On the other hand, if we insert (2.3) into (A2) we get

$$\langle U(-it)\rangle_0=1+\sum_{n=1}^{\infty}(-gt)^n\left\{\frac{S_n}{n!}+\frac{1}{2!}\sum_{\alpha_1+\alpha_2=n}\frac{S_{\alpha 1}}{\alpha_1!}\frac{S_{\alpha 2}}{\alpha_2!}+\cdots\frac{S_1{}^n}{n!}\right\}. \quad \text{(A6)}$$

Since both (A5) and (A6) are convergent in the neighborhood defined, the coefficients of $(-gt)^n$ can be equated to yield

$$a_n=S_n. \quad \text{(A7)}$$

We therefore can conclude that (A6) and at the same time (2.12) is absolutely convergent. This justifies the change of order of summation following (2.12).

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Radiation from Fast Particles Moving through Magnetic Materials*

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The problem of the generation of a changing magnetic field due to the interaction of a fast particle with a magnetic medium is studied. This combined Čerenkov-spin wave effect is shown to give rise to a "ringing" of the spin system under certain conditions of frequency and angle of observation, at least within an approximate evaluation of the general Green's function for the problem. Some striking differences from the usual Čerenkov effect are discussed and possibilities of using this effect as a neutral magnetic moment detector or as a probe of magnetic materials are mentioned briefly.

I. INTRODUCTION

CONSIDER the lecture-demonstration apparatus consisting of a two-dimensional array of compass needles suspended on pins, over which one passes a bar magnet. Then the short-range magnetic forces between the bar magnet and individual needles, combined with the interaction of the needles among themselves, gives rise to waves of motion of the needles, which could be used, for example, as a signal that a bar-magnet passed by.[1]

In the above demonstration, the waves are short ranged, and the oscillations rapidly die out. However, if we consider the microscopic problem of a point magnetic dipole moving through a nonmagnetic medium, we would expect a long-range effect, namely Čerenkov radiation,[2] although the intensity would be very low. Now the question arises, would it not be possible to have the best of both cases, namely, the long-range Čerenkov radiation, and a reasonably large magnetic field change due to cooperative waves in a magnetic material? In other words, could not, at least at certain angles, the magnetic fields in Čerenkov radiation generate spin waves, which in turn give changing magnetic fields, etc., so that the combined particle-spin system "rings"?

In order to answer these questions, we must study the equations for the radiation field, and for the spin system for magnetic materials. In Sec. II we write the separate Maxwell and spin-wave equations in a form convenient for obtaining their joint solutions. In Sec. III we formally solve these equations for arbitrary external sources, and see to what the intuitive feeling expressed in the above question corresponds. In Sec. IV we give an approximate evaluation of the complete Green's function for the problem, while in Sec. V we consider special cases of this approximate Green's function and look at the possibilities, in this approximation, for "ringing" the system. We also consider certain other effects which arise in this problem and which are different from the usual radiation in nonmagnetic materials. In Sec. VI we consider the special cases of point charges and magnetic moments, while finally in Sec. VII we summarize the results and consider some applications of these effects.

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[1] Playing with this simple device shows the marked dependence of the amplitude of the waves on the velocity and orientation of the little magnet. (The magnet had better not be big, or the needles jump off their pins.)

[2] See W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), Chap. 19, particularly problem 8 on p. 313.

II. THE MAXWELL EQUATIONS AND THE SPIN-WAVE EQUATION

Consider a general medium of dielectric constant ϵ and conductivity σ, in which there may be a net magnetization and through which an external electric charge current $\mathbf{j}_0$ moves. Then Maxwell's equations giving the electric and magnetic fields,[2]

$$\begin{aligned}&\nabla\cdot\mathbf{D}=0, &&\nabla\cdot\mathbf{B}=0,\\ &\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t, &&\nabla\times\mathbf{H}=\mathbf{j}+\partial\mathbf{D}/\partial t,\end{aligned} \quad (1)$$

are to be combined with

$$\mathbf{D}=\epsilon\mathbf{E},\quad \mathbf{j}=\mathbf{j}_0+\sigma\mathbf{E}, \quad (2)$$

and in particular with

$$\mathbf{H}=\mathbf{B}/\mu_0-\mathbf{M}, \tag{3}$$

to give the equation governing the change in magnetic field:

$$\nabla^2\mathbf{H}-\epsilon\mu_0\left[\frac{\partial}{\partial t}+\frac{\sigma}{\epsilon}\right]\frac{\partial\mathbf{H}}{\partial t} = -\nabla\times\mathbf{j}_0+\epsilon\mu_0\left[\frac{\partial}{\partial t}+\frac{\sigma}{\epsilon}\right]\frac{\partial\mathbf{M}}{\partial t}-\nabla(\nabla\cdot\mathbf{M}). \tag{4}$$

If, more generally, there is also an external magnetic moment current $\mathbf{M}_{\text{ext}}$, then the last term in Eq. (4) can be replaced by $\mathbf{J}$, where

$$\mathbf{J}=-\nabla\times\mathbf{j}_0+\epsilon\mu_0\left(\frac{\partial}{\partial t}+\frac{\sigma}{\epsilon}\right)\frac{\partial\mathbf{M}_{\text{ext}}}{\partial t}-\nabla(\nabla\cdot\mathbf{M}_{\text{ext}}). \tag{5}$$

In magnetic materials, the link is provided by Eq. (3) to the equation which governs the magnetization,[3] the spin-wave equation:

$$d\mathbf{M}/dt=(\mathfrak{D}/M_0)\mathbf{M}\times\nabla^2\mathbf{M}+\gamma\mathbf{M}\times\mathbf{H}+(\alpha/M_0)\mathbf{M}\times(d\mathbf{M}/dt). \tag{6}$$

In Eq. (6), we have assumed a uniform net magnetization M_0 when there are no currents $\mathbf{J}$. The constants $\mathfrak{D}$, α, and γ are, respectively, the exchange constant,[4] the damping constant,[5] and the gyromagnetic ratio.

The two equations (4) and (6) are to be solved simultaneously, subject to the existence of a constant, uniform magnetic field H_0 and magnetization M_0 when there are no external currents $\mathbf{J}$. We take H_0 and M_0 along the x axis (M_0 may point in the negative x direction).

