Very Low-Temperature Fermi Gas*

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The momentum distribution in a low-temperature Fermi gas is investigated using the methods of quantum statistics developed by Lee and Yang together with the linked-pair expansions of the previous paper. It is shown that in order to determine the momentum distribution at very low temperatures two coupled integral equations must be considered, one in momentum variables and due to Lee and Yang, and the other in temperature variables. It is also shown that the dominant low-temperature behavior of the momentum distribution can be extracted in terms of a certain function $\nu'(\mathbf{k})$. For a low-density Fermi gas with strong, short-range, two-body interactions, it is shown to third order in the scattering parameters of the interaction that at $T=0$ the function $\nu'(\mathbf{k})$ is equal to the free particle momentum distribution. Also, the energy and other thermodynamic quantities are expressed in terms of $\nu'(\mathbf{k})$, so that the theory permits a generalization of perturbation theoretic results to nonzero temperatures. The ground-state energy, momentum distribution, and thermodynamic potential are calculated to third order in the scattering parameters.

INTRODUCTION

THE momentum distribution in a very low-temperature Fermi system is a quantity of considerable interest to physicists. In the first place, it is a measureable quantity which is quite sensitive to the nature of the forces which the Fermi particles experience. In the second place, the momentum distribution plays a central role in most many-body theories and can greatly affect the calculation of other thermodynamic quantities.

A Fermi system of especial theoretical interest is the very low-temperature Fermi gas, which serves as a model, for free electrons in a conductor, for liquid He^3, and for nucleons inside of a very large nucleus. In this paper, we are thinking mainly of the latter two problems, for in our considerations we always assume that there are only two-body forces which are strong and of short range. However, the interesting case of attractive forces which are sufficiently strong to cause binding in the many-body system is not considered in this paper.

In recent years, the importance of knowing the momentum distribution in the ground state (i.e., the zero-temperature state) of a Fermi gas, has heightened, because of the assumption in many-body theories that the true momentum distribution resembles the free particle distribution to first approximation. For example, this assumption is basic to the Brueckner-Goldstone formulation of perturbation theory.[1] In a recent paper, Kohn and Luttinger[2] have directed attention to this assumption by examining the Brueckner-Goldstone formalism from the point of view of quantum statistics. In a subsequent paper, Luttinger and Ward[3] have shown that it is correct to calculate the ground-state energy of a Fermi system for spin-$\frac{1}{2}$ particles using the free-particle momentum distribution (i.e., that the Brueckner-Goldstone formalism is correct), provided that certain terms (i.e., anomalous or improper diagrams) are omitted in the calculation.

The work of Luttinger and Ward employs the methods of field theory. Although such an approach is quite fashionable in current literature, the particular application by Luttinger and Ward has the drawback that it can only be applied to weak interactions and not to the more realistic case of forces which include a hard core repulsion. It is therefore the purpose of the present paper to derive and apply methods for calculating the ground-state energy and momentum distribution of a Fermi gas in which the forces are both strong and of short range.

The present treatment of a Fermi system is based on the quantum statistical method of Lee and Yang[4] and it does not rely on the methods of field theory. The analysis stems from an integral equation for the momentum distribution, first derived by Lee and Yang,[5] into which the linked-pair expansion of the vertex functions[6] is substituted. The derivation of this equation, which is in terms of momentum coordinates, is reviewed in Sec. I. It is then shown in Sec. II that in order to determine the leading corrections to the momentum distribution of a Fermi gas in the temperature-density region $\rho|a_s|\lambda_T^2\gg1$ (ρ=density, a_s=scattering length, λ_T=thermal wavelength) a second integral equation, which is in terms of (inverse) temperature variables, must be coupled to the Lee-Yang equation. It is shown that in the scattering length approximation $\rho^{\frac{1}{3}}|a_s|\ll1$ the first, as well as the zeroth, order solution to the coupled set of integral equations is the free-particle momentum distribution.

