Coordinate Invariance and Energy Expressions in General Relativity

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The invariance of various definitions proposed for the energy and momentum of the gravitational field is examined. We use the boundary conditions that $g_{\mu\nu}$ approaches the Lorentz metric as $1/r$, but allow $g_{\mu\nu,\alpha}$ to vanish as slowly as $1/r$. This permits "coordinate waves." It is found that none of the expressions giving the energy as a two-dimensional surface integral are invariant within this class of frames. In a frame containing coordinate waves they are ambiguous, since their value depends on the location of the surface at infinity (unlike the case where $g_{\mu\nu,\alpha}$ vanishes faster than $1/r$). If one introduces the prescription of space-time averaging of the integrals, one finds that the definitions of Landau-Lifshitz and Papapetrou-Gupta yield (equal) coordinate-invariant results. However, the definitions of Einstein, Møller, and Dirac become unambiguous, but not invariant.

The averaged Landau-Lifshitz and Papapetrou-Gupta expressions are then shown to give the correct physical energy-momentum as determined by the canonical formulations Hamiltonian involving only the two degrees of freedom of the field. It is shown that this latter definition yields that inertial energy for a gravitational system which would be measured by a nongravitational apparatus interacting with it. The canonical formalism's definition also agrees with measurements of gravitational mass by Newtonian means at spacial infinity. It is further shown that the energy-momentum may be invariantly calculated from the asymptotic form of the metric field at a fixed time.

I. INTRODUCTION[1]

A GREAT number of different forms for the stress-tensor of the gravitational field have been given since Einstein's original pseudotensor.[2] While it may be argued whether the energy-momentum density is meaningful locally, one would at least expect that the integrated value, i.e., the total energy-momentum P^μ of a system, would have an unambiguous significance. The total P^μ of an isolated system should be a physically measurable quantity; for such a system, space is asymptotically flat, and one can then measure P^μ by conventional methods entirely at spatial infinity. On the other hand, one does not have any clear-cut definition for P^μ in situations where one cannot introduce asymptotically rectangular coordinate frames. Thus, we shall require $g_{\mu\nu}$ to approach the Lorentz metric $\eta_{\mu\nu}$ at spatial infinity. More precisely, we shall require here that $g_{\mu\nu}-\eta_{\mu\nu}\sim 1/r$, since in that case we shall see that the energy is finite. [Note that we also forbid a behavior of the type $g_{\mu\nu}-\eta_{\mu\nu}\sim t/r$ since $g_{\mu\nu}$ would not approach $\eta_{\mu\nu}$ at spatial infinity in every Lorentz frame. Similarly, t/r^2 is considered to go as $1/r$.] Further, one desires that P^μ be invariant under *all* coordinate transformations which preserve these boundary conditions and do not involve Lorentz transformations at infinity. Of course, under rigid Lorentz rotations of the asymptotic frame, P^μ should transform as a four-vector. Our boundary conditions have not stated that $g_{\mu\nu,\alpha}$ should behave as $1/r^2$ at infinity; in fact, it is most natural simply to require that $g_{\mu\nu,\alpha}$ also vanish as $1/r$. For example, we thus allow the metric to decrease as $(e^{iqx})/r$.

The asymptotic domain, as employed here, is defined by letting r approach infinity for fixed time, and is a region in which there is a negligible flux of gravitational radiation. By contrast, in a previous paper, IVb, a wave zone was defined in which asymptotic relations were also studied. There, however, since the flux of radiation was the object of interest, one had to verify that asymptotic expressions were good approximations for values of r where radiation was significant. To achieve desired accuracy, it was often required to move out along the light cone, which meant increasing t as well as r. In this paper we are dealing with the region beyond the wave front, this boundary representing the largest distance at which the flux is appreciable. Such a boundary must, of course, exist in order that the energy of the system be finite.

Let us now examine the various proposed expressions[2–8] for P^μ within the above framework. A common characteristic of all[9] these is that they can be cast into the

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[1] Previous papers by the authors will be referred to in text by Roman numerals: R. Arnowitt *et al.*, Phys. Rev. **117**, 1595 (1960) (III); **118**, 1100 (1960) (IV); Nuovo cimento (to be published) (IVa); Phys. Rev. **121**, 1556 (1961) (IVb). This paper is IVc.

[2] See reference (6) for the surface integral form of the Einstein expression.

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[8] The canonical formalism's expression derived in IV.

[9] We do not treat in this paper energy expressions which do not obey the differential conservation law $T^{\mu\nu}{}_{,\mu}=0$. An expression with covariant differential conservation has been given by L. Bel, Compt. rend. **248**, 1297 (1959).

form of a two-dimensional surface integral at spatial infinity.[10]

$$P^{\mu}=\oint dS_i\, H^{\mu i}(g_{\alpha\beta},g_{\nu\sigma,\tau}). \tag{1.1}$$

Physically, this property expresses the fact that the total energy can be measured by gravitational means, e.g., from the Newtonian potential at infinity. This surface integral form allows one to discuss invariance questions using only the asymptotic forms of the transformations. Similarly, $H^{\mu i}$ may be expanded in terms of the asymptotic values of $g_{\mu\nu}$ and $g_{\mu\nu,\alpha}$. The leading term is, in all cases, linear in $g_{\mu\nu,\alpha}$. Clearly, this term must go as $1/r^2$ in order that P^{μ} be at all defined. In Sec. II, we shall show that, in spite of the fact that $g_{\mu\nu,\alpha}\sim 1/r$, the linear term in most of the proposed expressions does behave as $1/r^2$ (the $1/r$ parts having cancelled out). This will be seen to occur as a consequence of the field equations. However, one must clearly also examine the quadratic terms in $H^{\mu i}$ since they too contribute $1/r^2$ terms in the surface integral. These quadratic terms are important in investigating the invariance of P^{μ}. We shall find that *none* of the proposed surface integral expressions are then invariant. However, it will be seen that a prescription can be given for *some* of these expressions which does make them yield an invariant value of P^{μ} in any allowed frame. This prescription will involve averaging over spatial (and in most cases also time) regions at infinity so that the effects of "coordinate waves" (which we will see $g_{\mu\nu}-\eta_{\mu\nu}\sim e^{iqr}/r$ introduces) are eliminated and the integrals become well-defined. Section 3 is devoted to showing that the P^{μ} thus defined is indeed *the* correct energy-momentum vector of the system. We shall start from the basic definition of the energy of any physical system, which is provided by the numerical value of the generator of time translations for a given field configuration. For such a generator to represent the energy of a system, it must, of course, be given in a "Heisenberg representation." This last term is used in the same sense as in quantum mechanics to indicate that the field variables carry all the time development of the system, in contrast to the Hamilton-Jacobi ("Schrödinger") representation. (Thus, in the latter, where $H=0$, one does not associate the Hamiltonian with the energy.) A Heisenberg representation is normally defined in terms of the variables appearing in the original Lagrangian, i.e., those which have primary measurable significance (for example, in terms of rods and clocks). In general relativity, these variables are the components $g_{\mu\nu}$ of the metric (and not some canonically transformed variables). However, since $g_{\mu\nu}$ involves information about the coordinate frame (see III) as well as the dynamical variables, the idea of a Heisenberg representation in general relativity is more complicated (and, in fact, allows certain types of coordinate waves to be present). In IVa, such questions were discussed in detail. It was shown there that in all Heisenberg representations, the numerical values of the respective Hamiltonians (which then represent the energy) agree for a given physical situation. From this basic definition of the energy and momentum, we will see here that an invariant expression for energy and momentum can be obtained directly from the asymptotic form of the metric. It is then shown that this P^{μ} is indeed the same as that given by the surface integral expressions which are invariant when the averaging prescription is imposed, justifying the validity of the latter.