We now Fourier-analyze the equations, with the convention

$$A(\mathbf{r},t)=(2\pi)^{-2}\int d^3k d\omega\, A(\mathbf{k},\omega)\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]. \tag{7}$$

Equation (4) becomes (with $c=1$ for convenience)

$$(-k^2+n^2\omega^2)\mathbf{H}(\mathbf{k},\omega) =\mathbf{J}(\mathbf{k},\omega)-n^2\omega^2\mathbf{M}(\mathbf{k},\omega)+\mathbf{k}[\mathbf{k}\cdot\mathbf{M}(\mathbf{k},\omega)], \tag{8}$$

where

$$n^2=\epsilon\mu_0\left(1+\frac{i\sigma}{\omega\epsilon}\right). \tag{9}$$

In component form, in which only terms linear in the changing $\mathbf{M}$ and $\mathbf{H}$ are kept, Eq. (6) becomes

$$\begin{aligned} M_z(\mathbf{k},\omega)&=\frac{(k^2\mathfrak{D}+\omega_0)\omega_s H_z-i\omega\omega_s H_y}{(k^2\mathfrak{D}+\omega_0)^2-\omega^2},\\ M_y(\mathbf{k},\omega)&=\frac{(k^2\mathfrak{D}+\omega_0)\omega_s H_y+i\omega\omega_s H_z}{(k^2\mathfrak{D}+\omega_0)^2-\omega^2},\\ M_x(\mathbf{k},\omega)&\sim O(M^2,MH)\sim 0. \end{aligned} \tag{10}$$

In Eqs. (10),

$$\begin{aligned}\omega_0&=\omega_0'+i\alpha\omega=\gamma H_0+i\alpha\omega,\\ \omega_s&=\gamma M_0.\end{aligned} \tag{11}$$

Equations (10) may be written very neatly if, for a *vector* quantity $\mathbf{A}(\mathbf{k},\omega)$, we define the particular combinations of components

$$A_\pm=A_z\pm iA_y. \tag{12}$$

Then Eqs. (10) become

$$M_\pm=\frac{[(k^2\mathfrak{D}+\omega_0)\omega_s\mp\omega\omega_s]}{(k^2\mathfrak{D}+\omega_0)^2-\omega^2}H_\pm, \tag{13}$$

or, more succinctly,

$$M_\pm=(\omega_s/D_\pm)H_\pm. \tag{14}$$

Here, the denominator $D_\pm$ is *not* a combination of vector components, but is defined as

$$D_\pm=k^2\mathfrak{D}+\omega_0\pm\omega. \tag{15}$$

(No confusion is likely to arise in practice.)

Using the last of Eqs. (10) and the definition Eq. (12), we get

$$\mathbf{k}\cdot\mathbf{M}=\tfrac{1}{2}(k_+M_-+k_-M_+), \tag{16}$$

so that Eq. (4) may be written

Maxwell:

$$(-k^2+n^2\omega^2)H_\pm =J_\pm-n^2\omega^2M_\pm+\tfrac{1}{2}k_\pm(k_+M_-+k_-M_+). \tag{17}$$

Equations (13) and (17) are then the equations whose joint solutions we seek.

III. COMBINED SPIN-MAXWELL EQUATIONS

Before proceeding to combine Eqs. (13) and (17), we point out what corresponds to the intuitive feelings about "ringing" the system expressed in the introduction. If we start to solve Eqs. (13) and (17) by iteration, then the zeroth approximation to solution of Eq. (17) will be

$$H_\pm^{(0)}(\mathbf{k},\omega)=J_\pm(\mathbf{k},\omega)/(-k^2+n^2\omega^2). \tag{18}$$

As we will see in Sec. VI, this is just the usual radiation from a current, and for point particles gives the Čerenkov radiation. If we use this approximation in Eq. (13) ("driving" the spin system with the Čerenkov

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[3] See R. F. Soohoo, Phys. Rev. **120**, 1978 (1960). We would like to thank Dr. Soohoo for sending us the results of his investigation prior to publication. The literature on spin waves is very extensive, and we refer only to the recent review by J. Van Kranendonk and J. H. Van Vleck, Revs. Modern Phys. **30**, 1 (1958).

[4] See, for example, C. Kittel, Phys. Rev. **110**, 1295 (1958).

[5] See, for example, H. Suhl, Proc. Inst. Radio Engrs. **44**, 1270 (1956).

radiation), then the resulting change in magnetization is

$$M_{\pm}^{(1)}(\mathbf{k},\omega)=\frac{\omega_s}{(-k^2+n^2\omega^2)}\frac{J_{\pm}(\mathbf{k},\omega)}{D_{\pm}}. \tag{19}$$

If, now, this is put back in Eq. (17), then the "ringing" will occur in the "radiative" (i.e., $k^2=n^2\omega^2$) mode for H_- when

$$D_-|_{k^2=n^2\omega^2}=0=n^2\omega^2\mathfrak{D}+\omega_0-\omega. \tag{20}$$

Since $n^2\omega^2\mathfrak{D}$ is small ($\sim 10^{-3}$ in the kMc/sec region) the resonant frequency ω is almost ω_0. While the above iterative treatment of the spin-Maxwell system of equations is intuitively appealing and makes more plausible the original conjecture, as we will see below it is very misleading and should not be taken too seriously.

We now return to combining Eqs. (13) and (17); by straightforward algebra, we get

$$H_{\pm}(\mathbf{k},\omega)=[\Delta(\mathbf{k},\omega)]^{-1}[\Lambda_{\pm\pm}J_{\pm}+\Lambda_{\pm\mp}J_{\mp}], \tag{21}$$

where the denominator Δ is

$$\begin{aligned}\Delta(\mathbf{k},\omega)=(D_+D_-/\omega_s^2)(-k^2+n^2\omega^2)^2\\+[(D_++D_-)/\omega_s](-k^2+n^2\omega^2)\\\times(-\tfrac{1}{2}k_+k_-+n^2\omega^2)+n^2\omega^2(-k_+k_-+n^2\omega^2).\end{aligned} \tag{22}$$

The numerators are

$$\begin{aligned}\Lambda_{\pm\pm}(\mathbf{k},\omega)=(D_{\pm}/\omega_s)[(D_{\mp}/\omega_s)\\\times(-k^2+n^2\omega^2)+n^2\omega^2-\tfrac{1}{2}k_+k_-],\end{aligned} \tag{23}$$

and

$$\Lambda_{\pm\mp}(\mathbf{k},\omega)=\tfrac{1}{2}(k_{\pm})^2D_{\pm}/\omega_s. \tag{24}$$

These functions depend on the magnitude of **k** and *also* on the directions of **k** through the k_+k_- and $(k_{\pm})^2$; if the angles of **k** relative to the external axis, H_0, are θ_{kx}, φ_{kx}, then

$$\begin{aligned}k_+k_-=k^2\sin^2\theta_{kx},\\(k_{\pm})^2=(\pm)k^2\sin^2\theta_{kx}\exp[\mp(2i\varphi_{kx})].\end{aligned} \tag{25}$$

In order to put the complete answer, Eq. (21), in a form which makes the Čerenkov-like behavior obvious, we invert the momentum part of the Fourier transform according to Eq. (7). Then we get

$$\begin{aligned}H_{\pm}(\mathbf{r},\omega)=\int d^3r'\{G_{\pm\pm}(\mathbf{r},\mathbf{r}';\omega)J_{\pm}(\mathbf{r}',\omega)\\+G_{\pm\mp}(\mathbf{r},\mathbf{r}';\omega)J_{\mp}(\mathbf{r}',\omega)\}.\end{aligned} \tag{26}$$

