Theory of Strongly Coupled Many-Fermion Systems. I. Convergence of Linked-Cluster Expansions*

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A strongly coupled system—the limiting case of a highly degenerate many-fermion system for which the variation of the kinetic energy is neglected, and the interaction restricted to a region of momentum space neighboring the Fermi surface—has been analyzed in a manner not dependent upon assumptions about the convergence of power series expansions or on partial summations of infinite series. The vacuum expectation value of the resolvent operator, $\langle 1/(H-z)\rangle_0$, is expressed as the Laplace transform of the exponential of a function linearly dependent on the volume of the system. It is shown that the linked-cluster expansion of the vacuum expectation value of the resolvent operator has a zero radius of convergence as a power series in the coupling constant. The most serious physical consequence of this is that a nontrivial interaction never results in a "normal" system.

INTRODUCTION

IN spite of the complexity of the quantum mechanical many-body problem, progress has been made in recent years towards the understanding of both "normal" and "superfluid" systems. What has been achieved, however, has often been founded on approximations or conjectures which determine the qualitative nature of the resulting solutions from the outset, and whose validity is difficult to establish on a rigorous basis. The investigation described in this article has been undertaken in an attempt to approach the many-fermion problem in such a way that one retains the possibility of obtaining qualitative information as long as possible, and so that one can obtain information which does not depend on assumptions about the convergence of power series expansions, or on partial summations of infinite series.

A salient feature of the many-fermion system is its extreme degeneracy before the effects of interaction are taken into account. The system to be analyzed is the limiting case of the degenerate many-fermion system in which the degeneracy is so great (the Fermi energy is so large) and the interaction so weak, that the variation of the kinetic energy can be neglected in the zero-order approximation—the kinetic-energy operator being replaced by its constant expectation value. We thus begin with a system all of whose "unperturbed" levels have precisely the same energy, the interaction term producing the entire level structure. The first step is then to diagonalize the interaction and to remove the degeneracy.

In this sense the situation is similar to that in strong-coupling meson theory[1]; however, what is really at issue here is the tremendous degeneracy of the system and the relatively *weak* interaction which couples only single-particle states close to the Fermi surface so that the variation of the kinetic energy is in fact relatively unimportant.

Although the strongly coupled system is a simplification, it is still in its essence a many-body problem—one for which most of the intricacy of the many-body dynamics has been retained. We thus present a simplified, intrinsically many-body problem from which we can obtain detailed information about analytic properties of the perturbation type expansions used in the analysis of many-body systems, and from which we hope to obtain qualitative information concerning the spectrum of such systems. This limit is thought to be of special relevance in the theory of finite nuclei[2] and in the theory of superconductivity; in the latter case it has in fact been shown[3] that the strong-coupling limit gives both qualitative and quantitive results similar to the full theory. One might hope, therefore, that the analysis of the strongly coupled system could be used to clarify the underlying pairing approximation in the theory of superconductivity.[4]

In this paper we develop the theory of the strongly coupled system with special emphasis on the separation of the volume dependence in a manner not assuming convergence of any series, and obtain explicit information on the analytic properties of perturbation expansion used for many-body systems. It seems likely that some of the results can be generalized to the full many-fermion problem.

In the first section the strongly coupled system is defined and the resolvent operator introduced. The vacuum expectation value of this operator is evaluated in the second section; this is obtained as the Laplace transform of the exponential of a function proportional to the volume of the system. The analytic properties of the resolvent operator are discussed in the third section. In particular it is shown that the linked-cluster expansion of the vacuum expectation value of the resolvent operator has a zero radius of convergence as a power series in the coupling constant.

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[1] S. M. Dancoff and R. Serber, Phys. Rev. **63**, 143 (1943).
[2] See, for example, B. R. Mottelson, in *The Many-Body Problem*, edited by C. Dewitt and P. Nozières (John Wiley & Sons, Inc., New York, 1959), p. 283.
[3] D. J. Thouless, Phys. Rev. **117**, 1256 (1960).
[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

1. THE STRONGLY COUPLED SYSTEM

The Hamiltonian of a system of particles interacting via two-body forces is written

$$H=H_0+gV, \tag{1.1}$$

where the kinetic term, H_0, is

$$H_0=\sum_\alpha \int \psi_\alpha^*(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2\right)\psi_\alpha(\mathbf{r})d\mathbf{r}, \tag{1.2}$$

the potential term is given by

$$V=\tfrac{1}{2}\sum_{\alpha,\beta}\int d\mathbf{r}_1 d\mathbf{r}_2\, \psi_\alpha^*(\mathbf{r}_1)\psi_\beta^*(\mathbf{r}_2) \times v(\mathbf{r}_1-\mathbf{r}_2)\psi_\beta(\mathbf{r}_2)\psi_\alpha(\mathbf{r}_1), \tag{1.3}$$

and the coupling constant, g, is written explicitly to make the investigation of analytic properties more convenient.

