4.12)–(4.14) with a suitable subtraction or assumption about the behavior of F for large t.

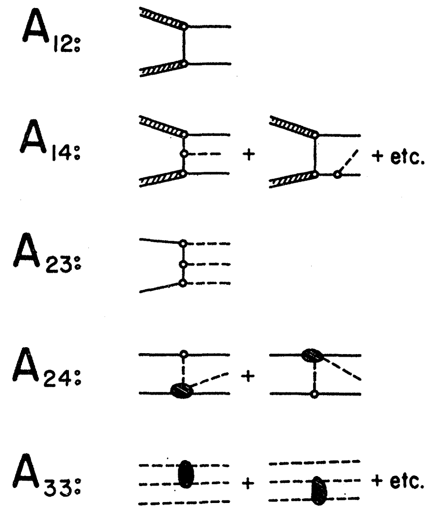

FIG. 1. Graphs contributing to the nucleon, deuteron, and three-pion form factors.

A sketch will now be made of how one might carry out an approximate but ambitious evaluation of this system of equations. We will assume that F_3 and F_4 are known. This implies a knowledge of D_{33}, D_{24}, and D_{44}. The choice of the matrix A^J is essentially a decision as to what types of graphs are to be considered in the calculation. The graphs that will be considered here are shown in Fig. 1. Thus, A^J is chosen to be of the form,

$$A^J=\begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ A_{12} & 0 & A_{23} & A_{24} \\ 0 & A_{23} & A_{33} & 0 \\ A_{14} & A_{24} & 0 & 0 \end{bmatrix}.$$

The superscript J will be omitted from now on. The explicit functional form of A is to be found by examining perturbation theory or the Mandelstam representation for the graphs considered.

The N's are therefore

$$N_{41}=\int\frac{dt'}{(t'-t)}[A_{41}D_{11}+A_{42}D_{21}],$$

$$N_{31}=\int\frac{dt'}{(t'-t)}[A_{32}D_{21}+A_{33}D_{31}],$$

$$N_{21}=\int\frac{dt'}{(t'-t)}[A_{21}D_{11}+A_{23}D_{31}+A_{24}D_{41}],$$

$$N_{32}=\int\frac{dt'}{(t'-t)}[A_{32}D_{22}+A_{33}D_{32}],$$

where the summation over intermediate variables in the three-particle states has been suppressed. When these relations are inserted in the definition of the D's,

coupled integral equations result which are difficult to solve analytically.

If the solution to Eq. (4.14) is used as a guide, approximate solutions for the D's involving the three-pion state are readily constructed:

$$D_{31}(t)=-\exp[\Delta(s_1)+\Delta(t_1)+\Delta(u_1)]\times\int_9^\infty dt'(t'-t)^{-1}\rho_3\int_{-\infty} dx(x-t')^{-1}A_{32}D_{21}(x),$$

$$D_{32}(t)=-\exp[\Delta(s_1)+\Delta(t_1)+\Delta(u_1)]\times\int_9^\infty dt'(t'-t)^{-1}\rho_3 N_{32}{}^0(t'),$$

where

$$N_{32}{}^0(t)=\int dx(x-t)^{-1}A_{32}(x),$$

and the variables (s_1,t_1,u_1) are the relative energies (squared) between the three pions. They satisfy the relation $s_1+t_1+u_1=3\mu^2+t$.

In this approximation, for example, the nucleon form factor becomes

$$F_2(t)=g_2(t)+\Sigma g_3\rho_3{}^{-1}|\exp[\Delta(s_1)+\Delta(t_1)+\Delta(u_1)]|^2\times\int dx(x-t)^{-1}\rho_3(x)N_{32}{}^0(x).$$

A similar but more complicated result holds for $F_1(t)$. The customary choice for g_1 is, of course, zero. One interesting point is that if $A_{32}(t)$ is strongly peaked about some point $t=t_0$, then to a good approximation,

$$D_{31}(t)=D_{32}(t)D_{21}(t_0).$$

If this is put into Eq. (4.12), then explicit reference to F_3 can be eliminated in favor of F_2 by using Eq. (4.13).