In this form, the Green's functions $G_{\dots}(\mathbf{r},\mathbf{r}';\omega)$ are given by

$$\begin{aligned}G_{\dots}(\mathbf{r},\mathbf{r}';\omega)=(2\pi)^{-3}\int d^3k\exp[i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})]\\\times\Lambda_{\dots}(\mathbf{k},\omega)/\Delta(\mathbf{k},\omega).\end{aligned} \tag{27}$$

In Eq. (27), the numerators $\Lambda_{\dots}(\mathbf{k},\omega)$ may be pulled out of the integral, and in the usual[2] way replaced by the appropriate differential operators. Thus,

$$\begin{aligned}G_{\dots}(\mathbf{r},\mathbf{r}';\omega)=(2\pi)^{-3}\Lambda_{\dots}(-i\nabla_\rho,\omega)\\\times\int d^3k\exp(i\mathbf{k}\cdot\boldsymbol{\rho})/\Delta(\mathbf{k},\omega),\end{aligned} \tag{28}$$

where we define

$$\boldsymbol{\rho}=\mathbf{r}'-\mathbf{r}. \tag{29}$$

We are really interested, therefore, in a "reduced" Green's function

$$\begin{aligned}g(\mathbf{r},\mathbf{r}';\omega)\equiv g(\rho,\Theta_{\rho x},\Phi_{\rho x};\omega)\\=(2\pi)^{-3}\int d^3k\exp(i\mathbf{k}\cdot\boldsymbol{\rho})/\Delta(\mathbf{k},\omega),\end{aligned} \tag{30}$$

where we have indicated explicitly that g depends on the angles $\Theta_{\rho x}$, $\Phi_{\rho x}$ of the vector $\boldsymbol{\rho}$ relative to the given external axis defined by H_0.

IV. EVALUATION OF THE REDUCED GREEN'S FUNCTION

The evaluation of the integral in Eq. (30) depends, of course, on the functional dependence of Δ on **k**. We rewrite the expression for Δ of Eq. (22) in a different form [using the first of Eqs. (25)] as

$$\Delta(\mathbf{k},\omega)=F_1(k^2)+F_2(k^2)f(\theta_{k\rho},\varphi_{k\rho};\Theta_{\rho x},\Phi_{\rho x}), \tag{31}$$

where $\theta_{k\rho}$ and $\varphi_{k\rho}$ are the angles of **k** relative to $\boldsymbol{\rho}$.[6] In particular, we choose the function f to be

$$\begin{aligned}f=\cos^2\theta_{kx}=[\cos\theta_{k\rho}\cos\Theta_{\rho x}\\+\sin\theta_{k\rho}\sin\Theta_{\rho x}\cos(\varphi_{k\rho}-\Phi_{\rho x})]^2.\end{aligned} \tag{32}$$

For reference, then,

$$\begin{aligned}F_1(k^2)=(D_+D_-/\omega_s^2)(-k^2+n^2\omega^2)^2\\+[(D_++D_-)/\omega_s](-k^2+n^2\omega^2)\\\times(-\tfrac{1}{2}k^2+n^2\omega^2)+n^2\omega^2(-k^2+n^2\omega^2),\end{aligned} \tag{33}$$

and

$$F_2(k^2)=[(D_++D_-)/\omega_s](-k^2+n^2\omega^2)(\tfrac{1}{2}k^2)+n^2\omega^2. \tag{34}$$

In this form, we are interested in

$$\begin{aligned}g=(2\pi)^{-3}\int k^2dkd(\cos\theta_{k\rho})d\varphi_{k\rho}\\\times\frac{e^{ik\rho\theta_{k\rho}}}{F_1(k^2)+F_2(k^2)f(\theta_{k\rho},\varphi_{k\rho};\Theta_{\rho x},\Psi_{\rho x})}.\end{aligned} \tag{35}$$

As it stands, the integrals in Eq. (35) are too complicated to perform. However, the form of the function f is such that when $\theta_{k\rho}$ is 0 or π,

$$f(0,\varphi_{k\rho};\Theta_{\rho x},\Psi_{\rho x})=f(\pi,\varphi_{k\rho};\Theta_{\rho x},\Phi_{\rho x})=\cos^2\Theta_{\rho x}, \tag{36}$$

[6] This choice simplifies the exponent. The other obvious choice, of measuring the angles of **k** from $\mathbf{H}_0$, simplifies Δ somewhat, but complicates the exponent.

and is independent of $\varphi_{k\rho}$ (and of $\Phi_{\rho x}$). This makes possible the usual[7] integration by parts for such Green's functions in order to get a series in powers of $1/\rho$. We show in Appendix I that the first term of such an expansion is given by

$$g(\rho,\Theta_{\rho x};\omega)=\frac{1}{4\pi}\sum_{i=1}^{4}\frac{e^{ik_i\rho}}{\rho}\left[\frac{k^2-k_i^2}{\Delta(k^2,\Theta_{\rho x};\omega)}\bigg|_{k^2=k_i^2}\right]. \quad (37)$$

(Note that g, in this approximation, does not depend on $\Phi_{\rho x}$.) In Eq. (37), the four k_i are the four zeroes of the equation

$$\Delta(k^2,\Theta_{\rho x};\omega)=0, \quad (38)$$

which lie in the upper-half of the complex k plane. Referring to Eqs. (22) and (15), we see that Eq. (38) is a fourth-order equation in k^2. (See also Soohoo, reference 3.) The important point here is that, in this approximation, the Δ in Eq. (37) is the same function as that in Eq. (22), except that the angle is now the polar angle between the vector $\boldsymbol{\varrho}$ and H_0.

Putting this approximation for the reduced Green's function in Eq. (28), we get

$$G_{\ldots}(\rho,\Theta_{\rho x};\omega)=\frac{1}{4\pi}\sum_{i=1}^{4}[\Lambda_{\ldots}(-i\nabla_\rho;\omega)e^{ik\rho}/\rho]\times\left[\frac{k^2-k_i^2}{\Delta(k^2,\Theta_{\rho x};\omega)}\bigg|_{k^2=k_i^2}\right]. \quad (39)$$

Now, when we put the $G_{\ldots}$ in Eq. (26), and we look at the fields for large distances $\mathbf{r}$, we use the usual[2] approximations

$$(1/\rho)\underset{r\to\infty}{\longrightarrow}(1/r),$$
$$\exp(ik_i\rho)\underset{r\to\infty}{\longrightarrow}\exp(ik_ir)\exp(-i\mathbf{k}_i\cdot\mathbf{r}'), \quad (40)$$

where the vector $\mathbf{k}_i$ is in the direction of observation, $\hat{r}$, and

$$\mathbf{k}_i=k_i\hat{r}. \quad (41)$$

An important simplification in this limit, in the same spirit as Eq. (41), is

$$\Theta_{\rho x}\underset{r\to\infty}{\longrightarrow}\Theta_{rx}, \quad (42)$$

so that the angle in the denominator of Eq. (39) becomes the polar angle Θ_{rx} of the direction of observation relative to the external magnetic field H_0.