For a nonrelativistic Fermion system, ψ is a two-component spinor which satisfies the anticommutation relations

$$\{\psi_\alpha(\mathbf{r}),\psi_\beta(\mathbf{r}')\}=0;\quad \{\psi_\alpha(\mathbf{r}),\psi_\beta^*(\mathbf{r}')\}=\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'). \tag{1.4}$$

The strongly coupled system is defined by neglecting the variation of the kinetic-energy term in the Hamiltonian. This is done by replacing H_0 by its expectation value, $\langle H_0\rangle=T$, the constant average kinetic energy of the system. In order that such a system have a ground state it is necessary to limit the region of interaction to some finite domain in momentum space, to provide the cutoff usually provided by the energy denominators. This can be done by using a momentum cutoff or, as we shall do, by limiting the interaction to a shell surrounding the Fermi surface. The last, which preserves the symmetry between particles and holes, is particularly germane since, if the strong coupling approximation is to make sense, the single-particle states involved must not vary widely in kinetic energy. This implies that they are restricted to a small region surrounding the Fermi surface.

These conditions are most conveniently formulated in momentum space. Expanding $\psi_\alpha(\mathbf{r})$,

$$\psi_\alpha(\mathbf{r})=\frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma}u_\alpha{}^\sigma e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{1.5}$$

where Ω is the volume of the system and $u_\alpha{}^\sigma$ is also a two-component spinor, we obtain [for a spin-independent $v(\mathbf{r}_1-\mathbf{r}_2)$]

$$H_{sc}=T+gV=T+\tfrac{1}{2}g\sum_{\substack{\mathbf{k},\mathbf{k}',\mathbf{q},\sigma,\sigma'\\ \mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{q},\mathbf{k}'-\mathbf{q}\,\epsilon\,\text{shell}}} c_{\mathbf{k}+\mathbf{q},\sigma}^*c_{\mathbf{k}'-\mathbf{q},\sigma'}^* \times c_{\mathbf{k}',\sigma'}c_{\mathbf{k},\sigma}v(\mathbf{q}), \tag{1.6}$$

with

$$v(\mathbf{q})=\frac{1}{\Omega}\int v(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r},$$

and where we assume that $v(\mathbf{r})$ is bounded. The commutation relations for the c operators are

$$\{c_{\mathbf{k},\sigma},c_{\mathbf{k}'\sigma'}\}=0;\quad \{c_{\mathbf{k},\sigma},c_{\mathbf{k}',\sigma'}^*\}=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}. \tag{1.7}$$

We will simplify the notation by setting the constant T equal to zero, thus shifting all energies by T. By the notation: $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}+\mathbf{q}$, $\mathbf{k}-\mathbf{q}$ ϵ shell we mean that the magnitudes of the vectors $\mathbf{k}$ and $\mathbf{k}'$ satisfy the condition:

$$k_f-\delta\leq|\mathbf{k}|\leq k_f+\delta, \tag{1.8}$$

where presumably $\delta/k_f\ll 1$, while the vector $\mathbf{q}$ ranges over all values such that $\mathbf{k}+\mathbf{q}$ and $\mathbf{k}'-\mathbf{q}$ also satisfy (1.8). This has the effect of allowing only those scattering processes which take a particle or hole in the shell defined by (1.8) into another particle or hole in this shell.

In our treatment we shall assume that the total number of single-particle "unperturbed" states of both spins in the shell defined by (1.8) is $4N$ and that the Fermi surface is symmetrically placed so that the total number of particles is $2N$. N, of course, is proportional to Ω for δ constant. The vacuum state of the unperturbed system is defined so that all single-particle levels below k_F are filled, while all single-particle levels above k_F are empty. All other "unperturbed" configurations $|\phi_i\rangle$ (there are a total of $(4N)!/[(2N)!]^2$ configurations) can be obtained by creating holes below and particles above the Fermi surface, keeping N constant. These are constructed from the vacuum by operating on the vacuum with the creation operators $c_{\mathbf{k}\sigma}^*$, $c_{\mathbf{l},\sigma}$ where $\mathbf{k}$ lies above the Fermi surface and $\mathbf{l}$ lies below:

$$|\phi_{\mathbf{k}_1\cdots\mathbf{k}_s;\mathbf{l}_1\cdots\mathbf{l}_s}\rangle=c_{\mathbf{k}_1}^*\cdots c_{\mathbf{k}_s}^*c_{\mathbf{l}_1}\cdots c_{\mathbf{l}_s}|\phi_0\rangle. \tag{1.9}$$

We thus are presented with a well defined problem—that of diagonalizing a finite dimensional matrix whose matrix elements are those of the operator gV between unperturbed states.

Creation and annihilation operators can be defined, as has been pointed out by Hugenholtz and Van Hove[5], for arbitrary definitions of the vacuum states. One can consider as the vacuum state any of the unperturbed states of the system, referring all other states to this vacuum by the addition of particles or holes. With respect to this arbitrarily defined vacuum, $c_\mathbf{k}^*$ is a creation operator if $\mathbf{k}$ is one of the states unoccupied in the vacuum, and it is an annihilation operator if $\mathbf{k}$ is one of the states occupied in the vacuum. On the other hand, $c_\mathbf{k}$ is considered a creation operator if $\mathbf{k}$ is one of the states occupied in the vacuum. Any results obtained for vacuum expectation values of operators not depending explicitly upon a particular choice of the vacuum state are equally valid for any diagonal matrix elements with all quantities suitably redefined.