The impulse approximation for scattering processes also arises naturally in the matrix formalism. Consider nucleon-deuteron elastic scattering, and neglect the pionic effects. Setting $D_{11}=\delta_{11}$, an excellent approximation, yields

$$M_{21}(t)=N_{21}(t)-\Sigma M_{22}(t)\rho_2{}^{-1}D_{21}(t). \qquad (4.16)$$

Now this must be continued to the region where t is space-like and s describes the energy of the scattering. The usual formulation of the impulse approximation follows by assuming that M_{22} is a slowly varying function of the energy s and the crossed energy u. If this is the case, then the factor M_{22} can be taken outside the integration, and one achieves the form

$$M_{21}(s,t,u)=N_{21}(s,t,u)+M_{22}(s_1,t,u_1)F_D(t), \qquad (4.17)$$

where

$$F_D(t)=-\Sigma\rho_2{}^{-1}D_{21}(t),$$

and s_1 is an appropriately chosen energy variable with the restriction that

$$s_1+u_1+2M_D{}^2=s+u+2m^2.$$

F_D is recognized as the form factor of the deuteron. The term N_{21} is conventionally dropped, but there is no compulsion to do so here. This term is necessary in order to satisfy unitarity in the s reaction. In addition, N_{21} also contains the exchange graphs which involve both intermediate nucleons in an inseparable manner. Notice that no ambiguities associated with off-energy-shell effects arise in this approximation. These effects, of course, show up in higher mass states, which correspond to the shorter range forces.

V. DISCUSSION

The generalized matrix formulation of unitarity described here allows the coupled nonlinear unitarity conditions to be replaced by a set of coupled linear equations. These equations involve the discontinuity of the relevant scattering amplitudes across their crossed cuts and above the highest mass intermediate state considered explicitly. Thus in this sense it is similar to the program of Symanzik[21] and Zimmerman.[22] The main point which must be understood before these relations form a complete dynamics concerns the location of the crossed singularities of the multiparticle amplitudes and the expression of the resultant discontinuities in terms of physical processes. In spite of these shortcomings, the use of these methods in an approximate treatment of the inelastic contributions to scattering amplitudes is very appealing. A model field theory involving inelastic reactions was formulated and solved in the Appendix as an illustration of our method.

The possibility of breaking away from a partial-wave expression of unitarity suggests a new approximation scheme which was applied in this paper to electroproduction of pions from pions and pion–K-meson scattering. The solutions thus obtained seem to have an interesting structure and the agreement near threshold with the partial-wave treatment is quite good.

Finally, an impulse approximation for the coupled form factor and scattering amplitude problem was formulated and discussed by means of an example, the nucleon and deuteron system.

It is clear that the generalized matrix approach does not make the three-particle contributions trivial to evaluate. It does seem, however, to build into a calculation the purely geometrical restrictions of unitarity. It therefore serves as a convenient base for approximate calculations.

ACKNOWLEDGMENT

I wish to thank Professor M. L. Goldberger for many interesting discussions.

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APPENDIX

In order to illustrate the utility of the matrix method and to clarify its physical content, let us attempt to solve a model field theory in which no more than three particles are allowed in any intermediate state. One such example is supplied by the Lee model in the $(V-\Theta)$ sector. The solution for $(V-\Theta)$ scattering has recently been given by Amado.[23] Rather than solve this problem, which can be done with the matrix approach, it is perhaps equally instructive to solve a model field theory in the sense of Zachariasen.[24] In this type of model, all particles are treated relativistically but crossing symmetry is completely destroyed.

The model theory will consist of three types of particles with masses M, M', and μ. Since the particles will be scalar, we are free to introduce the following four types of point interactions:

Interaction	Coupling constant
$2M' \rightleftharpoons 2M'$	a
$2M \rightleftharpoons 2M$	b
$2M \rightleftharpoons 2M'+\mu$	c
$2M'+\mu \rightleftharpoons 2M'+\mu$	d

The renormalized coupling constants will be defined in terms of the appropriate scattering amplitude at zero energy.