II. SURFACE INTEGRALS FOR P^{μ}

As mentioned in Sec. I, all expressions for P^{μ} can be written in the surface integral form, Eq. (1.1), where the integrand $H^{\mu i}$ must be considered up to quadratic terms in $g_{\mu\nu}-\eta_{\mu\nu}$. We begin, therefore, by examining the coordinate transformation properties of the linear terms, restricting ourselves to P^0 for simplicity. All but Møller's definition yield, to this order,

$$P^0=\oint dS_i(g_{ij,j}-g_{jj,i}), \tag{2.1}$$

while Møller's expression is

$$P^0=\oint dS_i(g_{00,i}-g_{0i,0}). \tag{2.2}$$

Under the coordinate transformation $x'^{\mu}=x^{\mu}+\xi^{\mu}(x)$, the metric, to first order, changes to $g_{\mu\nu}'=g_{\mu\nu}-\xi_{\mu,\nu}-\xi_{\nu,\mu}$ (where $\xi_{\mu}=\eta_{\mu\nu}\xi^{\nu}$). Thus, the integral (2.2) remains unchanged for transformations involving only x^i, which was Møller's purpose in constructing it. However, for transformations involving the time as well, expression (2.2) can be altered by arbitrary amounts even if the $g_{\mu\nu,\alpha}$ in the two frames are asymptotically $O(1/r^2)$. The change in Eq. (2.1) to first order is

$$\oint dS_i(-\xi_{i,j}+\xi_{j,i})_{,j}=\int d^3r(-\xi_{i,j}+\xi_{j,i})_{,ij}\equiv 0, \tag{2.3}$$

so that for any ξ^{μ}, Eq. (2.1) is unaltered. To first order, it is not necessary to transform the ∂_{μ} in Eqs. (2.1), (2.2), since they contribute quadratic terms only. However, to consider correctly the transformation properties of even the linear formula (2.1), it is necessary to include quadratic terms in the transformation law. The rigorous transformation is given by

$$g_{\mu\nu}=g_{\mu\nu}'+h_{\mu\alpha}'\xi^{\alpha}{}_{,\nu}+h_{\alpha\nu}'\xi^{\alpha}{}_{,\mu}+\xi^{\alpha}{}_{,\mu}\xi_{\alpha,\nu}+\xi_{\mu,\nu}+\xi_{\nu,\mu}, \tag{2.4}$$

[10] Greek indexes run from 0 to 3, Latin from 1 to 3; ordinary differentiation is denoted by a comma or the symbol ∂_{μ}. We use units such that $16\pi\gamma c^{-4}=1=c$, where γ is the Newtonian constant. Notation differs in one respect from that of previous papers: If it is necessary to avoid ambiguity between three- and four-dimensional quantities, metric quantities will be marked as in ${}^3g^{ij}$ (which means the reciprocal of the three-dimensional metric g_{ij}) rather than marking all four-dimensional quantities as in other papers.

where $h_{\mu\nu}'=g_{\mu\nu}'-\eta_{\mu\nu}$. The boundary conditions on $g_{\mu\nu}$ and $g_{\mu\nu}'$ require that $\xi^{\mu}{}_{,\nu}=O(1/r)$ asymptotically.

Before discussing the effects of quadratic terms in P^0, we will show that the surface integrals for P^0 are finite even in frames with $g_{\mu\nu,\alpha}\sim 1/r$. Clearly the quadratic contributions to $H^{\mu i}$ (which go as $1/r^2$) give finite surface integrals. It is necessary, therefore, to consider only the linear terms given by Eq. (2.1). The right-hand side of Eq. (2.1) is invariant to first order, and the quadratic terms in the coordinate transformations give at most $1/r^2$ contributions (and thus again finite effects in the surface integral). Hence, one need only prove the finiteness of the right-hand side of Eq. (2.1) in one frame to establish the theorem. We use as the reference frame with coordinates $x^{\mu\prime}$, one whose properties have been studied in previous papers. It is defined by the coordinate conditions[11]

$$g_{ij,j}'=0=\pi^{ii}{}_{,jj}'-\pi^{ij}{}_{,ij}', \tag{2.5a}$$

where

$$\pi^{ij}\equiv(-{}^4g)^{\frac{1}{2}}[{}^4\Gamma^0{}_{lm}-g_{lm}\,{}^4\Gamma^0{}_{pq}\,{}^3g^{pq}]{}^3g^{il}\,{}^3g^{jm} \tag{2.5b}$$

is essentially $g_{ij,0}$. In terms of the orthogonal decomposition (see, for example, III)

$$h_{ij}=h_{ij}{}^{TT}+h_{ij}{}^{T}+(h_{i,j}+h_{j,i}), \tag{2.6}$$

and $h_{ij}{}^T=\frac{1}{2}[\delta_{ij}h^T-\nabla^{-2}h^T{}_{,ij}]$, the coordinate conditions (2.5) read $h_i'=0=\pi^{T\prime}$. The constraint equations $G^0{}_\mu\equiv R^0{}_\mu-\frac{1}{2}\delta^0{}_\mu R=0$ determine $h^{T\prime}$ according to

$$-\nabla^2 h^{T\prime}=\tfrac{1}{4}[h'{}_{ij}{}^{TT}{}_{,k})^2+(\pi^{ijTT\prime})^2]+\cdots, \tag{2.7}$$

where cubic and higher terms are indicated by $\cdots$. Inverting ∇^2 in this equation, it is clear that the component $h^{T\prime}$ of the metric cannot fulfill the $1/r$ boundary conditions unless $h_{ij}{}^{TT\prime}$ and $\pi^{ijTT\prime}$ fall off faster than $1/r^{\frac{3}{2}}$. Then, Eq. (2.7) shows that $h^{T\prime}$ is static[12] and $\sim 1/r$, and that $h^T{}_{,i}'\sim 1/r^2$ but not necessarily static (for detailed proof, see Appendix A of IVb). If we now insert Eq. (2.6) into Eq. (2.1), we find that the right-hand side reduces to $-\oint dS_i' h^T{}_{,i}'$ which is finite. Note that for the class of frames in which $g_{\mu\nu,\alpha}$ goes faster than $1/r$ (as has been assumed in most previous discussions) the quadratic parts of $H^{\mu i}$ will not contribute to P^{μ}; because one factor in each such term is differentiated, all these parts of $H^{\mu i}$ go faster than $1/r^2$. Further, all the definitions of P^0 reducing to Eq. (2.1) are coordinate-independent *within* this class of frames. For the $\xi^{\mu}{}_{,\nu}$ which enter in these transformations go faster than $1/r$, so that quadratic terms in the transformation of Eq. (2.1) are also negligible, while the linear invariance has already been demonstrated.

In treating coordinate transformations through quadratic order, it is necessary to consider not only the quadratic terms in the transformation law (2.4) and in the various expressions $H^{\mu i}$, but also transport terms which arise in taking the integrals of $H^{\mu i}$ over different surfaces in different frames. In order to consider these systematically, it is helpful to first outline the calculation for a general $H^{\mu i}$. We wish to compare the expression for P^{μ} in an arbitrary frame x^{μ}, with that in the reference frame (2.5), denoted by $x^{\mu\prime}=x^{\mu}+\xi^{\mu}(x)$. In the general frame, P^{μ} is given by

$$P^{\mu}=\oint dS_i\, H^{\mu i}, \tag{2.8a}$$

where the surface is taken over the sphere of radius $r=R$ at time $t=0$. This surface may be specified by the equation $x^{\mu}=f^{\mu}(\theta,\varphi)$. In the reference frame, one has

$$P^{\mu\prime}=\oint dS_i'\, H^{\mu i\prime}, \tag{2.8b}$$

where the integration is over the surface $x^{\mu\prime}=f^{\mu}(\theta,\varphi)$ which is again a sphere of radius $r'=R$ at time $t'=0$. When the equations of the respective surfaces are substituted into the integrals (2.8), one sees that $dS_i(\theta,\varphi)=dS_i'(\theta,\varphi)$. The integrands can be related as follows:

$$H^{\mu i}(x=f)=H^{\mu i\prime}(x'=f+\xi)+(\Delta H)^{\mu i}, \tag{2.9a}$$

where $(\Delta H)^{\mu i}$ stands for all tensor transformation terms through quadratic order. We shall view it as a function of $g_{\mu\nu}'$ and ξ^{μ}. Note that all functions in Eq. (2.9a) are evaluated at the same point, i.e., $x^{\mu}=f^{\mu}$ which is also $x^{\mu\prime}=f^{\mu}+\xi^{\mu}$. Since, in Eq. (2.8b), $H^{\mu i\prime}(x'=f)$ appears, we expand to obtain[13]

$$H^{\mu i}(x=f)=H^{\mu i\prime}(x'=f)+\xi^{\alpha}H^{\mu i}{}_{,\alpha}'(x'=f)+(\Delta H)^{\mu i}. \tag{2.9b}$$

Inserting Eq. (2.9b) into (2.8a), one finds

$$P^{\mu}=P^{\mu\prime}+\oint dS_i\, H^{\mu i}{}_{,\alpha}'\xi^{\alpha}+\oint dS_i(\Delta H)^{\mu i}. \tag{2.10}$$

Thus one has an invariant P^{μ} only if the last two terms in Eq. (2.10) vanish.

In computing the terms in Eq. (2.10), we will find it convenient to distinguish two classes of functions, "static" and "oscillatory." These terms denote, respectively, functions whose derivatives go faster, on the same order, respectively, as the function itself, e.g., t/r or e^{iqx}/r. In particular, the requirement $\xi^{\mu}{}_{,\nu}\sim 1/r$ is consistent either with a static ξ^{μ} such as[14] $\xi^{\mu}\sim x^{\mu}/r$ or

[11] As was shown in III, this frame may be constructed at least in an iteration expansion.

[12] That the coefficient of $1/r$ in $h^{T\prime}$ is independent of time follows from the fact that it represents the total Hamiltonian which is a constant of the motion (see III). An alternate derivation of this is given here at the end of Appendix C.