To analyze the strongly coupled system, we make use

[5] N. M. Hugenholtz and L. Van Hove, Physica **24**, 363 (1958).

of the resolvent operator, $R(z)$, defined by

$$R(z)\equiv\frac{1}{H-z}=\frac{1}{gV-z}. \qquad (1.10)$$

This operator has been discussed previously in connection with the many-body problem by Hugenholtz[6] and Van Hove.[7] An arbitrary matrix element of $R(z)$ can be written

$$\langle\phi_i|R(z)|\phi_j\rangle=\langle R(z)\rangle_{ij}. \qquad (1.11)$$

If we expand the states $\langle\phi_i|$ and $|\phi_j\rangle$ in eigenfunctions $|\psi_r\rangle$ of the Hamiltonian, gV, where

$$gV|\psi_r\rangle=E_r|\psi_r\rangle, \qquad (1.12)$$

so that

$$|\phi_i\rangle=\textstyle\sum_n a_n{}^i|\psi_n\rangle, \qquad (1.13)$$

we have

$$\langle R(z)\rangle_{ij}=\sum_n(a_n{}^i)^*a_n{}^j\frac{1}{E_n-z}. \qquad (1.14)$$

The analytic properties of $\langle R(z)\rangle_{ij}$ are immediately evident from this last expression, since the sum is finite. The matrix elements of the resolvent operator are analytic everywhere except for poles on the real axis, and these poles are bounded both above and below by the maximum and minimum energy levels $E_0\leq E\leq E_m$. Such maximum and minimum energies exist, since the strongly coupled system is finite and $v(r)$ is bounded. This will be discussed further in Sec. 3. An arbitrary matrix element of the resolvent operator, $\langle R(z)\rangle_{ij}$, may be expanded in powers of the coupling constant to give

$$\langle R(z)\rangle_{ij}=-\frac{1}{z}\sum_n(g/z)^n\langle V^n\rangle_{ij}, \qquad (1.15)$$

where $\langle V^n\rangle_{ij}$ is independent of g or z. From (1.14) we know that $\langle R(z)\rangle_{ij}$ is analytic in $1/z$ for $|z|>|E_{\max}|$ and so can conclude that (1.15) has a nonzero radius of convergence and that the radius of convergence is determined by the energy level of maximum absolute value

$$|z_g|=|E_{\max}(g)|. \qquad (1.16)$$

It is further true that the poles of matrix elements of $R(z)$ reveal the energy levels of the interacting system. However, it is clear that the absence of a pole from a particular matrix element at a point on the real axis does not indicate the absence of an energy level there. It may only indicate that a particular coefficient a_m does not appear in the expansion of the wave function. If, for example, we were considering the poles of $\langle\phi_0|R(z)|\phi_0\rangle$ as is done in order to obtain the Goldstone formula[8] and if the "true" ground state $|\psi_0\rangle$ had a different symmetry character from the state $|\phi_0\rangle$, then the coefficient a_0 would be zero and this pole would not appear. This may be related to the difficulty uncovered by Kohn and Luttinger.[9]

2. EVALUATION OF THE RESOLVENT OPERATOR

To evaluate the resolvent operator we make an expansion in powers of the coupling constant

$$R(z)=-\frac{1}{z}\sum_{n=0}^{\infty}(g/z)^nV^n, \qquad (2.1)$$

having established that such an expansion has a finite radius of convergence. We will be interested primarily in diagonal elements of the resolvent operator and, keeping in mind the possibility of defining any state as the vacuum state, we consider the vacuum expectation value of the resolvent operator $\langle\phi_0|R(z)|\phi_0\rangle\equiv\langle R(z)\rangle_0$:

$$\langle R(z)\rangle_0=-\frac{1}{z}\sum_{n=0}^{\infty}(g/z)^n\langle V^n\rangle_0. \qquad (2.2)$$

The crux of the problem lies in the evaluation of vacuum expectation values of powers of the interaction V.

For an interaction of the form (1.6) the evaluation of matrix elements like $\langle V^n\rangle_0$ has been treated in detail in many places. In a form particularly convenient for this discussion one can refer to Hugenholtz.[6] In brief, these matrix elements are evaluated using Wick's theorem.[10] Each matrix element is given by a sum over contractions of the operators and can be put into one-to-one correspondence with Feynman-like diagrams. There are two basic types of diagrams; the first, called connected diagrams, are proportional to the volume of the system; the second, or disconnected diagrams, have a volume dependence depending upon the number of connected diagrams of which they are composed. A connected diagram is defined as one which cannot be divided into two separate diagrams without cutting at least one line. A disconnected diagram can be so divided.

A traditional problem in the perturbation analysis of many-body systems is the volume dependence of higher order terms in the perturbation expansion. Both Goldstone[8] and Hugenholtz[6] have solved this problem by showing that the interesting physical quantities can be obtained from "linked-cluster" expansions which have a linear volume dependence, but their proofs depend upon the assumptions of convergence and existence which we would like to verify. For the strongly coupled system the volume dependence can be sorted out in a particularly transparent way and in such a way that no unwarranted assumptions concerning the analytic properties of the functions introduced need be made.

The matrix element $\langle V^n\rangle_0$ is the sum of all vacuum-

[6] N. M. Hugenholtz, Physica **23**, 481 (1957). See also N. M. Hugenholtz, in reference 2, p. 1.

[7] L. Van Hove, Physica **21**, 901 (1955); and **22**, 343 (1956).

[8] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).