The notation to be followed is that the momenta of the M' particles are to be called k_i, that of the M particle p_i, and the μ momenta q_i. The scattering amplitudes on the energy shell are introduced as:

$$M_{11}=\langle k_1'|\, j_k\,| k_1k_2{}^+\rangle(\cdots)^{\frac{1}{2}},$$
$$M_{22}=\langle p_1'|\, j_p\,| p_1p_2{}^+\rangle(\cdots)^{\frac{1}{2}},$$
$$M_{23}=\langle p_1'|\, j_p\,| k_1k_2q^+\rangle(\cdots)^{\frac{1}{2}},$$
$$M_{23}=\langle {}^-p_1'p_2'|\, j_q{}^+| k_1k_2{}^+\rangle(\cdots)^{\frac{1}{2}},$$
$$M_{33}=\langle {}^-k_1'k_2'q'|\, j_q{}^+| k_1k_2{}^+\rangle(\cdots)^{\frac{1}{2}}.$$

Now the N's must be chosen in such a way as to be consistent with unitarity. If the k_1 particle in the second form for M_{23} is contracted, the unitarity relation leads immediately to the conclusion that $M_{23}(s,w^2)$, as a function of the variable $w^2=-(k_1+k_2)^2$, has the phase discontinuity of $(M'+M')$ scattering. Thus w^2 is a convenient variable to hold fixed in the discussion of unitarity in s. A similar statement can be made about the w^2 and w'^2 dependence of $M_{33}(s,w'^2,w^2)$.

We choose the N's and thus define the model as follows:

$$N_{11}=a,$$
$$N_{22}=b,$$
$$N_{23}=N_{32}(s,w^2)=c/D_{11}(w^2),$$
$$N_{33}=d/D_{11}(w'^2)D_{11}(w^2).$$

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The definition of the D's lead to

$$D_{11}(s)=1-saI_2(s,M'),$$
$$D_{22}(s)=1-sbI_2(s,M),$$
$$D_{23}(s,w)=-sN_{23}(w)I_2(s,M),$$
$$D_{32}(s,w)=-sN_{32}(w)I_3(s,w),$$
$$D_{33}(s)=\delta(w'-w)-sN_{33}(w',w)I_3(s,w'),$$

where

$$I_2(s,M)=\int_{4M^2}^{\infty} ds'\rho_2(s',M)[s'(s'-s)]^{-1},$$

$$I_3(s,w)=\int_{(2M'+\mu)^2}^{\infty} ds'\rho_3(s',w)[s'(s'-s)]^{-1}.$$

The ρ_i here differ by trivial factors of 4π from the ones introduced in the text. These factors come from the angular integrations.

The function ρ_3 is the three-particle phase-space factor, including the theta functions, which can be found from Eq. (2.9) by using (in the center-of-mass system of s) the relation

$$w^2=s+\mu^2-2q^0s^{\frac{1}{2}}.$$

If the matrix system for the M's are written out, the integral equations in the variable w are directly solvable because the kernels are separable and only s waves interact. This is, of course, what the model was designed to do. The solutions are

$$M_{33}(s;w',w)=[N_{33}(w',w)-M_{32}(s,w')D_{23}(s,w)]D_3^{-1}(s),$$
$$M_{23}(s;w)=[N_{23}(w)-M_{22}(s)D_{23}(s,w)]D_3^{-1}(s),$$
$$M_{22}(s)=N_{22}\left\{D_3(s)\left[\left(1-\frac{c^2}{db}\right)D_3(s)+\frac{c^2}{db}\right]^{-1} -1+D_{22}(s)\right\}^{-1},$$
$$M_{11}(s)=N_{11}/D_{11}(s),$$

where

$$D_3(s)=1-sdI_4(s),$$

$$I_4(s)=\int_{(2M'+\mu)^2}^{\infty}[s'(s'-s)]^{-1} \times\int dw\left(\frac{dq^0}{dw}\right)\rho_3(s',w)|D_{11}(w)|^{-2}.$$

The finite range of the w integration arising from the theta functions in ρ_3 is from $2M'$ to $(s'^{\frac{1}{2}}-\mu)$. The interested reader can easily check that these functions satisfy unitarity by taking the s discontinuity of $M_{33}D_3$, for example, assuming that M_{32} satisfies unitarity, and then rearranging to yield the discontinuity of M_{33}.