[13] More precisely, for "static" ξ^{μ} (defined in the next paragraph), which do not vanish at infinity, one should keep all terms in the Taylor expansion, since the transport is now a finite distance. However, the higher terms can be treated precisely like the first order term retained in Eq. (2.9b). Alternately, one can treat directly the finite quantity $H^{\mu i\prime}(f+\xi)-H^{\mu i\prime}(f)$.

[14] For example, the type $\xi^i\sim x^i/r$ occurs in transforming between isotropic and Schwarzschild coordinates.

$\ln r$, or with oscillatory ξ^μ like e^{iqx}/r which is the prototype of a coordinate wave. Upon transforming away from the reference frame (2.5), where $h_{i,j}'=0=\pi^{T\prime}$, one has

$$h_{i,j}\sim\xi_{i,j},\quad \pi^T\sim-2\xi^0{}_{,ll},$$

which shows how coordinate waves may make a $1/r$ contribution in $g_{\mu\nu,\alpha}$.

As the first example, we consider the Einstein pseudotensor,[2] where the full H^{0i} is

$$(H^{0i})_E=(-g)^{-\frac{1}{2}}g_{0\lambda}[g(g^{\lambda 0}g^{i\sigma}-g^{\lambda i}g^{0\sigma})]_{,\sigma}. \qquad (2.11)$$

We start with the $(\Delta H)^{\mu i}$ term. Our procedure will be to begin in the reference frame (2.5) and transform out of it. The quadratic terms arising upon transforming g_{ij} in the linear part (2.1) from the frame (2.5) to any other frame have the form of a divergence, e.g., $(\xi^\alpha{}_{,i}\xi_{\alpha,j})_{,j}$. For static ξ^μ, such terms are clearly $1/r^3$. For oscillatory ξ^μ, one would obtain $(e^{iqx}/r^2)_{,j}$ which is $\sim 1/r^2$ for $q\neq 0$. Such a coordinate wave term makes a nonvanishing, finite, but ambiguous contribution in P^0. This contribution is not well defined because P^0 depends on the space-time position of the surface at infinity. However, if one invokes the *ad hoc* prescription (to be justified in Sec. 3) that one averages over oscillatory terms, these terms are set to zero. Of course, oscillatory ξ^μ may have a nonoscillatory contribution in the product $(\xi^\alpha{}_{,i}\xi_{\alpha,j})$ due to the cancellation of phases, but then the outside derivative makes the term $\sim 1/r^3$ (this is exemplified by the case $q=0$ above). Thus averaging will remove all quadratic divergence terms. However, there are quadratic terms which are *not* divergences. The simplest of these arise upon transforming $\partial_\mu{}'$ in $H^{\mu i\prime}$. Here we need only consider the linear parts of $H^{0i\prime}$, since the transformation produces a coefficient $\xi^\mu{}_{,\nu}\sim 1/r$. However, the linear part of $H^{0i\prime}$ is $h^T{}_{,i}{}'\sim 1/r^2$, so that these terms are always negligible. The remaining nondivergence terms of $(\Delta H)^{0i}$ arise from the quadratic nondivergence parts of H^{0i}:

$$\tfrac{1}{2}h_{00}(h_{ll,j}-h_{lj,l})-h_{0j}h_{ll,0}+h_{0l}h_{lj,0} +h_{0j}h_{0l,l}-\tfrac{1}{2}h_{ii}h_{lj,l}. \qquad (2.12)$$

Before discussing the transformation of these terms, we note that they do not contribute to $P^{0\prime}$ in the frame (2.5). As we saw previously, $h_{ij}{}^{TT\prime}$ goes faster than $1/r^{\frac{1}{2}}$ and $h^T{}_{,i}{}'\sim 1/r^2$, so $h'_{ij,k}$ goes faster than $1/r$. As shown in Appendix C, $h_{0i}{}'_{,\mu}$, $h_{00}{}'_{,i}$, and $h_{ij}{}'_{,0}$ also go faster than $1/r$. Hence, all $g_{\mu\nu}{}'_{,\alpha}$ of frame (2.5) go faster than $1/r$ (at infinity) with the exception of $g_{00}{}'_{,0}$. Consequently, the terms of Eq. (2.12) go faster than $1/r^2$ in the frame (2.5). However, there exist (oscillatory) transformations leading out of the reference frame which give a well-defined nonzero value to this part of $\oint dS_i\,\Delta H^{0i}$. For example, choose $\xi^0=0$ and

$$\xi_i\sim r^{-1}[a_i\cos\mathbf{k}\cdot\mathbf{r}+k_i(\mathbf{a}\cdot\mathbf{r})r^{-1}\sin\mathbf{k}\cdot\mathbf{r}], \qquad (2.13)$$

with $\mathbf{a}\cdot\mathbf{k}=0$. In the next paragraph we will see that the third term (the transport term) of Eq. (2.10) does not contribute to P^μ upon averaging. Thus, under coordinate wave transformations, the Einstein pseudotensor does not give an invariant P^μ, even after averaging, due to the presence of terms (2.12).

Before turning to the other P^μ definitions, we prove that the transport terms do not contribute in any of the cases under consideration. From the general form $\oint\xi^\alpha H^{\mu i}{}_{,\alpha}{}'dS_i$, we first see that the quadratic terms in $H^{\mu i}{}_{,\alpha}{}'$ cannot contribute. For $H^{\mu i}{}_{,\alpha}{}'$ is a divergence, and hence is $1/r^3$ if the quadratic part of $H^{\mu i}$ is static, or $1/r^2$ if oscillatory. In the latter case, only an oscillatory ξ^α could prevent the product from averaging to zero, but such a ξ^α is itself $1/r$ (e.g., e^{iqx}/r). The linear terms in $H^{\mu i\prime}$ are always derivatives of $g_{\mu\nu}{}'$, but for the cases we are considering [the right-hand side of Eq. (2.1) and the corresponding forms for P^i] never involve $g_{00}{}'_{,0}$; hence by Appendix C this part of $H^{\mu i\prime}$ goes faster than $1/r$. Consequently, if $\xi^\alpha\sim 1/r$, the product again vanishes faster than $1/r^2$. If, on the other hand, ξ^α is of order unity (e.g., $\xi^\alpha\sim x^\alpha/r$), it must be static; in this case the oscillatory terms in the linear part of $H^{\mu i}{}_{,\alpha}{}'$ vanish upon averaging while the static terms of this part of $H^{\mu i}{}_{,\alpha}{}'$ are derivatives (∂_α) of static $g_{\mu\nu}{}'_{,\sigma}$ and hence go faster than $1/r^2$. In Appendix A, an alternate proof that the transport term gives no contribution is given, based on the behavior of the linear terms of $H^{\mu i\prime}$ as determined by the field equations.

We turn next to the definition of P^μ given by Landau and Lifshitz[3]:

$$(H^{\mu i})_{\rm LL}=[(-g)(g^{\mu 0}g^{\alpha i}-g^{\mu i}g^{0\alpha})]_{,\alpha}. \qquad (2.14)$$

Here, we note that $H^{\mu i}$ is itself a divergence, so that its quadratic terms all vanish on averaging by the foregoing discussion. Note that a total time derivative is as effective as a spatial divergence for this purpose. Thus the P^μ of Landau-Lifshitz is indeed coordinate invariant upon averaging.

The above discussion shows that, when $H^{\mu i}$ is a total derivative, the quadratic terms do not contribute to the averaged P^μ, so that the total P^μ it defines is invariant provided the linear parts are invariant. This is a case for the definition of Papapetrou-Gupta[4] (P-G) for the P^0 derived by the canonical formalism[8] (C), as well as for the Landau-Lifshitz definition (2.14). We list the first two:

$$(H^{\mu i})_{\rm P\text{-}G}=[(-g)^{\frac{1}{2}}(g^{\mu 0}\eta^{\alpha i} +g^{\alpha i}\eta^{\mu 0}-g^{0\alpha}\eta^{\mu i}-g^{\mu i}\eta^{0\alpha})]_{,\alpha}, \qquad (2.15)$$

$$(H^{0i})_C=g_{ij,j}-g_{ii,j}=-h^T{}_{,i}. \qquad (2.16a)$$

The canonical formalism's H^{ji} expression,

$$(H^{ji})_C=-2\pi^{ij} \qquad (2.16b)$$

is not a total derivative. However, upon transforming it from the frame (2.5) according to its definition, one finds that the averaged $(P^i)_C$ is invariant, as is shown in

Appendix B. The last-mentioned three definitions of P^μ, when averaged, are all equal.[15]