[9] W. Kohn and J. M. Luttinger, Phys. Rev. **118**, 41 (1960).

to-vacuum graphs of order n. Some of these are connected and thus are linearly dependent on N; others are disconnected and have a dependence upon N proportional to the number of disconnected graphs of which they are composed. In order to make the volume dependence of the matrix elements explicit, we construct all graphs of a given order from connected graphs only. This is done by observing that if a number of connected graphs are combined to make a disconnected graph the weight of the disconnected graph is just the product of the weights of the connected graphs. This differs from the situation in the usual case in that one here does not have to consider the combinations of the energy denominators. One has only to consider the number of different ways connected graphs can be combined in order to produce disconnected graphs.

To explicitly decompose $\langle V^n\rangle_0$ into products of connected graphs we introduce S_n, which is the sum over all connected vacuum-to-vacuum graphs of order n; letting $S_0\equiv 0$, we can write

$$\langle V^n\rangle_0=S_n+\sum_{\alpha+\beta=n} S_\alpha S_\beta C_{\alpha\beta}{}^n+\sum_{\alpha+\beta+\gamma=n} S_\alpha S_\beta S_\gamma C_{\alpha\beta\gamma}{}^n+\cdots+S_1{}^n C_{11\cdots 1}{}^n. \quad (2.3)$$

Determining the coefficients, $C_{\alpha\beta\gamma\cdots}{}^n$ is a simple combinatorial problem. The number of graphs which can be made of s connected graphs of order α_1, $\alpha_2\cdots\alpha_s$, where $\alpha_1\neq\alpha_2\neq\alpha_3\cdots\neq\alpha_s$, is just

$$(\alpha_1+\alpha_2+\cdots\alpha_s)!/(\alpha_1!\alpha_2!\cdots\alpha_s!). \quad (2.4)$$

Taking into account duplications that occur if any of the α_i are equal, and those that occur in the summation over $\alpha_1+\alpha_2+\cdots\alpha_s=n$, we obtain

$$C_{\alpha_1\cdots\alpha_s}{}^n=\frac{1}{s!}\frac{n!}{\alpha_1!\cdots\alpha_s!}. \quad (2.5)$$

Inserting (2.5) into (2.3) and (2.3) into (2.2), we can write the vacuum expectation of the resolvent operator

$$\langle R(z)\rangle_0=-\frac{1}{z}-\frac{1}{z}\sum_{n=1}^{\infty}\left(\frac{g}{z}\right)^n \times\left\{S_n+\frac{1}{2!}\sum_{\alpha_1+\alpha_2=n} S_{\alpha_1}S_{\alpha_2}\frac{n!}{\alpha_1!\alpha_2!}+\cdots+S_1{}^n\right\}. \quad (2.6)$$

It will be shown later that the term proportional to the first power of the volume in (2.6), which is related to the Goldstone series,[8] does not exist as a power series in g. This is so because (2.6) is not absolutely convergent. To proceed further, we introduce an integral representation of the gamma function:

$$\frac{\Gamma(n+1)}{Z^{n+1}}=\frac{n!}{Z^{n+1}}=-\int_0^\infty dt\, e^{tz}(-t)^n, \quad (2.7)$$

which is defined for $\mathrm{Re}z<0$; we can then rewrite (2.2) to get

$$\langle R(z)\rangle_0=\sum_{n=0}\int_0^\infty dt\frac{(-gt)^n}{n!}e^{tz}\langle V^n\rangle_0. \quad (2.8)$$

The domain of definition is now the left half-plane with the semicircle $|z|<|z_g|$ omitted.

The integration and summation can be interchanged if the sequence $\{u_n(t)\}$ is uniformly and absolutely convergent, where

$$u_n(t)=\frac{(-gt)^n}{n!}e^{tz}\langle V^n\rangle_0, \quad (2.9)$$

since $\langle R(z)\rangle_0$ as defined by (2.8) exists in the domain defined above. Using the asymptotic expansion of the gamma function, it is easy to show that

$$|u_n(t)|\leq|(g^n/z^n)\langle V^n\rangle_0|, \quad (2.10)$$

so that, since the series (2.2) has a finite radius of convergence, we conclude that $\{u_n(t)\}$ is absolutely and uniformly convergent.

We can then write

$$\langle R(z)\rangle_0=\int_0^\infty e^{tz}dt\sum_{n=0}^{\infty}\frac{\langle V^n\rangle_0(-gt)^n}{n!}; \quad (2.11)$$

inserting (2.3) into (2.11) and using (2.5), we have

$$\langle R(z)\rangle_0=\int_0^\infty e^{tz}dt\left[1+\sum_{n=1}^{\infty}\left\{\frac{S_n}{n!}(-gt)^n+\frac{1}{2!}\sum_{\alpha+\beta=n}\frac{S_\alpha}{\alpha!}(-gt)^\alpha\frac{S_\beta}{\beta!}(-gt)^\beta+\cdots+\frac{S_1{}^n}{n!}(-gt)^n\right\}\right]. \quad (2.12)$$