Also, when we look at Eq. (39) for large $\mathbf{r}$, we replace the differential operators $(-i\nabla_\rho)$ in $\Lambda_{\ldots}$ by the corresponding operators $(-i\nabla_r)$. But we can go even further: since we only want the terms in $1/r$, we restrict these differential operators to act only on the exponential factor.[8] This has the practical effect of recovering the explicit forms for the $\Lambda_{\ldots}$ of Eqs. (23) and (24), where now we have replaced the k^2, k_+k_-, etc., in those two equations by k_i^2, $k_{i+}k_{i-}$, etc. [In other words, $\mathbf{k}\to\mathbf{k}_i$, where $\mathbf{k}_i$ is defined in Eq. (41).]

If we define some two-dimensional matrices by

$$(H)=\begin{pmatrix}H_+\\H_-\end{pmatrix}, \quad (43)$$

$$(J)=\begin{pmatrix}J_+\\J_-\end{pmatrix}, \quad (44)$$

and

$$(\Lambda)=\begin{pmatrix}\Lambda_{++} & \Lambda_{+-}\\ \Lambda_{-+} & \Lambda_{--}\end{pmatrix}, \quad (45)$$

we can put the results of all the above approximations and manipulations in the form

$$(H(\mathbf{r},\omega))=\sum_{i=1}^{4}\frac{e^{ik_ir}}{4\pi r}\left[\frac{k^2-k_i^2}{\Delta(k^2,\Theta_{rx};\omega)}\bigg|_{k^2=k_i^2}\right](\Lambda(\mathbf{k}_i,\omega))\times\int d^3r'\exp(-i\mathbf{k}_i\cdot\mathbf{r}')(J(\mathbf{r}',\omega)). \quad (46)$$

This is the desired approximate solution of the problem.

V. FUNCTION $\Delta(k^2,\theta_{rx};\omega)$

The behavior of the magnetic fields as a function of the frequency ω depends on the residues of the reduced Green's functions, i.e., on the $\Delta(k^2,\Theta_{rx};\omega)$ in Eq. (46).[9] In this section we consider some special properties of this function, which properties will be independent of the external currents $\mathbf{J}$.

We rewrite Δ in the form

$$\begin{aligned}\Delta(k^2,\Theta;\omega)&=(D_+D_-/\omega_s^2)(-k^2+n^2\omega^2)^2\\&+[(D_++D_-)/\omega_s](-k^2+n^2\omega^2)(-k^2\sin^2\Theta+n^2\omega^2)\\&+n^2\omega^2(-k^2\sin^2\Theta+n^2\omega^2). \quad (47)\end{aligned}$$

(a) Then it is clear that the "radiative pole" $(-k^2+n^2\omega^2)$ which we encountered in the iterative solution of Eqs. (18) and (19), contrary to what might be expected, is not present in general. Rather, from Eq. (47) it can only be present, and contribute one of the modes k_i in Eq. (46), when we observe at right angles

[7] J. Sucher, "Lectures on Relativistic Quantum Mechanics, Volume 1," University of Maryland Physics Department Technical Report No. 192, p. 22 (unpublished).

[8] For the explicit term in $1/\rho^2$, Eq. (A.12), one can use the Riemann-Lebesgue theorem, or its extension by Hobson to show that the integrals themselves will go asymptotically to zero as $\rho\to\infty$. [See E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford, 1948), 2nd ed.] The two theorems are on p. 11 and p. 238, respectively. Even more powerful theorems concerning the asymptotic behavior of Fourier transforms are proven in S. Bochner and K. Chandrasekharan, *Fourier Transforms* (Princeton University Press, Princeton, New Jersey, 1949), p. 31.

[9] Some of the results in this section were independently arrived at by Soohoo (see reference 3 above).

to the applied field, i.e., $\Theta_{rx}=90°$. [We will see in the next section that for this mode, and this angle, the various numerators in Eq. (46) vanish for point charges or point magnetic moments.]

(b) Next we point out that the "magnetostatic mode"[10] where one of the $k_i=0$, occurs in Eq. (47) when

$$[(\omega_0^2-\omega^2)/\omega_s^2+2\omega_0/\omega_s+1](n^2\omega^2)^2=0, \tag{48}$$

or

$$\omega^2=(\omega_0+\omega_s)^2. \tag{49}$$

Since ω is real, referring to Eq. (11), the presence of this mode, wherein the whole spin system moves as a unit, requires that the damping constant α be zero, and is independent of the value of the exchange constant $\mathfrak{D}$.

We now go on to consider the possibility of "ringing" the system. What this corresponds to in the complete solution of Eq. (46) is having one of the factors in the square bracket get very large. In other words, for some one of the modes k_i, we look for the possibility of

$$(k^2-k_i^2)^{-1}\Delta(k^2,\Theta_{rx};\omega)|_{k^2=k_i^2}=0, \tag{50}$$

for some value of the frequency ω, which we might call the resonant frequency.[11]

(c) If we consider the special case of $\Theta_{rx}=0°$, then from Eq. (47), the function Δ factors into $(k_+=k_-=0)$:

$$\begin{aligned}\Delta(k^2,0°;\omega)&\\ =[(D_+/\omega_s)(-k^2+n^2\omega^2)+n^2\omega^2]&\\ \times[(D_-/\omega_s)(-k^2+n^2\omega^2)+n^2\omega^2]&\\ =(\omega_s^2/D_+D_-)\Lambda_{++}(k^2,0°;\omega)\Lambda_{--}(k^2,0°;\omega),&\end{aligned} \tag{51}$$

and the solution, Eq. (46), reduces to

$$H_\pm(0°)=\frac{1}{4\pi}\sum_{i_\pm=1}^{2}\frac{e^{ik_ir}}{r}\left[(k^2-k_i^2)\times\frac{D_\pm/\omega_s}{\{(D_\pm/\omega_s)(-k^2+n^2\omega^2)+n^2\omega^2\}}\bigg|_{k^2=k_i^2}\right]\times\int d^3r'\exp(-i\mathbf{k}_i\cdot\mathbf{r}')J_\pm(\mathbf{r}',\omega). \tag{52}$$

Because Δ factors, [Eq. (51)], the sum over modes in Eq. (52) for $H_\pm$ is restricted to those two k_i^2 which solve

$$(D_\pm/\omega_s)(-k^2+n^2\omega^2)+n^2\omega^2=0, \tag{53}$$

respectively.[12] Thus, the four modes of propagation are given by

$$k_i^2\mathfrak{D}=\tfrac{1}{2}[n^2\omega^2\mathfrak{D}-(\omega_0\pm\omega)]\pm\tfrac{1}{2}\{[n^2\omega^2\mathfrak{D}-(\omega_0\pm\omega)]^2+4n^2\omega^2\mathfrak{D}(\omega_0+\omega_s\pm\omega)\}^{\frac{1}{2}}. \tag{54}$$