The Dirac[7] H^{0i},

$$(H^{0i})_D = -({}^3g)^{-\frac{1}{2}}({}^3g\,{}^3g^{ij})_{,j}, \tag{2.17}$$

clearly is not a divergence. By an analysis similar to the one performed on the Einstein H^{0i}, one sees that the quadratic terms here can also contribute arbitrary coordinate-dependent amounts *after averaging*. This may again be shown by the coordinate transformation example of Eq. (2.13). Goldberg[6] has recently discussed generalizations of the Einstein and Landau-Lifshitz definitions which involve different density properties. For the Einstein class, one again finds that the quadratic structures are not invariant. For the Landau-Lifshitz class, the generalized $(H^{0i})_{\mathrm{LL}}$ linearizes to

$$h^T{}_{,i} + \frac{n}{2}(h_{jj} - h_{00})_{,i}, \tag{2.18}$$

(where n is any number) and is coordinate-dependent linearly,[16] except for the original Landau-Lifshitz definition, $n=0$. One may also generalize the Papapetrou stress tensor to an infinite class. The $\mathcal{T}^{\mu\nu}$ corresponding to Eq. (2.15) is simply $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} R$, linearized in terms of $\mathfrak{g}^{\mu\nu} \equiv (-g)^{\frac{1}{2}} g^{\mu\nu}$. One can see immediately that this expression must be conserved by considering a frame where $g_{\alpha\beta,\sigma}=0$ at a point; then the Bianchi identities $G^{\mu\nu}{}_{;\nu} \equiv 0$ imply that the linear terms in $G^{\mu\nu}$ have identically vanishing ordinary divergence. Thus by linearizing $G^{\mu\nu}$ in terms of $\bar{g}^{\mu\nu} \equiv (-g)^{n/2} g^{\mu\nu}$, one obtains the conserved expressions $\mathcal{T}_{(n)}{}^{\mu\nu} = H_{(n)}{}^{[\mu\beta][\nu\alpha]}{}_{,\alpha\beta}$, where

$$H_{(n)}{}^{[\mu\beta][\nu\alpha]} = \bar{g}^{\mu\nu}\eta^{\alpha\beta} + \bar{g}^{\alpha\beta}\eta^{\mu\nu} - \bar{g}^{\mu\alpha}\eta^{\nu\beta} - \bar{g}^{\nu\beta}\eta^{\mu\alpha} + \left(\frac{1-n}{1-2n}\right)(\bar{g}^{\rho\sigma}\eta_{\rho\sigma}) \times (\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}), \quad n \neq \tfrac{1}{2}. \tag{2.19}$$

All these tensors give rise to P^μ having the same coordinate invariance properties as Papapetrou's ($n=1$), since their $H_{(n)}{}^{\mu i} = H_{(n)}{}^{[0i][\mu\alpha]}{}_{,\alpha}$ are divergences, and have the same linear form.

Let us summarize the essential arguments of this section. For the Einstein and Dirac surface integral expressions, an oscillatory coordinate transformation (2.13) has been given which maintains the rectangular boundary conditions but which, in flat space, leads to a nonzero result for these averaged surface integrals (due to the form of the quadratic terms in their $H^{\mu i}$). For the Papapetrou and Landau-Lifshitz surface integrals, the quadratic terms in $H^{\mu i}$ are divergences and so vanish upon space-time averaging. The linear terms of their $H^{\mu i}$ are the entire $H^{\mu i}$ of the canonical expressions, and their finiteness and invariance after averaging are therefore established together with those of the canonical expressions.

[15] The analysis given in IV established the equality (without averaging) of various energy expressions in frames without coordinate waves. It did not consider the more general coordinate systems treated in this paper.

[16] It is clear that the reason Eq. (2.18) yields incorrect Schwarzschild energies in isotropic coordinates (reference 6) is that the additional terms there are not coordinate-invariant even linearly.

III. IDENTIFICATION OF ENERGY-MOMENTUM

In the previous section it was seen that the usual surface-integral expressions for P^μ were neither well defined nor invariant under admissible coordinate transformations leaving $g_{\mu\nu} - \eta_{\mu\nu} \sim 1/r$. However, the Landau-Lifshitz, Papapetrou, and canonical formalism definitions could be made invariant (and equal) if one eliminated the ambiguous terms by a space-time averaging at infinity. We shall now see that with this averaging prescription, the above definitions of P^μ give, in fact, the physical energy and momentum of a system.

As stated in Sec. I, the energy of a system is the numerical value of the Hamiltonian (i.e., the generator of time translations) in a Heisenberg representation. At the end of this section, we shall see that the total energy of the full theory as defined in this way agrees with the energy as measured by experiments which can be analyzed in Newtonian terms.[17] As was shown in IVa, all Heisenberg representations yield the same value for the energy of a system. In particular, we shall arrive at the energy in terms of the frame (2.5), which, with the canonical variables $g_{ij}{}^{TT}$ and π^{ijTT} gives a Heisenberg representation. In this canonical formalism, the Hamiltonian density $-\mathcal{T}^0{}_0$ is $-\nabla^2 h^T$ when the latter quantity is expressed in terms of the canonical variables by means of the constraint equations $G^0{}_\mu = 0$ (see III). Thus, $h^T = +(1/\nabla^2)\mathcal{T}^0{}_0 \sim (1/4\pi r)\int d^3r(-\mathcal{T}^0{}_0)$, and one can read the energy $E = \int d^3r(-\mathcal{T}^0{}_0)$ from the asymptotic form of h^T according to

$$h^T \sim E/4\pi r. \tag{3.1}$$

Note that the constraint equation leads to expression (3.1) since $\mathcal{T}^0{}_0$ falls off faster than $1/r^3$ [see discussion of Eq. (2.7)] and guarantees the coefficient of $1/r$ to be angle-independent[12]. Having now one unambiguous method for computing the energy, i.e., using Eq. (3.1) after transforming to the frame of Eq. (2.5), we next wish to see how this same energy may be obtained without going to this particular frame. We see immediately from Eq. (2.4) that the $1/r$ part of h^T is unaffected by any coordinate transformation preserving the asymptotic boundary conditions $g_{\mu\nu} - \eta_{\mu\nu} \sim 1/r$ (even though $g_{\mu\nu,\alpha} \sim 1/r$ in the new frame). Thus Eq. (3.1) provides an *invariant way* of computing the energy. (As mentioned in the introduction, frames where $g_{\mu\nu,\alpha} \sim 1/r$ are of physical interest since there are Heisenberg representations where coordinate waves exist with this boundary condition. An example is given in Sec. 2 of IVa.)

It is now easy to establish that those surface-integral

[17] That it also gives the correct inertial definition of energy is shown in Sec. IV.

expressions for P^0 which are invariant when averaged, correctly give the energy E. By invariance, one need only check this in the frame (2.5). As was shown in Sec. II, all these expressions then reduce to Eq. (2.1). The latter can be converted to a volume integral:

$$\int d^3r(g_{ij,ij}-g_{ii,jj})=\int d^3r(-\nabla^2 h^T)=\int d^3r(-\mathcal{T}^0{}_0), \quad (3.2)$$

and the last member is correctly the energy.

We next establish that the momentum, as defined by the generator of spatial translations may be obtained in any coordinate frame (with $g_{\mu\nu}-\eta_{\mu\nu}\sim 1/r$) from the asymptotic form of π^i. As for the energy, this will then allow us to justify that those definitions of P^i which are invariant when averaged correctly give the momentum. In III it was seen that the momentum density $\mathcal{T}^0{}_i$ and π^{ij} are related, in the frame (2.5), by

$$-2\pi^{ij}{}_{,j}\equiv -2\pi^i{}_{,jj}-2\pi^j{}_{,ij}=\mathcal{T}^0{}_i[g^{TT},\pi^{TT}]. \quad (3.3)$$

Equation (3.3) is also obtained by solving the constraint equations, $G^0{}_\mu=0$. This equation for π^i is essentially a Poisson equation, and the asymptotic solution is determined by the monopole moment of the source, $\int d^3r\ \mathcal{T}^0{}_i\equiv P^i$. As was the case for the energy, the momentum P^i is also a constant of the motion. One finds

$$\pi^i\sim(1/4\pi r)(1/12)[5P^i+Q_{ij}P^j], \quad (3.4)$$

where $Q_{ij}=\frac{1}{2}(3x^ix^jr^{-2}-\delta_{ij})$. Thus in frame (2.5), the momentum is read off from the $1/r$ part of π^i. However, it is shown in Appendix B that π^i is invariant to order $1/r$, and hence Eq. (3.4) holds in any frame where $g_{\mu\nu}-\eta_{\mu\nu}\sim 1/r$. Next, to show that the averaged surface integrals of Sec. II yield the momentum, we need again merely to establish this in the frame (2.5). In this frame, the quadratic terms of any H^{ji} are negligible, and all the H^{ji} expressions which lead to invariant P^μ's when averaged differ from $-2\pi^{ij}$ by quadratic terms. We have then

$$\oint dS_j(-2\pi^{ij})=\int d^3r(-2\pi^{ij}{}_{,j})=\int d^3r\ \mathcal{T}^0{}_i. \quad (3.5)$$

In Sec. II, the invariance of the averaged P^μ was established for all transformations not involving Lorentz rotations at spatial infinity, while in this section we have identified P^μ with energy-momentum. However, the Landau-Lifshitz and Papapetrou P^μ are manifestly Lorentz four-vectors. This ensures that the energy-momentum [defined by Eqs. (3.1), (3.4)] has the proper Lorentz transformation properties.