The series above is shown in the Appendix to be absolutely convergent so that the orders of summation now can be interchanged. We then obtain

$$\langle R(z)\rangle_0=\int_0^\infty e^{tz}dt\left[1+B_0+\frac{1}{2!}B_0{}^2+\frac{1}{3!}B_0{}^3+\cdots\right], \quad (2.13)$$

where

$$B_0(-gt)=\sum_{n=1}^{\infty}\frac{S_n}{n!}(-gt)^n. \quad (2.14)$$

The bracketed term in (2.13) sums to an exponential yielding

$$\langle R(z)\rangle_0=\int_0^\infty e^{tz}dt\,\exp[B_0(-gt)]. \quad (2.15)$$

It is convenient at this point to recall that

$$S_n=\sum_{\text{connected graphs}}\langle\phi_0|V^n|\phi_0\rangle,$$

and that each such graph is linearly dependent on N as $N\to\infty$. To make this dependence explicit, we write

$$S_n=\sigma_n N, \tag{2.16}$$

where σ_n is a number which contains the details of the diagrammatic sums. Using this, we can rewrite (2.15) as

$$\langle R(z)\rangle_0=\int_0^\infty dt\, e^{tz}\exp\left[N\sum_{n=1}^{\infty}\frac{\sigma_n}{n!}(-gt)^n\right]. \tag{2.17}$$

We have in this way expressed $\langle R(z)\rangle_0$ as the Laplace transform of a function proportional to N. This function is a sum over connected diagrams and will be shown shortly to be convergent. We can invert (2.15) to obtain[11]

$$\exp[B_0(-gt)]=\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\langle R(z)\rangle_0 e^{-tz}dz, \tag{2.18}$$

where $\eta=\eta^*<-|E_{\max}|$. This is defined for $t>0$ but can be analytically continued into the entire t plane. If we now insert (1.14) into (2.18) and do the integration, we are left with

$$\exp[B_0(-gt)]=\sum|a_m|^2e^{-E_mt}. \tag{2.19}$$

The right-hand side of (2.19) can be identified as the vacuum expectation value of the U matrix,

$$\langle U(-it)\rangle_0=\textstyle\sum_m|a_m|^2e^{-E_mt}, \tag{2.20}$$

so that

$$\exp[B_0(-gt)]=\langle U(-it)\rangle_0. \tag{2.21}$$

It is clear from (2.14) and (2.21) that the ground state is proportional to N and to g. This is a direct consequence of the restrictions implicit in the strongly coupled system.

3. ANALYTIC PROPERTIES

The analytic simplicity of the strongly coupled system is due largely to the fact that an expansion in powers of the coupling constant, g, is at the same time an expansion in powers of $1/z$ so that analyticity at $g=0$ is equivalent to analyticity at $z=\infty$. We now proceed to an investigation of some of the analytic properties of the strongly coupled system.

It was stated in Sec. 1 that the expansion of $\langle R(z)\rangle_{ij}$ in powers of g/z had a finite radius of convergence. This depended upon the existence of a bound on the energy levels of the system. Now using a device similar to that employed by Yennie and Gartenhaus,[12] we can obtain an upper bound on the absolute value of any eigenstate—albeit an almost uselessly large upper bound—and thus establish explicitly the convergence of the power series expansion of $\langle R(z)\rangle_{ij}$.

[10] G. C. Wick, Phys. Rev. **80**, 268 (1950).
[11] D. V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, New Jersey, 1946). See especially theorem 7.6a, p. 69.
[12] D. R. Yennie and S. Gartenhaus, Nuovo cimento **9**, 59 (1958).

We note first that the vacuum expectation value of products of fermion operators satisfies the inequality

$$|\langle\phi_0|\cdots c\cdots c^*\cdots\;\cdots|\phi_0\rangle|\leq 1. \tag{3.1}$$

In fact, such vacuum expectation values will usually be zero. Referring to (1.6), we can then write

$$|\langle V^n\rangle_{ij}|\leq(\tfrac{1}{2})^n\sum_{\substack{k_1\cdots k_n;\,k_1'\cdots k_n';\\ \sigma_1\cdots\sigma_n;\,\sigma_1'\cdots\sigma_n'}}\Big|\sum_{q_1\cdots q_n}v(q_1)\cdots v(q_n)\Big|$$
$$=(8N^2)^n|\textstyle\sum_q v(q)|^n. \tag{3.2}$$

Defining

$$|\textstyle\sum_q v(q)|=v, \tag{3.3}$$

we have

$$|\langle V^n\rangle_{ij}|\leq(8vN^2)^n, \tag{3.4}$$

where v is independent of g, z, or N.

We can now introduce a comparison series

$$T(z)=\sum_{n=0}^{\infty}(g/z)^n(8vN^2)^n, \tag{3.5}$$

which is larger in absolute value term by term than $|z\langle R(z)\rangle_{ij}|$ defined using (1.15) as $(8vN^2)^n\geq|\langle V^n\rangle_{ij}|$. The series (3.5) can be summed to give

$$T(z)=-\frac{1}{8vN^2}\left(\frac{1}{g/z-1/8vN^2}\right), \tag{3.6}$$

so that $T(z)$ has a pole at $g/z=1/8vN^2$ and is analytic elsewhere. Since $|z\langle R(z)\rangle_{ij}|$ is bounded term by term by $T(z)$, (1.15) must be analytic for $g/z<1/8vN^2$ so that

$$|E_{\max}|\leq|8gvN^2|. \tag{3.7}$$

In spite of the fact that this upper bound has the wrong volume dependence it is sufficient to prove that, for N finite and for v bounded, a ground state exists and the expansion of $\langle R(z)\rangle_{ij}$ in powers of the coupling constant has a finite radius of convergence.

The fact that the coupling constant expansion of the resolvent operator has a nonzero radius of convergence does not guarantee that the related linked-cluster expansion which is linear in the volume of the system has a finite radius of convergence. We show below, on the contrary, that the linked-cluster expansion related to the resolvent operator expansion is not analytic in g at $g=0$.