Since, in the region of interest, $n^2\omega^2\mathfrak{D}$ is small, we may approximate these roots by

$$k_{1\pm}^2\simeq-(\omega_0\pm\omega)/\mathfrak{D}, \tag{55}$$

and

$$k_{2\pm}^2\simeq n^2\omega^2\left[\frac{\omega_0+\omega_s\pm\omega}{\omega_0\pm\omega}\right]. \tag{56}$$

Consider first the modes for H_+, k_{1+}^2, and k_{2+}^2. If we set the damping constant $\alpha=0$ and the conductivity $\sigma=0$ in Eq. (11) for simplicity, then ω_0 and n are real and positive. Then, k_{1+}^2 is negative and this mode gives a decaying exponential in Eq. (52), so is of no further interest. The mode k_{2+}^2 is real, positive, and almost "radiative." The condition for "ringing" in this mode (or any mode in this special case) is when the square root in Eq. (54) vanishes, so for H_+ we need

$$[n^2\omega^2\mathfrak{D}-(\omega_0+\omega)]^2+4n^2\omega^2\mathfrak{D}(\omega_0+\omega_s+\omega)=0. \tag{57+}$$

If ω_s is positive, this is never accomplished for real ω. If $\omega_s<0$ (i.e., M_0 points along the negative x direction, H_0 along the positive x direction), then there is a positive, real solution of Eq. (57+), but only for very large ω (due to the smallness of $\mathfrak{D}$).

The case for H_- is more interesting. For $\omega<\omega_0$, k_{1-}^2 corresponds again to a decaying exponential in Eq. (52), and the analysis for k_{2-}^2 is much the same as above for k_{2+}^2. For $\omega>\omega_0$, however, both modes become normal propagation modes, and now the condition for ringing, Eq. (57+), becomes for H_-

$$[n^2\omega^2\mathfrak{D}-(\omega_0-\omega)]^2+4n^2\omega^2\mathfrak{D}(\omega_0+\omega_s-\omega)=0. \tag{57−}$$

Again for $\omega_s>0$, no resonant frequency is found in the region of interest, but for $\omega_s<0$, we get such a frequency where (to order $\mathfrak{D}$)

$$(\omega-\omega_0)^2\simeq4n^2\omega_0^2\mathfrak{D}|\omega_s|. \tag{58}$$

(This is the condition for either mode, k_{1-} or k_{2-}.)

Thus, when we observe along the direction of H_0, and look at the polarization H_-, we expect from Eq. (52) a very rapidly varying and large response in the frequency range around ω_0 given by Eq. (58).

(d) Next consider the other angle of special interest, $\Theta_{rx}=90°$. Then Δ becomes

$$\begin{aligned}\Delta(k^2,90°;\omega)&\\ =(-k^2+n^2\omega^2)\{(D_+D_-/\omega_s^2)(-k^2+n^2\omega^2)&\\ +[(D_++D_-)/\omega_s](-\tfrac{1}{2}k^2+n^2\omega^2)+n^2\omega^2\}.&\end{aligned} \tag{59}$$

Thus the "radiative mode" $(-k^2+n^2\omega^2)$ appears explicitly, as discussed in (a) above. For this mode, the residues or denominators in square brackets in Eq. (46)

[10] L. R. Walker, Phys. Rev. **105**, 390 (1957); J. Appl. Phys. **29**, 318 (1958).

[11] Actually, what we call here resonant frequencies really correspond, in Soohoo's language (reference 3) to double zeroes of the fourth-order secular determinant for k_i^2, i.e., double zeroes of Eq. (38). Or, they are double poles in the reduced Green's function, Eq. (35), for particular angles Θ_{rx} and for particular frequencies ω.

[12] The other two modes give zero from the $(k^2-k_i^2)$ factor in Eq. (52).

go through zero (for real ω_0) when

$$n^2\omega^2\mathfrak{D}+\omega_0+\omega_s=0. \tag{60}$$

We get a magnetostatic solution ($\omega=0=k$) when ($\omega_0+\omega_s$) is zero [see (b) above], and we get a resonance when $\omega_s<0$. We do not pursue this "radiative mode" further for reasons, pointed out in passing in (a) above, which we will discuss in the next section.

For the other modes, we set the curly bracket in Eq. (59) equal to zero and solve for k_i^2. To approximate this, we ignore the term in $\mathfrak{D}^2$ from D_+D_-, and get

$$[(\omega_0^2-\omega^2)/\omega_s^2](-k^2+n^2\omega^2) \\ +[(k^2\mathfrak{D}+\omega_0)/\omega_s](-k^2+2n^2\omega^2)+n^2\omega^2=0. \tag{61}$$

The roots are given by

$$k_i^2\mathfrak{D}=-\tfrac{1}{2}[(\omega_0^2-\omega^2+\omega_0\omega_s)/\omega_s-2n^2\omega^2\mathfrak{D}] \\ \pm\tfrac{1}{2}\{[(\omega_0^2-\omega^2+\omega_0\omega_s)/\omega_s-2n^2\omega^2\mathfrak{D}]^2 \\ +(4n^2\omega^2\mathfrak{D}/\omega_s)[(\omega_0+\omega_s)^2-\omega^2]\}^{\frac{1}{2}}. \tag{62}$$

Expanding the square root, the modes are given to lowest order in $\mathfrak{D}$ by

$$k_1^2\simeq[\omega^2-\omega_0(\omega_0+\omega_s)]/\omega_s\mathfrak{D}, \tag{63}$$

$$k_2^2\simeq n^2\omega^2\left[\frac{\omega^2-(\omega_0+\omega_s)^2}{\omega^2-\omega_0(\omega_0+\omega_s)}\right], \tag{64}$$

and, of course, from inspection of Eq. (59),

$$k_3^2=n^2\omega^2. \tag{65}$$

Once again we will "ring" the system when the square root in Eq. (62) vanishes, which occurs when (to order $\mathfrak{D}$)

$$[(\omega_0^2-\omega^2+\omega_0\omega_s)/\omega_s]^2\simeq-4n^2\omega^2\mathfrak{D}(\omega_0+\omega_s). \tag{66}$$

This can only occur when

$$\omega_s<0; \quad |\omega_s|>\omega_0. \tag{67}$$

Then the resonant frequencies ω are given by

$$\omega=|\omega_s|n[\mathfrak{D}(|\omega_s|-\omega_0)]^{\frac{1}{2}} \\ \pm[(|\omega_s|-\omega_0)(n^2\omega_s^2\mathfrak{D}-\omega_0)]^{\frac{1}{2}}, \tag{68}$$

and are only real and positive (assuming again $\alpha=\sigma=0$ for simplicity) when

$$0\leq\omega_0\leq n^2\omega_s^2\mathfrak{D}, \tag{69}$$

which is an impractically small range.

Therefore, at $\Theta_{rx}=90°$, the three modes are not of interest: The "radiative" mode we will see gives no contribution; the other two modes Eqs. (64) and (65), while they are proper modes of radiation for the conditions of Eq. (68), do not show the resonance behavior under realistic circumstances.