We turn now to the relation between the P^μ defined by the canonical formalism and asymptotic measurements of energy and momentum performed according to Newtonian definitions, since the latter must be correct in the weak-field region at infinity. Our method is to use the acceleration of a test particle at infinity to measure the gravitational mass of the system, which, also within the Newtonian framework is equal to the inertial mass. In any frame, the acceleration of a test particle in general relativity is governed by the geodesic equation,

$$d^2x^\mu/d\tau^2+\Gamma^\mu{}_{\alpha\beta}(dx^\alpha/d\tau)(dx^\beta/d\tau)=0. \quad (3.6)$$

To analyze this motion by Newtonian means, several restrictions must be imposed, however. First, as is the case for our discussion, the metric must be sufficiently weak, which is a restriction not only on the physical situation but also on the coordinate system (since the Newtonian formulation presupposes a rectangular background frame). Furthermore, the time derivatives of the field must be much smaller than space derivatives, since Newtonian theory is a static approximation. Next, in Newtonian theory, the Poisson equation states that, asymptotically, the field of any system is spherically symmetric. The last requirement is that the velocity of the test particle be small, i.e., $dx^i/d\tau\cong 0$. The geodesic equation now reduces to

$$d^2x^i/dt^2=(-g_{0i,0}+\tfrac{1}{2}g_{00,i})\cong\tfrac{1}{2}g_{00,i}, \quad (3.7)$$

since $g_{0i,0}$ is negligible by the static requirement. Writing $-g_{00}\sim 1-m/8\pi r$, we see that the mass is the coefficient of $1/8\pi r$ in g_{00}. The quantity g_{00} which yields the energy according to the Newtonian definition is indeed invariant under the class of transformations maintaining Newtonian conditions. Its change is (asymptotically) $2\xi^0{}_{,0}$, so that ξ^0 would have to be $\sim t/r$ to affect g_{00} to $\sim 1/r$. This is not permitted, since then $g_{0i,0}$ would be $O(1/r^2)$, i.e., the same size as space derivatives of the metric. However, a wider class of asymptotic transformations (which we are considering here) is permissible in relativity. We therefore exhibit a quantity which agrees with g_{00} to order $1/r$ in Newtonian frames and is invariant under the wider class; this quantity will be seen to be the invariant h^T. Under the above assumptions that the metric is static and spherically symmetric, it follows that the asymptotic form of the metric is the Schwarzschild solution in a static frame. Since g_{00} is invariant in all Newtonian frames, we relate it to h^T in a particular one, namely isotropic coordinates, where the Schwarzschild metric is asymptotically

$$g_{ij}\sim\delta_{ij}(1+m/8\pi r),\quad g_{00}\sim -1+m/8\pi r,\quad g_{0i}\sim 0. \quad (3.8)$$

Since h^T is defined to be $h_{ii}-(1/\nabla^2)\partial^2{}_{ij}h_{ij}$ (see III), we find for h^T asymptotically[18] $h^T\sim m/4\pi r\sim 2(g_{00}+1)$. This establishes that the canonical formulation's definition of energy [Eq. (3.1)] gives the mass as measured by Newtonian experiments. By the equivalence principle,

[18] It is shown in Appendix B of IVb that the operator $(1/\nabla^2)\partial^2{}_{ij}$ occurring in the definition of h^T preserves the leading asymptotic form of h_{ij}. Thus one can get h^T asymptotically by inserting the asymptotic form of g_{ij} in the formula for h^T.

this mass is the total energy of the system, and must therefore be, by special relativity, the fourth component of the energy-momentum vector. Since this mass is also the E of Eq. (3.1), which we have seen to be the fourth component of the vector P^μ [with P^i defined by (3.4)], P^i is thus identified as the physical momentum. The above identification of the canonical P^μ with that obtained from Newtonian definitions was made for systems which had asymptotically Newtonian metrics; it is an interesting question whether every metric which goes to $\eta_{\mu\nu}$ as $1/r$ is asymptotically Newtonian in a suitable frame.

IV. DISCUSSION

A number of different expressions for "energy" have been proposed in general relativity in the past. These have been based on the desire for a simple conservation law of the form $(\mathcal{T}_G{}^{\mu\nu}+\mathcal{T}_M{}^{\mu\nu})_{,\mu}=0$ in which $\mathcal{T}_M{}^{\mu\nu}$ is the stress tensor of the sources which appears in the Einstein equations, while $\mathcal{T}_G{}^{\mu\nu}$ is a quantity which does not involve matter variables.

The richness of Riemannian geometry allows many possibilities which exhibit various features of energy in Lorentz-covariant theory. Thus, requirements such as symmetry, time derivatives entering only in first order, expressions having more coordinate invariance than just Lorentz invariance, and use only of the metric (without explicit dependence on the Lorentz metric), have all been factors in defining such $\mathcal{T}_G{}^{\mu\nu}$. However, such requirements do not of themselves lead to an expression which arises from the basic physical meaning of the energy of a system. Physically, the energy of the gravitational field must agree with the definition of energy of all other nongravitational systems. Thus, if two systems interact, and in the first gravitation is negligible before and after the interaction, then the change in energy of the other system (including all gravitational effects) must agree with the change in energy of the first (test) system. Since the meaning of the word "energy" in the test system is unambiguous (because it is in flat space before and after its interaction), one is led to an unambiguous definition of the gravitational system's energy based on this *physical* conservation condition.

A necessary demand in the definition of energy has thus been that, asymptotically, space is flat (and so, of course, that the gravitational system is bounded). Since the reference system's energy is defined by Lorentz-covariant concepts, it is not, for example, invariant under arbitrary curvilinear transformations out of the Cartesian frame, and thus one cannot require this for the gravitational energy. [Of course, Lorentz-covariant formulas can be written in an appropriately modified form so as to hold in curvilinear coordinates; however, this corresponds to no increased generality of the definition since it does not preserve form-invariance.]

This unambiguous physical definition of energy can be translated into a formal one: One may define the energy to be the numerical value of the generator of time translations in a Heisenberg representation of the dynamics of the gravitational field. This means that, once the field has been cast into canonical form in terms of its true (unconstrained) degrees of freedom, its energy is just the Hamiltonian. The requirement that one use a Heisenberg representation, is, of course, equally vital for all other dynamical systems, to exclude, for example, Hamilton-Jacobi representations in which the Hamiltonian is *not* the energy. For general relativity, Heisenberg canonical formation is defined by a relation of the form $g_{\mu\nu}(t)=g_{\mu\nu}[p(t),q(t)]$ not depending explicitly on time, and such that the field equations then become Hamilton equations for $q(t)$ and $p(t)$, which are therefore true canonical variables of the gravitational field. The lack of explicit time dependence in $g_{\mu\nu}[p,q]$ is essential in excluding Hamilton-Jacobi like representations. As in other physical systems, all Heisenberg representations yield the same numerical value of the Hamiltonian for a given physical situation and this value is conserved (see IVa). It is now easy to see that the above construction of the Hamiltonian properly corresponds to our physical energy. For, when a nongravitational system is coupled to the field, the coupling leaves the Heisenberg relations $g_{\mu\nu}=g_{\mu\nu}(p,q)$ unaltered (since the canonical variables are, of course, unchanged —see IV and V), while the addition of the Hamiltonian is the generally covariant generalization of the matter Hamiltonian; i.e., $H_M[p_M,q_M;\eta_{\mu\nu}]$ becomes $H_M[p_M,q_M;g_{\mu\nu}]$. The total Hamiltonian is therefore still conserved, precisely because $g_{\mu\nu}[p,q]$ brings in no explicit time dependence. Before and after the interaction of the matter system with the gravitational field, the former may be assumed to be at spatial infinity and thus the total (conserved) Hamiltonian is $H_G+H_M[p_M,q_M;\eta_{\mu\nu}]$ (bearing in mind the asymptotically flat boundary conditions on our system). Thus H_G represents the gravitational energy. Concretely, in the Heisenberg representation governed by the coordinate conditions (2.5) with q, p corresponding to $g_{ij}{}^{TT}$, π^{ijTT}, we have seen that H_G is $-\int d^3r\ \mathcal{T}^0{}_0[g^{TT},\pi^{TT}]$ and thus $E=-\int d^3r\ \nabla^2 h^T$ in this frame.