To demonstrate this it is sufficient to consider the linked vacuum expectation value of the resolvent operator which is denoted after Hugenholtz[6] by $B_0(z)$

$$\langle\phi_0|R(z)|\phi_0\rangle_c\equiv B_0(z)$$
$$=-\frac{1}{z}\sum_{n=0}^{\infty}(g/z)^n\langle V^n\rangle_{0\text{ connected}}. \tag{3.8}$$

This function is used by Hugenholtz in his derivation of the Goldstone expression for the shift in ground-state

energy of the many-fermion system, and closely related functions are fundamental to his further conclusions concerning the many-fermion system.

Using the notation defined immediately preceding (2.3), we can write

$$B_0(z) = -\frac{1}{z}\left[1+\sum_{n=1}^{\infty} S_n(g/z)^n\right]. \qquad (3.9)$$

We will show that (3.9) has a zero radius of convergence by demonstrating that the related series $B_0(-gt)$ defined by (2.14) has a finite radius of convergence. It then follows that (3.9) has a zero radius of convergence.[13]

We can show that $B_0(-gt)$ has a finite radius of convergence as a power series in gt as follows. From (2.21) we see that $\langle U(-it)\rangle_0$ can have singularities or zeros at points in the complex t plane only if $B_0(-gt)$ becomes singular at these points. We conclude from (2.20) that $\langle U(-it)\rangle_0$ is an entire function of t, but from (2.20) we can also show that in all but trivial cases $\langle U(-it)\rangle_0$ must be zero at some point in the finite t plane. In fact we can show that either

$$\langle U(-it)\rangle_0 = Ae^{Bt}, \qquad (3.10)$$

where A and B are constants, or $\langle U(-it)\rangle_0$ has at least one zero in the finite t plane.

The last statement is established by using Hadamard's factorization theorem[14] which in this case asserts that an integral function of order ρ with no zeros can be written in the form

$$f(z) = e^{Q(z)}, \qquad (3.11)$$

where $Q(z)$ is polynomial of degree not greater than ρ. Since $\sum |a_m|^2 e^{-E_m t}$ can increase as $t\to\infty$ no faster than $Ae^{|E_{\max}|t}$, it is of order one. Therefore

$$Q(z) = Bz + C, \qquad (3.12)$$

so using (3.12) and (3.11) we see that if $\langle U(-it)\rangle_0$ has no zeros in the complex t plane it must have the form (3.10).[15]

It follows that $B_0(-gt)$ has an infinite radius of convergence if and only if the system has only a single energy level. We can therefore conclude that for the strongly coupled system, if the energy spectrum contains more than one energy level the expansion of $B_0(z)$, which is the connected diagram expansion of $\langle R(z)\rangle_0$, has a zero radius of convergence as a power series in the coupling constant.

[13] By the Cauchy test, the radius of convergence, R, of a series $\Sigma_n a_n X^n$ is given by $R=\lim_{n\to\infty}|a_n/a_{n+1}|$. Suppose two series, denoted by S_1 and S_2, are given by $S_1=\Sigma\, b_n X^n$, and $S_2=\Sigma (b_n/n!)X^n$. Then, if the radius of convergence of S_2 is finite so that $\lim_{n\to\infty}(n+1)|b_n/b_{n+1}|=R_2<M$, it follows that $R_1\leq\lim_{n\to\infty}M/(n+1)=0$.

[14] E. C. Titchmarsh, *The Theory of Functions* (Clarendon Press, 1939), 2nd ed. See especially Chap. 8.

[15] The author wishes to express his appreciation to Dr. C. Davis and Dr. J. Werner for pointing out the relevance of Hadamard's theorem to the problem of the zeros of $\langle U(-it)\rangle_0$.

The linked-cluster expansion, $B_0(z)$, is most closely related to zero-temperature perturbation expressions. Related finite-temperature perturbation expansions, such as those used by Matsubara[16] or by Bloch and De Dominicis[17] involve the grand partition function whose analytic properties are similar to those of the U matrix, whose diagonal elements are expressed by (2.21) as the exponential of a linked-cluster expansion $B_0(-gt)$.

We can show that the power series expansion of $B_0(-gt)$ given by (2.14) has a finite radius of convergence by combining (2.14) and (2.19) to get

$$\exp\left(N\sum_{n=0}^{\infty}\frac{\sigma_n}{n!}(-gt)^n\right) = \langle U(-it)\rangle_0. \qquad (3.13)$$

Since $\langle U(-it)\rangle_0$ is, as noted above, an entire function of t, singularities of $B_0(-gt)$ must correspond to its zeros. However $\langle U(0)\rangle_0=1$ and there must exist some neighborhood surrounding $t=0$ for which $\langle U(-it)\rangle_0\neq 0$. Therefore $B_0(-gt)$ must be analytic in the neighborhood mentioned above. It then follows that its series expansion has a finite radius of convergence. Such quantities as the ground state can be obtained using (2.19) as

$$E_0 = -\lim_{t\to\infty}\left[\frac{B_0(-gt)}{t}\right], \qquad (3.14)$$

but they cannot be calculated as a power series in g since the series (2.14) has only a finite radius of convergence in gt.