In Sec. III, *master graphs* are introduced, which enable one to write down the above set of coupled integral equations in a simple manner. Possible ap-

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[3] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960).
[4] T. D. Lee and C. N. Yang, Phys. Rev. **113**, 1165 (1959).
[5] T. D. Lee and C. N. Yang, Phys. Rev. **117**, 22 (1960), hereafter referred to as LY IV.
[6] F. Mohling, preceding paper [Phys. Rev. **122**, 1043 (1961)], hereafter referred to as I.

proximations methods for determining the solutions to these equations are reviewed in Sec. IV, and it is then shown that the dominant low-temperature behavior of the solution can be extracted by means of a temperature and momentum dependent transformation which is called the Λ *transformation*. Before the Λ transformation is performed, the iterated solution to the integral equations is in terms of a function $\nu(\mathbf{k})$, where

$$\nu(\mathbf{k})=\exp[\beta(g-\omega_{\mathbf{k}})]/\{1+\exp[\beta(g-\omega_{\mathbf{k}})]\},$$
$$[\omega_{\mathbf{k}}=\hbar^2\mathbf{k}^2/2m],$$

and g is the thermodynamic potential. At zero temperature ($\beta=\infty$), the function $\nu(\mathbf{k})$ is a step function with the step at $\omega_{\mathbf{k}}=g$. After the Λ transformation, the iterated solution is in terms of a function $\nu'(\mathbf{k})$, where

$$\nu'(\mathbf{k})=\frac{\exp[\beta(g-\omega_{\mathbf{k}}')]}{\{1+\exp[\beta(g-\omega_{\mathbf{k}}')]\}},\quad [\omega_{\mathbf{k}}'=\omega_{\mathbf{k}}+\Delta(\mathbf{k},\beta)],$$

and the temperature dependence of $\Delta(k,\beta)$ is negligible at very low temperatures, i.e., $\Delta(\mathbf{k},\beta)\rightarrow\Delta(\mathbf{k})$ as $T\rightarrow 0$. [As mentioned above, the leading term of $\Delta(\mathbf{k})$ in the scattering length approximation is also momentum independent.]

The functions $\nu(\mathbf{k})$ and $\nu'(\mathbf{k})$ reduce to the free-particle momentum distribution in the absence of interactions, since $\Delta(\mathbf{k},\beta)=0$ for free particles. On the other hand, $\nu(\mathbf{k})$ has nothing to do with the momentum distribution for interacting particles, whereas it is shown (to third order in $\rho^{\frac{1}{3}}\cdot a_s$) in Secs. V and VI that for a Fermi gas the integral of $\nu'(\mathbf{k})$ over all momentum space gives the density ρ. Therefore, when the quantity $g-\omega_{\mathbf{k}}'$ has only one zero at $T=0$, this zero must occur at $k=k_F$ (or $\omega_{\mathbf{k}}=E_F$), where

$$\rho\equiv[(2J+1)/6\pi^2]k_F^3 \quad\text{and}\quad E_F\equiv\hbar^2k_F^2/2m.$$

Thus, it is shown (to third order) that for a gas of interacting Fermi particles at $T=0$, the quantity $\nu'(\mathbf{k})$ is the free-particle momentum distribution, and that the true momentum distribution differs from $\nu'(\mathbf{k})$ only by terms which are $O(k_Fa_s)^2$.

An important consequence of the Λ transformation is that the grand potential $f(=\beta\times\text{pressure})$, and therefore also the energy, of a Fermi system can be calculated in terms of the quantity $\nu'(\mathbf{k})$. This justifies (to third order for a Fermi gas) the oft-made assumption that the ground-state energy can be calculated in terms of the free particle momentum distribution. The result is perhaps not too surprising. On the other hand, the prescription given in Sec. VI for making the calculation is valid at nonzero temperatures as well as at $T=0$.

The ground-state energy of a Fermi gas is calculated to third order in scattering parameters in Sec. VI, and the result is in agreement with previous calculations which have appeared in the literature. Curves are also presented, which show the orders of magnitudes of the various terms in the expansion of the energy for the cases of hard-core repulsions with and without weak attractive forces.