Having established the correct *definition* for the energy, one may then ask the (secondary) question of how it may *conveniently* be read off in any frame (which is asymptotically flat). One may also then investigate to what extent various expressions proposed for "energy" agree with the basic one—in particular, in what class of frames this may be the case. The most convenient way of obtaining the energy (i.e., the most invariant) was then found to be from the coefficient of the leading $1/r$ term in the asymptotic expansion of h^T. Due to the invariance of h^T to order $1/r$, any frame in which $g_{\mu\nu}-\eta_{\mu\nu}$ and $g_{\mu\nu,\alpha}$ go as $1/r$ may be used here. Another useful expression is the surface integral form, $\oint h^T{}_{,i}dS_i$, which is invariant for frames with $g_{\mu\nu,\alpha}$ going faster than $1/r$; for the more general case, however, it was found that the effect of "coordinate waves" could be removed

only upon performance of an averaging procedure at infinity. It can be shown that *spatial* averaging is sufficient for the canonical energy-momentum integrals (2.1), (2.16). Note that both prescriptions share the property of extracting the energy from knowledge of the field configuration on a single space-like surface. With respect to the earlier prescriptions proposed by various authors (all of which are of the surface-integral type), it was found that they varied considerably in applicability. We discuss here the energy-momentum expressions, but for simplicity use the word energy to stand for P^μ. Thus, in one case it was found that the expression was invariant only under static coordinate transformations not involving the time, and that it yielded the energy only for systems which admit of Newtonian frame, in which $g_{\mu\nu,i} \gg g_{\alpha\beta,0}$ so that $(g_{00}+1)\sim\frac{1}{2}h^T$. The original definition of Einstein was invariant, and gave the energy, only within frames with $g_{\mu\nu,\alpha}$ going faster than $1/r$. For coordinate waves, the expression could not be kept invariant; the same was found for Dirac's expression. The surface-integral prescriptions of Landau-Lifshitz and Papapetrou closely resemble the $\oint dS_i\, h^T{}_{,i}$ one, and therefore are invariant linearly. In the presence of coordinate waves, however, invariance can be achieved here only if a *time* as well as a spatial averaging was performed. In this sense, their prescriptions require more than the field configuration at a given time. Similar results were found for the momentum vector, with π^i playing the role of h^T.

Finally, we mention that two required properties of the energy were shown explicitly to hold for our definition. The first, conservation, $\partial_t \oint dS_i\, h^T{}_{,i}=0$ follows from the asymptotic form of the $h^T{}_{,0}$ field equation (Appendix C). The second is that the inertial definition of energy discussed above is equivalent to the definition of energy in terms of Newtonian measurements at spatial infinity for situations in which the latter is well defined.

APPENDIX A

We give here an alternate proof that the transport terms in Eq. (2.10),

$$\oint dS_i\, \xi^\alpha H^{\mu i}{}_{,\alpha}{}', \qquad \text{(A.1)}$$

do not contribute to $P^\mu - P^{\mu\prime}$. As mentioned in text, the quadratic terms in $H^{\mu i}{}_{,\alpha}{}'$ give negligible contributions in the frame (2.5) since they are divergences. We first consider only the energy contribution, $\mu=0$ in (A.1). All $H^{0i\prime}$ (which are linearly invariant) reduce to $g_{ij}{}'{}_{,j} - g'{}_{jj,i} \equiv -h^T{}_{,i}{}'$ linearly. We must therefore show that $h^T{}_{,i\alpha}{}'$ goes faster than $1/r^2$ since ξ^α need not vanish at infinity. The constraint equations $G^0{}_\mu=0$ determine $h^{T\prime}$ according to $\nabla^2 h^{T\prime} = \mathcal{T}^0{}_0{}'$ (see III). In the frame (2.5), $\int d^3r(-\mathcal{T}^0{}_0{}')$ is the Hamiltonian, and hence is bounded. Consequently, the equation has the solution[19]

$$h^{T\prime} \sim \left(-\int \mathcal{T}^0{}_0{}' d^3r'\right)\frac{1}{4\pi r} + O[f(t)/r^{1+\epsilon}]. \qquad \text{(A.2)}$$

In the gradient $h^T{}_{,i}{}'$, therefore, the higher terms $O[f(t)/r^{1+\epsilon}]$ are negligible, and so $h^T{}_{,i}{}' \sim (x^i/4\pi r^3) \times \int \mathcal{T}^0{}_0 d^3r$. It is clear, therefore, that $h^T{}_{,ij}{}'$ goes faster than $1/r^2$, and one need only consider

$$h^T{}_{,i0}{}' \sim (x^i/4\pi r^3)\partial_0\left(\int \mathcal{T}^0{}_0 d^3r\right). \qquad \text{(A.3)}$$

It was shown in III, however, that the Hamiltonian was conserved in time in this frame, so that this last term is zero (an alternate proof of conservation is given in Appendix C).

The transport contribution to the momentum integrals vanishes in the same fashion, due to the conservation of $P^i = \int \mathcal{T}^0{}_i\, d^3r$ in the canonical theory. All linearly invariant H^{ii} differ from $(H^{ii})_C = -2\pi^{ij}$ by quadratic terms which are negligible in the transport integral. We need, therefore, only examine $\pi^{ij}{}_{,\alpha}{}'\xi^\alpha$. In the breakup of $\pi^{ij\prime}$ into $\pi^{ijTT\prime} + (\pi^i{}_{,j}{}' + \pi^j{}_{,i}{}')$, Eq. (3.4) shows that $\pi^i{}_{,j}{}'$ goes as $P^i/r^2 + O(r^{-2-\epsilon})$. Since P^i is a constant of the motion, the contribution of $(\pi^i{}_{,j}{}' + \pi^j{}_{,i}{}')_{,\alpha}$ is negligible (faster than $1/r^2$). In the remaining part, $\pi^{ijTT}{}_{,\alpha}{}'\xi^\alpha$, the factor $\pi^{ijTT\prime}$ can go no slower than $r^{-\frac{3}{2}-\epsilon}$ [see discussion following Eq. (2.7)]. Since oscillatory ξ^α go as $1/r$, only the static type need be considered. For static $\pi^{ijTT\prime}$, its derivatives are $1/r^{\frac{5}{2}+\epsilon}$ and are negligible. Finally, oscillatory $\pi^{ijTT}{}_{,\alpha}{}'$ cancel upon averaging with the static ξ^α. [Note that the averaging definition being used here,

$$\lim_{R,T\to\infty} \frac{1}{2T}\int_{t-T}^{t+T} dt \frac{1}{R}\int_R^{2R} dr \oint dS_i\, f_i, \qquad \text{(A.4)}$$

gives zero for any oscillatory f_i which vanishes faster than $1/r$.]

APPENDIX B

In this Appendix we show that, although $(H^{ii})_C = -2\pi^{ij}$ is not a divergence, its surface integral is invariant upon averaging. This is due to its special structure. We shall also show, as required in Sec. III, that π^i is invariant to $O(1/r)$.

In examining the invariance of the surface integral $\oint \pi^{ij} dS_j$, we proceed as usual, transforming from the reference frame (2.5). The transport terms are negligible as mentioned in Appendix A, and one need only consider the transformation of π^{ij} itself through quadratic order. This is obtained from the definition (2.5b) of π^{ij} in terms of the metric. The linear contribution, $\oint dS_j(\xi^0{}_{,ij} - \delta_{ij}\xi^0{}_{,ll})$, is identically zero as may be seen

[19] The terms of higher order than $1/r$ are not necessarily the usual multipole series since $\mathcal{T}^0{}_0{}'$ was not assumed to vanish outside a finite region. The general form of the higher order terms is discussed in Appendix A of IVb.

by converting it to a volume integral. The quadratic contributions are of two types: those involving the product of a ξ^μ times a metric quantity, and those quadratic in ξ^μ. The former may be simply disposed of. They have the characteristic form $\pi^{lm\prime}\xi^\mu{}_{,\alpha}$ or $h_{lm}{}'\xi^0{}_{,ij}$ (this last type arising from the inhomogeneous term in the $\Gamma^0{}_{ij}$ transformation). In the $\pi^{lm\prime}\xi^\mu{}_{,\alpha}$ example, $\xi^\mu{}_{,\alpha}$ goes as $1/r$, while $\pi^{lm\prime}$ goes faster than $1/r^{\frac{3}{2}}$. This behavior of $\pi^{lm\prime}$ is obtained from the fact that $\pi^{lm\prime}=\pi^{lmTT\prime}+(\pi^l{}_{,m}{}'+\pi^m{}_{,l}{}')$; the $\pi^{lmTT\prime}$ must vanish faster than $1/r^{\frac{3}{2}}$ [see discussion following Eq. (2.7)], while $\pi^l{}_{,m}{}'\sim 1/r^2$ from Eq. (3.4). Thus this type does not contribute. In the $h_{lm}{}'\xi^0{}_{,ij}$ type, $\xi^0{}_{,ij}$ again can be $\sim 1/r$ and oscillatory or $1/r^2$ and static. The latter case does not contribute, $h_{lm}{}'$ being at least $\sim 1/r$. To treat the oscillatory $\xi^0{}_{,ij}$, we note that $h_{lm}{}'=h_{lm}{}^{TT\prime}+h_{lm}{}^{T\prime}$. Again $h_{lm}{}^{TT\prime}$ may be oscillatory, but goes at least as $1/r^{\frac{3}{2}+\epsilon}$, and hence cannot produce a contribution. The $h_{lm}{}^{T\prime}$ part is static to order $1/r$ [from Eq. (3.1)] and hence, its product with the oscillatory $1/r$ part of ξ^μ vanishes when averaged.