VI. PARTICULAR CURRENTS

We consider here only currents consisting of point charged particles, or point particles with a magnetic moment. Before proceeding to consider such currents in our general magnetic medium, we review briefly the usual Čerenkov effect as a special case of our problem.

It must be true, of course, that in the limit where our medium becomes nonmagnetic, we should recover the usual Čerenkov effect for point particles. We do indeed recover this result, as we shall see, from either of two directions. (In what follows, we generally imply, for clarity, that $\alpha=\sigma=0$.)

The limit of the medium becoming nonmagnetic corresponds to the limit of $\omega_s\to0$ in our equations. Then in the complete algebraic solution of our problem, Eq. (21), we see that

$$H_\pm(\mathbf{k},\omega)\underset{\omega_s\to0}{\longrightarrow}J_\pm(\mathbf{k},\omega)/(-k^2+n^2\omega^2), \tag{70}$$

since

$$\Delta(\mathbf{k},\omega)\underset{\omega_s\to0}{\longrightarrow}(D_+D_-/\omega_s^2)(-k^2+n^2\omega^2)^2, \tag{71}$$

$$\Lambda_{\pm\pm}(\mathbf{k},\omega)\underset{\omega_s\to0}{\longrightarrow}(D_\pm D_\mp/\omega_s^2)(-k^2+n^2\omega^2), \tag{72}$$

and

$$\Lambda_{\pm\mp}(\mathbf{k},\omega)\underset{\omega_s\to0}{\longrightarrow}\tfrac{1}{2}(k_\pm)^2D_\pm/\omega_s. \tag{73}$$

Equation (70) is, of course, just the solution of Maxwell's equation (17) with no magnetic material, and will be shown below to give the usual Čerenkov effect.

The other approach is to take the limit $\omega_s\to0$ in the approximate solution in $\mathbf{r}$ space, Eq. (46). Then from Eq. (71) in this limit the four modes k_i are found from the equation

$$D_+D_-(-k^2+n^2\omega^2)^2=0. \tag{74}$$

However, from Eqs. (72) and (73) it is clear that three of these modes give zero contribution in Eq. (46), and we are left with the "radiative mode," so that Eq. (46) becomes

$$H_\pm(\mathbf{r},\omega)\underset{\omega_s\to0}{\longrightarrow}-\frac{1}{4\pi}\frac{e^{ikr}}{r}\int d^3r'e^{-i\mathbf{k}\cdot\mathbf{r}'}J_\pm(\mathbf{r}',\omega), \tag{75}$$

where

$$k=n\omega. \tag{76}$$

This is just the inverse transform[13] of Eq. (70). The fact that we arrive at the same (correct) result via either the complete solution Eq. (21), or the approximate solution Eq. (46) gives us some confidence that the terms neglected in arriving at the approximate solution (see Appendix I) do not change our qualitative results.

Since the effects considered in the previous section are mostly independent of the currents $\mathbf{J}$, and are therefore just superimposed on the usual Čerenkov effect, we write down the rest of the derivation of the Čerenkov effect[2] for later reference. Using Eq. (75) [or actually, since the components are now uncoupled, the same equation for the vector components of $\mathbf{H}(\mathbf{r},\omega)$], and the definition of $\mathbf{J}$, Eq. (5), for a current due to an external

[13] Poles on the real k axis are handled by the usual limiting procedure (see reference 7) of adding an imaginary part, integrating by contours, and then letting the imaginary part go to zero.

charge, we have

$$\mathbf{j}_0(\mathbf{r},t)=e\mathbf{v}\delta(\mathbf{r}-\mathbf{v}t),$$
$$\mathbf{J}(\mathbf{r},t)=-\nabla\times\mathbf{j}_0(\mathbf{r},t). \tag{77}$$

Write

$$\delta(\mathbf{r}-\mathbf{v}t)=\delta_2(\mathbf{r}_\perp)\delta(\mathbf{r}\cdot\hat{v}-vt), \tag{78}$$

where $\delta_2(\mathbf{r}_\perp)$ is the two-dimensional δ function for those two components of the vector $\mathbf{r}$ perpendicular to the velocity $\mathbf{v}$ and $\hat{v}$ is the unit vector in the direction of the velocity. Then

$$\mathbf{j}_0(\mathbf{r},\omega)=(2\pi)^{-\frac{1}{2}}\int dt\, e^{i\omega t}\mathbf{j}_0(\mathbf{r},t)$$
$$=(2\pi)^{-\frac{1}{2}}e\hat{v}\delta_2(\mathbf{r}_\perp)\exp[i(\omega/v)\mathbf{r}\cdot\hat{v}]. \tag{79}$$

Integrating Eq. (75) by parts, we get

$$\mathbf{H}(\mathbf{r},\omega)=(-i/4\pi)(e^{ikr}/r)e\mathbf{k}\times\hat{v}\int d^2r_\perp' d(\mathbf{r}\cdot\hat{v})$$
$$\times e^{-i\mathbf{k}\cdot\mathbf{r}'}\delta_2(\mathbf{r}_\perp')\exp[i(\omega/v)\mathbf{r}'\cdot\hat{v}], \tag{80}$$

where we define

$$\mathbf{k}=n\omega\hat{r},$$
$$\hat{r}_\perp=\hat{r}\times\hat{v}. \tag{81}$$

Finally, then

$$\mathbf{H}(\mathbf{r},\omega)=(-i/2(2\pi)^{\frac{1}{2}})$$
$$\times(e^{ikr}/r)\hat{r}_\perp n\omega\sin\theta\delta(\omega/v-n\omega\cos\theta), \tag{82}$$

where the angle θ is defined by

$$\hat{r}\cdot\hat{v}=\cos\theta. \tag{83}$$

The δ function in Eq. (82) is, essentially, the Čerenkov-effect cone of radiation from a particle moving in a medium of index of refraction n with velocity v greater than the velocity of light in that medium, c/n. Finding the radiated intensity from Eq. (83) as usual, we get for the number N of photons emitted per unit path length L in an interval of frequency $d\omega$,

$$\delta N/\delta L=(1/137)(1-c^2/n^2v^2)d\omega/c, \tag{84}$$

so that, in the region of frequencies of interest here (kMc/sec), the number of photons emitted per centimeter is quite small ($\sim 10^{-2}$).

We now proceed to look at the case of magnetic materials, and of a current $J_\pm(\mathbf{r}',\omega)$ in Eq. (46) due to point charges or point magnetic moments. We show first, what was mentioned in passing in Sec. V, that the magnetic fields $H_\pm$ will vanish for $\Theta_{rx}=90°$ in the "radiative mode," $k^2=n^2\omega^2$.