The thermodynamic potential g at $T=0$ is calculated to third order in scattering parameters, and the momentum dependence of $\Delta(\mathbf{k})$ is explicitly given to second order in (k_Fa_s).

I. THE MOMENTUM DISTRIBUTION

The theory of the grand canonical ensemble is useful for calculating not only the thermodynamic quantities of a system, such as the energy and the pressure, but also for calculating distribution functions such as the momentum distribution and the two-body correlation function. Thus, the probability that any particle in a system has momentum $\mathbf{k}_0$ is simply[7]

$$\langle n(\mathbf{k}_0)\rangle=\sum_{N=1}^{\infty}(N!)^{-1}\sum_{\mathbf{k}_1\cdots\mathbf{k}_N}\langle\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N|\rho_N^{(s)}|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N\rangle\Big(\sum_{j=1}^{N}\delta_{\mathbf{k}_j,\mathbf{k}_0}\delta_{m_j,m_0}\Big),\tag{1}$$

where $\langle\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N|\rho_N^{(s)}|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N\rangle$ is a symmetrized (or antisymmetrized) matrix element of the N-particle density operator. Now, in I the grand partition function was written in the form

$$\exp(\Omega f^{(s)})=\sum_{N=0}^{\infty}z^N(N!)^{-1}\sum_{\mathbf{k}_1\cdots\mathbf{k}_N}\exp\Big(-\beta\sum_{i=1}^{N}\omega_i\Big)\langle\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N|W_N^{(s)}|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N\rangle,\tag{2}$$

where $\langle\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N|W_N^{(s)}|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N\rangle$ is a symmetrized matrix element of the operator $W_N(\beta)=\exp(\beta H_0^{(N)})\times\exp(-\beta H^{(N)})$.[8] It therefore follows[9] that the matrix elements of the density operator are

$$\langle\mathbf{k}_1\cdots\mathbf{k}_N|\rho_N^{(s)}|\mathbf{k}_1\cdots\mathbf{k}_N\rangle=z^N\exp(-\Omega f^{(s)})\exp\Big(-\beta\sum_{i=1}^{N}\omega_i\Big)\langle\mathbf{k}_1\cdots\mathbf{k}_N|W_N^{(s)}|\mathbf{k}_1\cdots\mathbf{k}_N\rangle,\tag{3}$$

and that the momentum distribution is given by

$$\langle n(\mathbf{k}_0)\rangle=\exp(-\Omega f^{(s)})\sum_{N=1}^{\infty}z^N(N!)^{-1}\sum_{\mathbf{k}_1\cdots\mathbf{k}_N}\exp\Big(-\beta\sum_{i=1}^{N}\omega_i\Big)\langle\mathbf{k}_1\cdots\mathbf{k}_N|W_N^{(s)}|\mathbf{k}_1\cdots\mathbf{k}_N\rangle\Big(\sum_{j=1}^{N}\delta_{\mathbf{k}_j,\mathbf{k}_0}\delta_{m_j,m_0}\Big).\tag{4}$$

[7] It is convenient to define $\langle n(\mathbf{k}_0)\rangle$ to be the momentum distribution for a fixed momentum and spin state. Therefore, in Eqs. (1) and (4) $\delta_{\mathbf{k}_j,\mathbf{k}_0}$ is multiplied by the factor δ_{m_j,m_0}, where m is the spin magnetic quantum number.

[8] We use the interaction representation throughout this paper.

[9] D. ter Haar, *Elements of Statistical Mechanics* (Rinehart and Company, Inc., New York, 1954), Chap. VII. See also footnote 7 in I.