The remaining transformation terms are then quadratic in ξ^μ. They are:

$$\begin{aligned}[\xi^0{}_{,ll}(\xi^i{}_{,j}+\xi^j{}_{,i})-(\xi^0{}_{,lj}\xi^i{}_{,l}+\xi^0{}_{,li}\xi^j{}_{,l})]\\+[\xi^0{}_{,ij}\xi^l{}_{,l}-\xi^0{}_{,l}\xi^l{}_{,ij}-(\xi^0{}_{,lj}\xi^l{}_{,i}+\xi^0{}_{,li}\xi^l{}_{,j})]\\-\delta_{ij}[\xi^0{}_{,ll}\xi^m{}_{,m}-\xi^0{}_{,l}\xi^l{}_{,mm}-2\xi^0{}_{,lm}\xi^l{}_{,m}].\qquad\text{(B.1)}\end{aligned}$$

In spite of the fact that $\pi^{ij\prime}$ is not a divergence, it is easy to see that each bracket in Eq. (B.1) is. All these terms, therefore, vanish upon averaging by the analysis of divergences following Eq. (2.11). It is interesting to note that a critical factor in this proof was the position of the indexes on $\pi^{ij\prime}$. Thus, if one had used either $\pi^i{}_j{}'$ or $\pi_{ij}{}'$ on the surface integral [these being numerically equal asymptotically in frame (2.5)], the analog of Eq. (B.1) would fail to be a divergence, and the averaging would not yield a coordinate-invariant result.

To show the invariance of π^i in order $1/r$, we write the transformation law for π^{ij} in the form (2.9a),

$$\begin{aligned}\pi^{ij}(x^\mu=a^\mu)=\pi^{ij\prime}(x^{\mu\prime}=a^\mu)+(\xi^0{}_{,ij}-\delta_{ij}\xi^0{}_{,ll})\,|_{x=a}\\+\pi^{ij}{}_{,\alpha}{}'(x'=a)\xi^\alpha+\{\pi^{ij\prime}\xi^\mu{}_{,\alpha}+h_{lm}{}'\xi^0{}_{,ij}\}\\+(\xi^0{}_{,m}\xi^k{}_{,n})_{,l}.\qquad\text{(B.2)}\end{aligned}$$

The terms in the brace are symbolic for quadratic terms which are products of metric quantities and ξ^μ, while $(\xi^0{}_{,m}\xi^k{}_{,n})_{,l}$ stands for the divergence (B.1). We now calculate how $\pi^i{}_{,j}$ transforms from (B.2); we will see that its change is oscillatory in order $1/r^2$, so that the change in π^i from these terms is still oscillatory and $1/r^2$, while the other terms in $\pi^i{}_{,j}$ are faster than $1/r^2$ and cannot affect π^i in order $1/r$. To obtain $\pi^i{}_{,j}$ from π^{ij}, one applies a linear operator as follows (see III):

$$\begin{aligned}\pi^i{}_{,j}=(1/\nabla^2)\partial^2{}_{kj}\pi^{ik}-\tfrac{1}{2}[(1/\nabla^2)\partial^2{}_{ij}][(1/\nabla^2)\partial^2{}_{lm}]\pi^{lm}\\=Q_{i,j}[\pi^{lm}].\qquad\text{(B.3)}\end{aligned}$$

Equation (B.2) relates functions of a^μ; all ξ^μ are taken at $x^\mu=a^\mu$, while the unspecified metric quantities are taken at $x'^\mu=a^\mu+\xi^\mu(a)$. The operator $Q_{i,j}$ constructed from $\partial_i=\partial/\partial a^i$, gives rise to $\pi^i{}_{,j}$ at $x^\mu=a^\mu$ when applied to $\pi^{ij}|_{x=a}$, and gives $\pi^i{}_{,j}{}'=\partial_j{}'\pi^{i\prime}$ at $x'=a$ when applied to $\pi^{ij\prime}|_{x=a}$. By inspection, this operator annihilates the term $(\xi^0{}_{,ij}-\delta_{ij}\xi^0{}_{,ll})$ of Eq. (B.2). As is proven in Appendix B of IVb, $Q_{i,j}$ maintains the leading asymptotic character of any power through $1/r^2$ and gives no $1/r^2$ terms from functions decreasing faster than $1/r^2$. By Appendix A, and the present Appendix, all the remaining terms in Eq. (B.2) are of order $1/r^{\frac{3}{2}+\epsilon}$ (or higher) and oscillatory through order $1/r^2$. One has then

$$\pi^i{}_{,j}=\pi^i{}_{,j}{}'+O(e^{iqx}/r^{\frac{5}{2}+\epsilon}),\qquad\text{(B.4)}$$

and thus $\pi^i=\pi^{i\prime}+O(1/r^{\frac{3}{2}+\epsilon})$ provided the O terms have spatial (and not merely time) oscillations. To see that this is the case, note that in Eq. (B.2), the terms $\pi^{ij\prime}\xi^\mu{}_{,\alpha}\sim 1/r^{\frac{5}{2}+\epsilon}$, while in $h_{lm}{}'\xi^0{}_{,ij}$ and $(\xi^0{}_{,m}\xi^k{}_{,n})_{,l}$, the derivatives of ξ^μ are all spatial and thus in the absence of net spatial oscillations in these terms they would fall off faster than $1/r^2$. Finally, in the transport terms, $\pi^{ij}{}_{,\alpha}{}'\xi^\alpha$, the contribution from $\pi^i{}_{,j}{}'$ is negligible (as in Appendix A), leaving only $\pi^{ijTT}{}_{,\alpha}{}'\xi^\alpha$. In $\pi^{ijTT\prime}$, time oscillation necessarily implies spatial oscillation since π^{ijTT} obeys the wave equation in leading order. The proof of this is identical to that given in IVb for the wave zone (where $\pi^{ijTT}\sim e^{ikx}/r$). In IVb the nonlinear terms in the Einstein equations were $O(1/r^2)$ while past the wave front they are shown in Appendix C to be $O(1/r^{\frac{5}{2}+\epsilon})$. With the replacement of $O(1/r^2)$ terms in IVb by $O(1/r^{\frac{5}{2}+\epsilon})$, then, the IVb proof can be applied to π^{ijTT} past the wave front directly.

APPENDIX C

It is established here that, in the frame of Eq. (2.5), all $g_{\mu\nu,\alpha}$ except $g_{00,0}$ go faster than $1/r$ asymptotically. We will also show directly that the canonical formulation's surface-integral definitions for P^μ are constants in time. It was seen in text that $g_{ij,k}$ goes faster than $1/r$; there remains to establish this property for $g_{ij,0}$, $g_{00,i}$ and $g_{0i,\alpha}$. In $g_{ij,0}=h_{ij}{}^{TT}{}_{,0}+h_{ij}{}^T{}_{,0}$, one knows that $h_{ij}{}^{TT}{}_{,0}$ goes as $1/r^{(\frac{3}{2}+\epsilon)}$ by the discussion following Eq. (2.7); further by Eqs. (A.2), (A.3), $h_{ij}{}^T{}_{,0}$ goes faster than $1/r$. The behavior of $g_{0\mu,\alpha}$ is determined by the coordinate conditions. We apply these to the field equations; for this purpose we revert to the three-dimensional notation of IVb:

$$g_{ij,0}=2N({}^3g)^{-\frac{1}{2}}(\pi^{ij}-\tfrac{1}{2}g^{ij}\pi)+\eta_{i|j}+\eta_{j|i},\qquad\text{(C.1a)}$$

$$\begin{aligned}\pi^{ij}{}_{,0}=-N({}^3g)^{\frac{1}{2}}({}^3R^{ij}-\tfrac{1}{2}g^{ij}\,{}^3R)\\+\tfrac{1}{2}N({}^3g)^{-\frac{1}{2}}g^{ij}(\pi^{ln}\pi_{ln}-\tfrac{1}{2}\pi^2)\\-2N({}^3g)^{-\frac{1}{2}}(\pi^{im}\pi^j{}_m-\tfrac{1}{2}\pi\pi^{ij})\\+({}^3g)^{\frac{1}{2}}(N^{|ij}-g^{ij}N^{|m}{}_{|m})\\+\pi^{ij}{}_{|m}\eta_m+(\pi^{ij}\eta^m{}_{|m}-\eta^i{}_{|m}\pi^{mj}-\eta^j{}_{|m}\pi^{mi}),\qquad\text{(C.1b)}\end{aligned}$$

where $\eta_i\equiv g_{0i}$, $N\equiv(-g^{00})^{-\frac{1}{2}}$, $\pi\equiv g_{ij}\pi^{ij}$ and the vertical bar means covariant differentiation with respect to the

three-metric g_{ij}. In using these equations to determine $\eta_{i,\alpha}$ and $N_{,\alpha}$, it is important to remember that π^{ijTT} goes as $1/r^{\frac{3}{2}+\epsilon}$ (and may be oscillatory in this order), while the remainder of π^{ij}, i.e., $\pi^{i}{}_{,j}$, is of order $1/r^2$ (see Appendix A). Separating out the linear part of Eq. (C.1a), one has

$$g_{ij,0}\equiv g_{ij}{}^{TT}{}_{,0}+g_{ij}{}^{T}{}_{,0}=2\pi^{ijTT}-2\delta_{ij}\pi^{l}{}_{,l} +(2\pi^{i}+\eta_i)_{,j}+(2\pi^{j}+\eta_j)_{,i}+O(r^{-\frac{5}{2}-\epsilon}), \quad \text{(C.2a)}$$

where the O represents the nonlinear terms. These are, typically, $(N-1)\pi^{ijTT}$ and $\eta_l\Gamma^{l}{}_{ij}$. Equation (C.2a) yields directly the stated behavior of η_i, since $g_{ij,0}$, π^{ijTT}, $\pi^{l}{}_{,j}$ are all faster than $1/r$. In fact, by applying the linear operator $Q_{i,j}$ of Appendix B, one finds

$$\eta_{i,j}+\eta_{j,i}=-2(\pi^{i}{}_{,j}+\pi^{j}{}_{,i})+Q_{i,j}[-2\delta_{lm}\pi^{k}{}_{,k}+O(r^{-\frac{5}{2}-\epsilon})].$$