Considering a charge, from Eq. (77) and integrating by parts in Eq. (46), we get

$$\int d^3r'\exp(-i\mathbf{k}_i\cdot\mathbf{r}')J_\pm(\mathbf{r}',\omega)$$
$$=-i(2\pi)^{\frac{1}{2}}e\delta(\omega/v-\mathbf{k}_i\cdot\hat{v})[\pm k_x v_\pm\mp k_\pm v_x]|_{k=k_i}. \tag{85}$$

If $k_i^2=n^2\omega^2$, then at $\Theta_{rx}=90°(k_{ix}=0)$ from Eqs. (23), (24), and (25)

$$\Lambda_{\pm\pm}(90°)|_{k_i=n\omega}=(D_\pm/\omega_s)\tfrac{1}{2}k_+k_-|_{k^2=n^2\omega^2},$$
$$\Lambda_{\pm\mp}(90°)|_{k_i=n\omega}=\tfrac{1}{2}(D_\pm/\omega_s)k_\pm^2|_{k^2=n^2\omega^2}, \tag{86}$$

so that

$$H_\pm\propto\Lambda_{\pm\pm}J_\pm+\Lambda_{\pm\mp}J_\mp\propto k_+k_-k_\pm+(k_\pm)^2(-k_\mp)=0. \tag{87}$$

Thus, observing at right angles to the field H_0, no radiation in the mode $k=n\omega$ should be observed, in contrast to the case when $\omega_s=0$ (if the velocity is so oriented that Θ_{rx} lies on the Čerenkov cone, of course).

For a point magnetic moment, from Eq. (5), the equations which correspond to Eq. (77) and (85) are

$$\mathbf{M}_{\mathrm{ext}}(\mathbf{r},t)=\boldsymbol{\mu}\delta(\mathbf{r}-\mathbf{v}t), \tag{88}$$

and

$$\int d^3r'\exp(-i\mathbf{k}_i\cdot\mathbf{r}')J_\pm(\mathbf{r}',\omega)$$
$$=-(2\pi)^{\frac{1}{2}}(\mu/v)\delta(\omega/v-\mathbf{k}_i\cdot\hat{v})[n^2\omega^2\hat{u}_\pm-ki_\pm(\mathbf{k}_i\cdot\hat{u})], \tag{89}$$

where $\hat{u}$ is a unit vector in the direction of $\boldsymbol{\mu}$. From Eqs. (86), at $\Theta_{rx}=90°$ $(k_x=0)$,

$$n^2\omega^2\hat{u}_\pm-k_\pm(\mathbf{k}\cdot\hat{u})|_{k=n\omega}=\tfrac{1}{2}k_+k_-\hat{u}_\pm-\tfrac{1}{2}(k_\pm)^2\hat{u}_\mp, \tag{90}$$

so that

$$H_\pm\propto k_+k_-[k_+k_-\hat{u}_\pm-k_\pm^2\hat{u}_\mp]$$
$$+k_\pm^2[k_+k_-\hat{u}_\mp-k_\mp^2u_\pm]=0. \tag{91}$$

Therefore, the same observations as follow Eq. (87) hold also for point magnetic moments.

For other angles, Eqs. (85) and (89) should be combined with the general solution Eq. (46). The Čerenkov cone,

$$\delta(\omega/v-\mathbf{k}_i\cdot\hat{v}), \tag{92}$$

is thus just a common multiplicative factor, and all the considerations of the last section about possible resonances are just superimposed on it. In particular, the considerations for the other special angle, $\Theta_{rx}=0°$, apply and merely describe the frequency spectrum of the disturbance.[14]

A final point to be made from Eqs. (92) and (46) is that the Čerenkov-effect cone, which tells the angle at which the disturbance is propagated, is a cone of constant polar angle of the direction of observation measured relative to the particle velocity; the modes k_i of propagation, and the conditions for "ringing" are determined by a cone of constant polar angle of the direction of observation measured relative to the external magnetic field H_0. Thus, only in the special case of motion along the x direction will the modes and possible resonances be independent of the azimuthal angle of the

[14] At $\Theta_{rx}=0^0(k_\pm=0)$ from Eqs. (85) and (89), the $H_\pm$ will, in general, not vanish unless $v_\pm$ (for charges) and $\mu_\pm$ (for magnetic moments) vanish; that is, unless $\mathbf{v}$ and $\boldsymbol{\mu}$ point along the x axis, in which case again $H_\pm=0$.

direction of observation. For the general case of velocity at some angle to the x axis, the frequency spectrum will depend on the azimuthal angle of direction of observation (measured relative to the direction of motion), and could be found from Eq. (46) by finding the modes and resulting resonances for all the angles Θ_{rx} which intersect the Čerenkov-effect cone, Eq. (92).

VII. SUMMARY AND CONCLUSIONS

We have shown, subject to the qualitative validity of our approximate solution, Eq. (46) (see Appendix I), that the intuitive idea of driving a magnetic material's spin system in resonance with the magnetic fields of the Čerenkov effect ("ringing" the system) seems to have some correspondence with the nature of the solutions of the coupled Maxwell-spin-wave equations. In particular, we have shown that these effects occur at one angle of particular simplicity, $\Theta_{rx}=0°$ (at which angle the resonant field is polarized, H_-) and expect them to occur at other angles also.

We have also shown that in the complete solution of the coupled Maxwell-spin equations, the usual mode of propagation for nonmagnetic materials, $k=n\omega$, is only allowed, for magnetic materials, at $\Theta_{rx}=90°$; and that at that angle it vanishes for currents due to a point charge, or point magnetic moment.

Finally, we have pointed out that for general motion of point particles at some nonzero angle relative to the external field H_0, the modes of propagation, the conditions for resonance, and therefore the frequency spectra of the disturbances due to the motion of fast particles in magnetic materials will depend on the azimuthal angle of the point of observation around the direction of motion.

We should point out that in all of the above, the actual vanishing of denominators, the reality of the modes of propagation, etc., usually required the (unphysical) condition $\alpha=\sigma=0$. While magnetic materials with $\sigma=0$ are known, all magnetic materials have a nonzero α. This means that resonance, in the sense of a vanishing denominator, cannot be achieved. However, the relaxation times in some cases can be fairly long,[15] and we therefore expect enhancements by many orders of magnitude.

This leads to the pleasant thought that if an insulating, magnetic material with sufficiently long relaxation time could be found (and if the charged-particle background is sufficiently well eliminated), then the possibility of detecting (electromagnetically) the passage of a neutral particle with a magnetic moment would exist. Such a detector would obviously have great use in high-energy particle physics.[16]

The other category of usefulness for these ideas would be in the use of charged particles as probes of the properties of magnetic materials. Such a probe would be contained in the experimental observation of the enhancement effect; in the variation of the frequency spectrum with the conditions of the material $(\omega_0,\omega_s,\alpha,n,\sigma)$; and in the variation of the spectrum with direction (say azimuthal angle about the direction of motion).