Upon comparing Eqs. (2) and (4) we see that, providing the matrix elements of $W_N(\beta)$ are expressed in the plane wave representation, $\langle n(\mathbf{k})\rangle$ may be obtained by a simple functional differentiation of the grand partition function:

$$\langle n(\mathbf{k})\rangle = -\beta^{-1}\exp(-\Omega f^{(s)})\left[\frac{\delta}{\delta\omega_{\mathbf{k}}}\exp(\Omega f^{(s)})\right]_W, \quad (5)$$

where the matrix elements of $W_N(\beta)$ are to be held fixed. According to I, however, the matrix elements of $W_N(\beta)$ can be expressed entirely in terms of certain functions $\langle \mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N|T_N|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N\rangle$, which are symmetric combinations of the Boltzmann cluster-functions. Moreover, the grand potential $f^{(s)}$ can also be written directly in terms of these T functions, and therefore Eq. (5) may be replaced by

$$\langle n(\mathbf{k})\rangle = -\beta^{-1}\left[\frac{\delta}{\delta\omega_{\mathbf{k}}}\Omega f^{(s)}\right]_T. \quad (6)$$

We now follow the development of Lee and Yang [LY IV], who have shown that $\Omega f^{(s)}$ can be written either as a sum over all primary 0-graphs or as a sum over all contracted 0-graphs. The $\omega_{\mathbf{k}}$-dependence of the primary 0-graphs occurs explicitly in factors $(\epsilon z\exp(-\beta\omega_{\mathbf{k}}))$ which are assigned to the (internal) lines of the graphs. On the other hand, in the contracted 0-graphs the $\omega_{\mathbf{k}}$-dependence is entirely contained in line factors $\epsilon\nu_{\mathbf{k}}$ ($\epsilon=+1$ for Bose-Einstein statistics, $\epsilon=-1$ for Fermi-Dirac statistics), where

$$\nu_{\mathbf{k}} = \frac{z\exp(-\beta\omega_{\mathbf{k}})}{1-\epsilon z\exp(-\beta\omega_{\mathbf{k}})} = \frac{\exp[\beta(g-\omega_{\mathbf{k}})]}{1-\epsilon\exp[\beta(g-\omega_{\mathbf{k}})]}. \quad (7)$$

The quantity g is the thermodynamic potential per particle. Since we shall be concerned only with contracted 0-graphs, it is convenient to perform the functional differentiation in Eq. (6) with respect to $\nu_{\mathbf{k}}$ instead of $\omega_{\mathbf{k}}$. Equation (6) then becomes

$$\langle n(\mathbf{k})\rangle = \nu_{\mathbf{k}} + \nu_{\mathbf{k}}(1+\epsilon\nu_{\mathbf{k}}) \times[\delta/\delta\nu_{\mathbf{k}}\textstyle\sum \text{(all contracted 0-graphs)}]_T, \quad (8)$$

where the first term is the free-particle contribution $\nu_{\mathbf{k}}$ to the momentum distribution.

The differentiation in Eq. (8) can be graphically performed by breaking open the lines of contracted 0-graphs. This leads to the expression

$$\langle n(\mathbf{k})\rangle = \nu_{\mathbf{k}} + \epsilon\nu_{\mathbf{k}}(1+\epsilon\nu_{\mathbf{k}})\textstyle\sum\text{(all contracted 1 graphs)}. \quad (9)$$

Contracted ζ graphs are formally defined in LY IV and in Sec. I of I.

An approximate evaluation of Eq. (9), in which only those contracted 1 graphs which have small numbers of vertices with each vertex representing a low-order T function, is obviously suggested. However, Lee and Yang have called attention to the fact that for a Bose-Einstein gas the resulting power series development is inapplicable near and below its transition temperature. As we shall see in the next section, such a power series development is also not applicable for a very low-temperature Fermi gas. It is therefore necessary to write Eq. (9) as an integral equation before its solution can be attempted. This is accomplished by first writing Eq. (9) as follows:

$$N(\mathbf{k}) = \nu_{\mathbf{k}}[1+\nu_{\mathbf{k}}\textstyle\sum\text{(all different contracted 1 graphs)}], \quad (10)$$

where the function $N(\mathbf{k})$ is defined by

$$N(\mathbf{k}) = \exp[\beta(g-\omega_{\mathbf{k}})][1+\epsilon\langle n(\mathbf{k})\rangle]. \quad (11)$$

Equation (10) for $N(\mathbf{k})$ can then be written as an integral equation after introducing the concept of reducible and irreducible graphs [see LY IV].