Nondiagonal matrix elements of the U matrix can be related to the linked-cluster expansion of such matrix elements by

$$\langle i|U|j\rangle = \langle i|U|j\rangle_L\, \langle 0|U|0\rangle. \qquad (3.15)$$

The diagrams of which $\langle i|U|j\rangle_L$ is composed contain no vacuum-to-vacuum components—that is, no components of the form $\langle\phi_0|A|\phi_0\rangle$—but they are not necessarily connected. Since the matrix element of any scattering process in the presence of the "real" vacuum is composed of linked graphs only, we must compute

$$\langle i|U|j\rangle_L = \langle i|U|j\rangle / \langle 0|U|0\rangle. \qquad (3.16)$$

From the previous arguments we can conclude that $\langle i|U|j\rangle$ is an entire function of g; the matrix element $\langle 0|U|0\rangle$ we know has zeros in the finite g (or t) plane, but there exists a neighborhood about $g=0$ for which $\langle 0|U|0\rangle\neq 0$. Therefore $\langle i|U|j\rangle_L$ has a finite region of analyticity about $g=0$ and a finite radius of convergence as a power series expansion in g. This last result is similar to one obtained by Yennie and Gartenhaus.[12]

4. CONCLUSION

The basic result we have obtained is that, for the strongly coupled system, the linked-cluster expansion

[16] T. Matsubara, Progr. Theoret. Phys. (Kyoto) **14**, 351 (1955).

[17] C. Bloch and C. De Dominicis, Nuclear Phys. **7**, 459 (1958).

of the vacuum expectation value of the resolvent operator has a zero radius of convergence as a power series in the coupling constant. The related linked-cluster expansion—that which occurs in the U matrix or the partition function—is, however, a convergent series for small enough values of the coupling constant. We therefore have available a method for calculating the properties of the strongly coupled system in terms of quantities which have a linear dependence on the volume.

A question that occurs immediately is whether these arguments can be generalized. Although this has not yet been done, comparison of the linked-cluster expansion of the resolvent operator for the strongly coupled system and the full many-body system seems to make it unlikely that one should converge if the other does not. The many-body resolvent operator is not likely in any case to have an expansion in powers of $1/z$, but this is not related so easily to an expansion in powers of g as in the strongly coupled case.

For those many-fermion systems for which the strongly coupled system is a good limiting approximation, the most serious implication of the divergence of the linked-cluster expansion (aside from the doubt it casts on any formal results obtained from a manipulation of this expansion) is that such many-fermion systems are not "normal" systems. This already appears to be quite generally true for attractive potentials where superconductivity occurs, and has also been suggested for repulsive potentials.[18] In the case of the superconductor it is known that the qualitative nature of the actual solution is entirely different from that of the "normal" solution. Whether or not this is the case for repulsive potentials is not known. The divergence of the linked-cluster expansion may be related to some phenomenon such as zero sound[19] while a physically interesting class of normal solutions still exists. On the other hand, it may indicate that even for repulsive potentials there are qualitative changes in the system that are not yet understood. At the very least, one should be cautious in drawing conclusions from calculations of energy shifts and other properties of many-body systems using expansions that may not be convergent. One has from the beginning guaranteed that no new qualitative features will be introduced in a situation where they may very well exist.

In spite of this, one can be somewhat sanguine about the use of even nonconvergent linked-cluster expansions as some manipulations do lead to correct results. Hugenholtz,[6] for example, has shown that the ground-state energy shift for the full system is given by $\lim_{z\to 0}[z^2 B_0(z)]$. This also can turn out to be the case for the strongly coupled system, but one must not expand $B_0(z)$ in powers of g; that expansion does not exist.

In conclusion we should like to suggest that the strongly coupled system treated here offers the possibility of obtaining information concerning the many-fermion system by approximations quite different from those usually made, approximations which may be more appropriate for many-body systems or which may serve to illuminate some of the qualitative properties of many-body systems. It also suggests a possible connection between the theory of entire functions of finite order, or the theory of Dirichlet series, and the many-fermion problem as the zeros and rate of growth of the U matrix will be intimately connected with the degeneracy and spacing of the energy spectrum.

Work is being continued on these and related questions.

APPENDIX. PROOF OF THE ABSOLUTE CONVERGENCE OF (2.12)

The U matrix is defined by

$$U(-it)=\exp(-gVt); \tag{A1}$$

its vacuum expectation value expanded in power of g is

$$\langle U\rangle_0=\sum_{n=0}^{\infty}\frac{(-gt)^n}{n!}\langle V^n\rangle_0. \tag{A2}$$

Since the radius of convergence of (2.2) is finite, that of (A2) is infinite,[13] so that $\langle U(-it)\rangle_0$ is an entire function of gt. Further, $\langle U(0)\rangle_0=1$, so there exists a neighborhood, $|gt|<\eta$, about $gt=0$ in the complex gt plane in which $\langle U(-it)\rangle_0\neq 0$.