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APPENDIX I. EXPANSION OF THE REDUCED GREEN'S FUNCTION

We wish to get an approximate evaluation of the integral of Eq. (35) in the text. So, we want to consider

$$\int k^2dkd(\cos\theta_{k\rho})d\varphi_{k\rho}\,e^{ik\rho\cos\theta_{k\rho}} \times[F_1(k^2)+F_2(k^2)f(\theta_{k\rho},\varphi_{k\rho};\,\Theta_{\rho x},\Phi_{\rho x})]^{-1},\quad\text{(A.1)}$$

where

$$\begin{aligned}f(\theta_{k\rho},\varphi_{k\rho};\,\Theta_{\rho x},\Phi_{\rho x})&=\cos^2\theta_{kx}\\&=[\cos\theta_{k\rho}\cos\Theta_{\rho x}\\&\quad+\sin\theta_{k\rho}\sin\Theta_{\rho x}\cos(\varphi_{k\rho}-\Phi_{\rho x})]^2,\end{aligned}\quad\text{(A.2)}$$

and

$$f(0,\varphi_{k\rho}\cdots)=f(\pi,\varphi_{k\rho}\cdots)=\cos^2\Theta_{\rho x}\equiv f_0.\quad\text{(A.3)}$$

We integrate by parts (henceforth we drop the subscripts on the angles),

$$\int d^3k\exp(i\mathbf{k}\cdot\rho)[F_1+F_2f]^{-1} =\int k^2dkd\varphi(ik\rho)^{-1}\left[\frac{2i\sin k\rho}{F_1+F_2f_0}+I_1\right],\quad\text{(A.4)}$$

$$I_1=\int_0^\pi d\theta\,e^{ik\rho\cos\theta}[F_1+F_2f]^{-2}\frac{df}{d\theta}F_2.\quad\text{(A.5)}$$

The problem with I_1 as it stands is that the integral is over $d\theta$ rather than $d(\cos\theta)$. To overcome this, we first

[15] J. A. Giordmaine (private communications).

[16] It was actually the search for such a detector (for Λ^0 particles, say) which led the author to the ideas of the Introduction and hence to spin waves. With presently available materials, the damping constants α are too large, and a detector for neutral magnetic moments does not yet seem feasible. For charged particles, however, using the material constants of reference 3, we estimate that several hundred microwave photons per centimeter would be emitted; such yields should be detectable with current or proposed maser techniques. [See J. Weber, Revs. Modern Phys. **31**, 681 (1959); and N. Bloembergen, Phys. Rev. Letters **2**, 84 (1959).] This estimate ignores the dependence of the effect on azimuthal angle. In order to take account of this dependence in a more careful estimate, a machine calculation of Eq. (50) would be necessary.

perform the φ integration by parts, to get

$$\int_0^{2\pi} d\varphi\, I_1 = \int d\varphi d(\cos\theta) e^{ik\rho\cos\theta} \times\left[A\frac{F_2}{(F_1+F_2f)^2}+B\frac{F_2{}^2}{(F_1+F_2f)^3}\right], \quad (A.6)$$

where the functions A and B are given by

$$A=-2\cos^2\Theta\cos\theta+2\sin^2\Theta\cos\theta\cos^2(\varphi-\Phi) -4\sin\theta\cos\Theta\sin\Theta\cos(\varphi-\Phi), \quad (A.7)$$

and

$$B=8\cos\theta\sin^2\Theta\sin^2(\varphi-\Phi) \times[\cos\theta\cos\Theta+\sin\theta\sin\Theta\cos(\varphi-\Phi)]. \quad (A.8)$$

We can then integrate again over $d(\cos\theta)$ by parts to get

$$\int d^3k \exp(i\mathbf{k}\cdot\boldsymbol{\rho})[F_1+F_2f]^{-1}$$

$$=\int k^2dkd\varphi\left\{\frac{2\sin k\rho}{k\rho}(F_1+F_2f_0)^{-1}-\frac{4\cos k\rho}{(k\rho)^2}\right.$$

$$\times\left[[-\cos^2\Theta+\sin^2\Theta\cos^2(\varphi-\Phi)]\frac{F_2}{(F_1+F_2f_0)^2}\right.$$

$$\left.+\sin^2 2\Theta\sin^2(\varphi-\Phi)\frac{F_2{}^2}{(F_1+F_2f_0)^3}\right]$$

$$+[1/(k\rho)^2]\int_0^\pi d\theta\, e^{ik\rho\cos\theta}\frac{d}{d\theta}$$

$$\left.\times\left[A\frac{F_2}{(F_1+F_2f)^2}+B\frac{F_2{}^2}{(F_1+F_2f)^3}\right]\right\}. \quad (A.9)$$

Recognizing that

$$F_1+F_2f_0=\Delta(k^2,\Theta_{\rho x};\omega), \quad (A.10)$$

we get, finally

$$g(\rho,\Theta_{\rho x},\Phi_{\rho x};\omega) =\frac{1}{2\pi^2\rho}\int_0^\infty dk\frac{k\sin k\rho}{\Delta(k^2,\Theta_{\rho x};\omega)}+K/\rho^2, \quad (A.11)$$

where, explicitly

$$K=\frac{2}{\pi^2}\int_0^\infty dk\cos k\rho\left\{\frac{(\frac{3}{2}\sin^2\Theta_{\rho x}-1)F_2(k^2)}{[\Delta(k^2,\Theta_{\rho x};\omega)]^2}+\frac{\frac{1}{2}\sin^2 2\Theta_{\rho x}F_2{}^2(k^2)}{[\Delta(k^2,\Theta_{\rho x};\omega)]^3}\right\}+(2\pi)^{-3}$$

$$\times\int dkd\theta_{k\rho}d\varphi_{k\rho}\, e^{ik\rho\cos\theta_{k\rho}}\frac{d}{d\theta_{k\rho}}\left[A\frac{F_2}{\Delta^2}+B\frac{F_2{}^2}{\Delta^3}\right]. \quad (A.12)$$

In the first term of Eq. (A.11), we can write [see Eq. (22) of the text]

$$\Delta=\text{const}\prod_{i=1}^4(k^2-k_1{}^2)=\text{const}\prod_{i=1}^4(k+k_i)(k-k_i), \quad (A.13)$$

so that the integral in the $1/\rho$ term can be done by contours, to get

$$\int_0^\infty kdk\sin k\rho/\Delta=\pi\sum_{i=1}^4 e^{ik_i\rho}\left(k\frac{k-k_i}{\Delta}\right)\bigg|_{k=k_i}. \quad (A.14)$$

In Eq. (A.14) we consider only the poles in the upper half of the complex k plane, so

$$k\frac{(k-k_i)}{\Delta}\bigg|_{k=+k_i}=\tfrac{1}{2}[\text{const}\prod_{j\neq i}(k^2-k_j{}^2)]^{-1} =\tfrac{1}{2}\frac{(k^2-k_i{}^2)}{\Delta}\bigg|_{k^2=k_i{}^2}. \quad (A.15)$$

This gives Eq. (37) of the text directly. As mentioned in reference 8, some general statements can be made about the asymptotic behavior of K in Eq. (A.12), but we do not pursue this further here.