A contracted 0 graph or 1 graph is called *reducible* if by cutting two of its (solid) internal lines open the entire graph can be separated into two (or more) disconnected contracted ζ graphs ($\zeta=1, 2$). An *irreducible* ζ graph is a contracted ζ graph which is not reducible, with its (solid) internal lines representing factors $\epsilon N(\mathbf{k})$ instead of $\epsilon\nu(\mathbf{k})$. With these definitions, it is shown in LY IV that

$$N(\mathbf{k}) = \nu(\mathbf{k})[1+N(\mathbf{k}) \times\textstyle\sum\text{(all different irreducible 1 graphs)}]. \quad (12)$$

The present paper is mainly concerned with the development of a systematic way of solving this complicated integral equation for a very low-temperature Fermi gas, using the linked-pair expansion introduced in I. We shall also consider the evaluation of the grand potential, which according to LY IV may be written

$$\begin{aligned}\Omega f^{(s)}(N,\Omega,\beta) = {} & \epsilon\textstyle\sum_{\mathbf{k}}\ln[1+\epsilon\langle n(\mathbf{k})\rangle] \\ & +\textstyle\sum\text{(all different irreducible 0-graphs)} \\ & -\epsilon\textstyle\sum_{\mathbf{k}}[\nu^{-1}(\mathbf{k})N(\mathbf{k})-1], \end{aligned}\quad (13)$$

and which completely determines the pressure, density, and energy in terms of the temperature T, the thermodynamic potential g, and the interaction parameters of the system.

II. LOW-TEMPERATURE MOMENTUM DISTRIBUTION IN A FERMI GAS; LOWEST ORDER CORRECTIONS

In I we showed how the contracted ζ-graph vertex factors, i.e., the T functions, could be expressed in terms of two-body "pair functions"

$$\begin{aligned} {}^{t_2}\!\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_1} &= \langle\mathbf{k}_1\mathbf{k}_2|R(t_2,t_1)|\mathbf{k}_3\mathbf{k}_4\rangle + \epsilon\langle\mathbf{k}_1\mathbf{k}_2|R(t_2,t_1)|\mathbf{k}_4\mathbf{k}_3\rangle, \\ R(t_2,t_1) &= -\frac{\partial}{\partial t_1}[\exp(t_2H_0^{(2)})\exp[-(t_2-t_1)H^{(2)}] \times\exp(-t_1H_0^{(2)})], \end{aligned}\quad (14)$$

by means of wiggly-line cluster graphs, and how this fact permits one to express the grand potential in terms of linked-pair 0-graphs with both solid and wiggly lines. Such a treatment of the T functions is applicable to irreducible ζ graphs as well as to contracted ζ graphs. We may therefore express the grand potential (13) and the momentum distribution (12) in terms of *irreducible linked-pair 0- and 1-graphs*, respectively, which are assigned symmetry numbers in accordance with the rule given with Eq. (I.58). Some examples of reducible and irreducible linked-pair 0-graphs are shown in Fig. 1.

We now introduce a "new" pair function

$$
{}^{t_1t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}=\begin{cases}\theta(t_1-t_2)\;{}^{t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}\theta(t_2-t_0)+\theta(t_2-t_1)\;{}^{t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}\theta(t_1-t_0) & \text{if } t_1\neq t_2\\ {}^{t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}\theta(t_1-t_0) & \text{if } t_1=t_2,\end{cases}\tag{15}
$$

where

$$
\begin{aligned}\theta(x)&=+1 \quad \text{if} \quad x>0\\ &=\;\;0 \quad \text{if} \quad x<0,\end{aligned}\tag{16}
$$

which may be substituted for each of the "old" pair functions (14) in a linked-pair ζ graph without changing its value. The variables t_1 and t_2 are the temperature labels at the heads of outgoing wiggly lines from the vertex t_0, and for a solid outgoing line (i) $t_i=\beta$ (see Fig. 8 of I). When we use the pair functions (15) in a linked-pair ζ graph we may extend the range of integration of the temperature variables at all of the vertices to the entire interval 0 to β. For this reason, we shall assume in what follows that unless otherwise indicated the pair functions (15) are being used in linked-pair ζ graphs.