We therefore have

$$\langle U(-it)\rangle_0=\exp[\mathcal{L}(-gt)], \tag{A3}$$

where $\mathcal{L}(-gt)$ is analytic in the neighborhood defined above, and can be written

$$\mathcal{L}(-gt)=\sum_{n=0}^{\infty}\frac{a_n}{n!}(-gt)^n, \tag{A4}$$

with $a_0=0$. Since (A4) is absolutely convergent we can manipulate the expansion of $\langle U(-it)\rangle_0$ to obtain

$$\begin{aligned}\langle U(-it)\rangle_0 &=1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\alpha_1\cdots\alpha_n}\frac{a_{\alpha_1}}{\alpha_1!}\cdots\frac{a_{\alpha_n}}{\alpha_n!}(-gt)^{\alpha_1}\cdots(-gt)^{\alpha_n}\\ &=1+\sum_{n=1}^{\infty}(-gt)^n\left\{\frac{a_n}{n!}+\frac{1}{2!}\sum_{\alpha_1+\alpha_2=n}\frac{a_{\alpha_1}}{\alpha_1!}\frac{a_{\alpha_2}}{\alpha_2!}+\cdots+\frac{a_1{}^n}{n!}\right\},\end{aligned} \tag{A5}$$

which is valid at least for $|gt|<\eta$.

[18] L. Van Hove, International Congress on Many-Particle Problems, Utrecht, June, 1960 (unpublished). During this conference Van Hove also presented an analysis of the ideal Fermi gas which showed that a linked-cluster expansion had poorer convergence properties than the corresponding expansion including all diagrams.

[19] L. D. Landau, Zhur. Eksp. i Teoret. Fiz. **32**, 59 (1957) [translation: Soviet Phys.—JETP **5**, 101 (1957)].

On the other hand, if we insert (2.3) into (A2) we get

$$\langle U(-it)\rangle_0=1+\sum_{n=1}^{\infty}(-gt)^n\left\{\frac{S_n}{n!}+\frac{1}{2!}\sum_{\alpha_1+\alpha_2=n}\frac{S_{\alpha 1}}{\alpha_1!}\frac{S_{\alpha 2}}{\alpha_2!}+\cdots\frac{S_1{}^n}{n!}\right\}. \quad \text{(A6)}$$

Since both (A5) and (A6) are convergent in the neighborhood defined, the coefficients of $(-gt)^n$ can be equated to yield

$$a_n=S_n. \quad \text{(A7)}$$

We therefore can conclude that (A6) and at the same time (2.12) is absolutely convergent. This justifies the change of order of summation following (2.12).

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Radiation from Fast Particles Moving through Magnetic Materials*

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The problem of the generation of a changing magnetic field due to the interaction of a fast particle with a magnetic medium is studied. This combined Čerenkov-spin wave effect is shown to give rise to a "ringing" of the spin system under certain conditions of frequency and angle of observation, at least within an approximate evaluation of the general Green's function for the problem. Some striking differences from the usual Čerenkov effect are discussed and possibilities of using this effect as a neutral magnetic moment detector or as a probe of magnetic materials are mentioned briefly.

I. INTRODUCTION

CONSIDER the lecture-demonstration apparatus consisting of a two-dimensional array of compass needles suspended on pins, over which one passes a bar magnet. Then the short-range magnetic forces between the bar magnet and individual needles, combined with the interaction of the needles among themselves, gives rise to waves of motion of the needles, which could be used, for example, as a signal that a bar-magnet passed by.[1]

In the above demonstration, the waves are short ranged, and the oscillations rapidly die out. However, if we consider the microscopic problem of a point magnetic dipole moving through a nonmagnetic medium, we would expect a long-range effect, namely Čerenkov radiation,[2] although the intensity would be very low. Now the question arises, would it not be possible to have the best of both cases, namely, the long-range Čerenkov radiation, and a reasonably large magnetic field change due to cooperative waves in a magnetic material? In other words, could not, at least at certain angles, the magnetic fields in Čerenkov radiation generate spin waves, which in turn give changing magnetic fields, etc., so that the combined particle-spin system "rings"?

In order to answer these questions, we must study the equations for the radiation field, and for the spin system for magnetic materials. In Sec. II we write the separate Maxwell and spin-wave equations in a form convenient for obtaining their joint solutions. In Sec. III we formally solve these equations for arbitrary external sources, and see to what the intuitive feeling expressed in the above question corresponds. In Sec. IV we give an approximate evaluation of the complete Green's function for the problem, while in Sec. V we consider special cases of this approximate Green's function and look at the possibilities, in this approximation, for "ringing" the system. We also consider certain other effects which arise in this problem and which are different from the usual radiation in nonmagnetic materials. In Sec. VI we consider the special cases of point charges and magnetic moments, while finally in Sec. VII we summarize the results and consider some applications of these effects.

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[1] Playing with this simple device shows the marked dependence of the amplitude of the waves on the velocity and orientation of the little magnet. (The magnet had better not be big, or the needles jump off their pins.)

[2] See W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), Chap. 19, particularly problem 8 on p. 313.

II. THE MAXWELL EQUATIONS AND THE SPIN-WAVE EQUATION

Consider a general medium of dielectric constant ϵ and conductivity σ, in which there may be a net magnetization and through which an external electric charge current $\mathbf{j}_0$ moves. Then Maxwell's equations giving the electric and magnetic fields,[2]

$$\begin{aligned}&\nabla\cdot\mathbf{D}=0, && \nabla\cdot\mathbf{B}=0,\\ &\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t, && \nabla\times\mathbf{H}=\mathbf{j}+\partial\mathbf{D}/\partial t,\end{aligned} \quad (1)$$

are to be combined with

$$\mathbf{D}=\epsilon\mathbf{E},\quad \mathbf{j}=\mathbf{j}_0+\sigma\mathbf{E}, \quad (2)$$