The property of $Q_{i,j}$ (derived in Appendix B of IVb), relevant to our present purpose, is that the asymptotic expansion of $Q_{i,j}[f^{lm}]$ can be obtained from the asymptotic expansion of f^{lm} to within terms of order $1/r^3$. One finds that, if $f^{lm}\sim 1/r^n$, then $Q_{i,j}[f]\sim 1/r^n$ for $3>n>0$; for $n=3$, $Q_{i,j}[f]\sim(\ln r)/r^3$, and for $n>3$, $Q_{i,j}[f]\sim 1/r^3$. Hence, $\eta_{i,j}$ goes as $1/r^2$, and to this order is determined by $\pi^{i}{}_{,j}$. Thus, we may write $\eta_i\approx P^i/r+O(r^{-1-\epsilon})$ where the first term indicates a formula like Eq. (3.4) but with slightly different numerical coefficients. Clearly, $\eta_{i,0}$ then goes faster than $1/r$, since P^i is a constant of motion. [We might note that this result for $\eta_{i,0}$ has been obtained for the minimal behavior for the canonical modes, i.e., $1/r^{(\frac{3}{2}+\epsilon)}$. If one has the slightly more rapid behavior of $1/r^2$, then $\eta_{i,0}$ also goes as $1/r^2$.] To obtain $N_{,\alpha}$, we turn to Eq. (C.1b), where we again separate linear terms:

$$\pi^{ijTT}{}_{,0}+(\pi^{i}{}_{,j}+\pi^{j}{}_{,i})_{,0} =\tfrac{1}{2}\nabla^2 g_{ij}{}^{TT}-[\delta_{ij}\nabla^2(N+\tfrac{1}{4}g^T) -(N+\tfrac{1}{4}g^T)_{,ij}]+O_1(r^{-\frac{5}{2}-\epsilon})+O_2(r^{-3-\epsilon}). \quad \text{(C.2b)}$$

The leading nonlinear terms in O_1, typified by $\pi^{ijTT}{}_{,m}\eta^m$, are $1/r^{\frac{5}{2}+\epsilon}$ and oscillatory. The leading static term, O_2, is of order $1/r^{3+\epsilon}$. Such terms can come from $(\pi^{ijTT})^2$ with cancellation of phases in oscillations. [The order of the term $(N-1)^3R_{ij}\sim(N-1)\nabla^2 g_{ij}{}^{TT}$ is *a priori* $1/r^{\frac{5}{2}+\epsilon}$ by boundary conditions on $N\equiv(-g^{00})^{-\frac{1}{2}}$. It could only be static if $g_{ij}{}^{TT}$ is oscillatory and $(N-1)$ is oscillatory in $1/r$. But this possibility for N is contradicted by the fact that the term $(\delta_{ij}\nabla^2-\partial^2{}_{ij})N$ would be order $1/r$ and no other term in (C.2b) is of this order.] Applying the operator $Q_{ij}{}^T$ defined by (see III)

$$Q_{ij}{}^T[f_{lm}]\equiv f_{ij}{}^T\equiv\tfrac{1}{2}[\delta_{ij}-(1/\nabla^2)\partial^2{}_{ij}] \times[f_{ll}-(1/\nabla^2)f_{lm,lm}] \quad \text{(C.3)}$$

to Eq. (C.2b), we obtain

$$0=(\delta_{ij}\nabla^2-\partial^2{}_{ij})(N+\tfrac{1}{4}g^T)+O_1(r^{-\frac{5}{2}-\epsilon})+O_2(r^{-3}), \quad \text{(C.4)}$$

since the operator $Q_{ij}{}^T$ shares the asymptotic properties of $Q_{i,j}$ (see also Appendix B of IVb), in that it leaves the behavior of the oscillatory $1/r^{\frac{5}{2}+\epsilon}$ terms unchanged and can produce $\sim 1/r^3$ terms from terms that are static and asymptotically $1/r^{3+\epsilon}$. Such $1/r^3$ terms, however, are necessarily of the quadrupole form Y_{2m}/r^3 (as follows from Appendix A of IVb) and so have vanishing monopole moment. Contracting on i and j and differentiating once yields

$$\nabla^2(N+\tfrac{1}{4}g^T)_{,i}=O_{1,i}+O_{2,i}. \quad \text{(C.5)}$$

Hence $(N+\frac{1}{4}g^T)_{,i}$ and thus $N_{,i}$ itself falls off faster than $1/r$. This follows from the fact that $1/\nabla^2$ of the oscillatory structure $O_{1,i}$ maintains its asymptotic order, while $1/\nabla^2$ of the static $O_{2,i}$ goes faster than $1/r$ since the monopole moment of the source $O_{2,i}$ vanishes. (A more detailed proof of this follows directly from Appendix A of IVb.) On the other hand, while $\frac{1}{4}g^T{}_{,0}$ does indeed go faster than $1/r$, (since $g^T\sim P^0/r$) it is clear that the above argument does not hold for $N_{,0}$, since $O_{2,0}$ is not a spatial divergence. As in the $N_{,i}$ case, $O_{1,0}$ produces no $1/r$ terms; $O_{2,0}$ can still have a $1/r^3$ part, if it is time-oscillatory. Due to the fact that this part is proportional to Y_{2m}, no $\ln r/r$ terms arise upon inverting ∇^2, as only a Y_{00} term could produce a $\ln r/r$ (see Appendix A of IVb). Thus, $(N+\frac{1}{4}g^T)_{,0}$ may start as $f(t)/r$ with $f(t)$ oscillatory; hence N differs asymptotically from $-\frac{1}{4}g^T$ by $f(t)/r$ terms. [Such a term, of course, would not destroy our boundary conditions on $g_{\mu\nu}-\eta_{\mu\nu}$ which an $f(t)\sim t$ would.] The $f(t)/r$ behavior of $N_{,0}$ was all that was required for any of the proofs in the text. It is interesting to note, however, that $f(t)/r$ can easily be removed by a coordinate transformation where $\xi^0\sim\int^t dt\, f(t)/r$. In this new frame one does have $N_{,0}$ going faster than $1/r$.

Finally, we show directly that the canonical formalism's expressions for energy-momentum in the frame (2.5) are constants of motion. One merely computes $\oint dS_i(-g^T{}_{,i})_{,0}$ and $\oint dS_j(-2\pi^{ij})_{,0}$. This is done easily from the asymptotic field equations. One has from Eq. (C.2a)

$$g^T{}_{,i0}=g_{kk,0i}=[2N({}^3g)^{\frac{1}{2}}\pi^{kk}-\tfrac{1}{2}g^{kk}\pi+2\eta_{k|k}]_{,i}. \quad \text{(C.6)}$$

From the above discussion, the third member of Eq. (C.6) goes at least as $1/r^{\frac{5}{2}+\epsilon}$, so that the surface integral vanishes. Similarly, one may evaluate $\pi^{ij}{}_{,0}$ from Eq. (C.2b). One has

$$\oint dS_j\,\pi^{ij}{}_{,0}=\oint\{\tfrac{1}{2}\nabla^2 g_{ij}{}^{TT}-[\delta_{ij}\nabla^2(N+\tfrac{1}{4}g^T) -(N+\tfrac{1}{4}g^T)_{,ij}]\}dS_j, \quad \text{(C.7)}$$

and the right-hand side vanishes identically upon converting to a volume integral, since both terms in it have vanishing divergence.[20]

[20] Note that the above proof is essentially the converse of the results of Appendix A, that $h^T{}_{,i0}$ goes faster than $1/r^2$ if P^μ is conserved; of course, the constancy of P^μ was not assumed in estimating the behavior of the right-hand side of Eqs. (C.6,7).