The structure of these solutions may well hold true in a more realistic field theory. Their form is certainly

correct if the crossed cuts in the N's were given functions. It is interesting to note that, depending on the coupling constants c and d, the inelastic contribution to M_{22} can have a very large effect on the position of a resonance, even if it has an energy considerably below the inelastic threshold. Another amusing point is the possibility of a bound state in the $(M'-M')$ system of mass M_0 which should yield a two-particle cut in M_{22} starting at $(M_0+\mu)^2$. Continuing analytically in the coupling constant a, one sees that this cut arises from the $I_4(s)$ terms in $M_{22}(s)$. The procedure for performing this continuation has been described elsewhere.[25] The essential point is that since D_{11} develops a zero at the bound-state mass, the s' integral in I_4 must be deformed to avoid the resulting singularity of the integrand. This procedure then yields an "anomalous threshold" beginning at the point $(M_0+\mu)^2$. By picking out the residue at M_0 in the variable w, one can calculate the processes $(M_0+\mu\rightarrow M_0+\mu)$ and $(M+M\rightarrow \mu+M_0)$ from M_{33} and M_{23}, respectively.

One final observation concerns the possibility of a three-particle bound state. This occurs when the denominator in M_{22} develops a zero below the point $4M^2$. The bound-state pole is also seen to be present in M_{23} and M_{33} due to the general structure of the solutions.

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PHYSICAL REVIEW VOLUME 122, NUMBER 3 MAY 1, 1961

Self-Consistent Field Theory of Nuclear Shapes*

MICHEL BARANGER
Carnegie Institute of Technology, Pittsburgh, Pennsylvania

(Received December 27, 1960)

The Hartree-Fock equations are generalized to include pairing effects on the same footing with field-producing effects. In addition to the Hartree potential, there enters a pairing potential. When applied to a spherically symmetric shell-model Hamiltonian, these equations may possess deformed solutions. Application is made to pairing plus quadrupole forces, with results identical to those of Belyaev and Kisslinger and Sorensen. The spherical shape becomes unstable when some collective vibration of the spherical nucleus reaches zero frequency.

1. INTRODUCTION

THE question to be considered here is the calculation of nuclear shapes starting from a spherically symmetric Hamiltonian. In a Hartree-Fock type of theory, all nuclei are deformed except at closed shells. It is well known that residual interactions tend to keep nuclei spherical for a while as one goes away from closed shells. Since a prominent effect of residual interactions is to produce pairing correlations, one might think that the inclusion of pairing[1] effects together with the Hartree-Fock field would be able to give realistic estimates of shapes.

This problem has been treated by Belyaev,[2] who gave it an approximate solution for the case where the single-particle levels are degenerate and the two-body force is a combination of pairing force and quadrupole force. In this paper, we would like to consider the more practical case of nondegenerate single-particle levels and also to develop a formalism which can be used with any two-body interaction. The inclusion of a general two-body interaction was also considered by Belyaev at the beginning of his paper, but his solution involves three successive canonical transformations, which makes it of limited practical value. In the present work, a single transformation is made, the generalized Bogolyubov transformation.[3] It differs from the more familiar Bogolyubov-Valatin transformation[4] in the following way. In the latter, a specific assumption is made about the bound state (the Cooper pair) into which pairs of particles are allowed to condense; for instance, in the application to spherical nuclei,[5,6] the two particles are assumed to have opposite angular momenta. On the other hand, in the generalized transformation, the bound-state wave function is left completely arbitrary and is determined by minimizing the

* This work was supported by the Office of Naval Research.

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