Corresponding to each irreducible linked-pair 1 graph we next introduce an L *graph* with exactly the same structure, but subject to the condition that we do not integrate over the temperature variable t_1 at the vertex to which the incoming external line attaches. Thus, if we define

$$L(\beta,t_1,\mathbf{k})\equiv\sum(\text{all } L \text{ graphs with solid external lines}),\tag{17}$$

then Eq. (12) may be replaced by

$$N(\mathbf{k})=\nu(\mathbf{k})\left[1+N(\mathbf{k})\int_0^\beta dt\, L(\beta,t,\mathbf{k})\right].\tag{18}$$

The advantage of the L graphs over the irreducible linked-pair 1 graphs is that the former permit a simple generalization which is useful for generating more complicated graphs from simpler graphs. Thus, we define L graphs with one or both of the solid external lines replaced by wiggly lines by merely specifying the temperature variable t_2 at the vertex to which the outgoing line is directed if that line is wiggly:

$$L(t_2,t_1,\mathbf{k})\equiv\sum(\text{all different } L \text{ graphs with given external lines}).\tag{19}$$

Equation (19) reduces to Eq. (17) for L graphs with external solid lines. For L graphs with outgoing wiggly lines, we note that we may have $t_2<t_1$ as well as $t_2>t_1$.

We next define a quantity $P(t_2,t_1,\mathbf{k})$:

$$P(t_2,t_1,\mathbf{k})\equiv\sum(\text{all } L \text{ graphs with given external lines which cannot be separated into two } L \text{ graphs by cutting one wiggly line}),\tag{20}$$

in terms of which we can write down a simple integral equation

$$L(t_2,t_1,\mathbf{k})=\int_0^\beta ds\, G(t_2,s,\mathbf{k})P(s,t_1,\mathbf{k}),\tag{21}$$

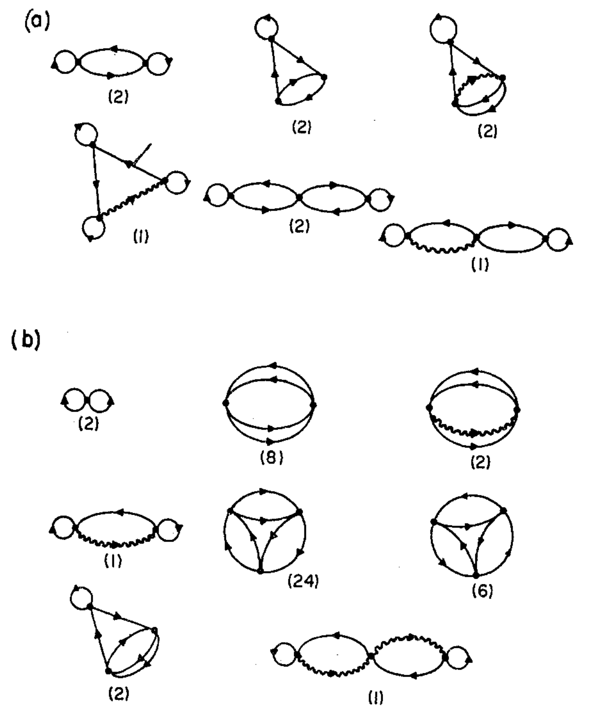

FIG. 1. (a) Reducible linked-pair 0-graphs. There are no such graphs with only one vertex, and there is only one with two vertices. Five of the eight 3-vertex 0-graphs are shown; (b) Irreducible linked-pair 0-graphs. There is only one 1-vertex 0-graph and there are three 2-vertex 0-graphs. Four of the twenty-one 3-vertex 0-graphs are shown. (We do not draw irreducible graphs with heavy black lines as is done in LY IV.) The symmetry number S has been included below each 0-graph. For convenience, the temperature labels of the vertices have been omitted.

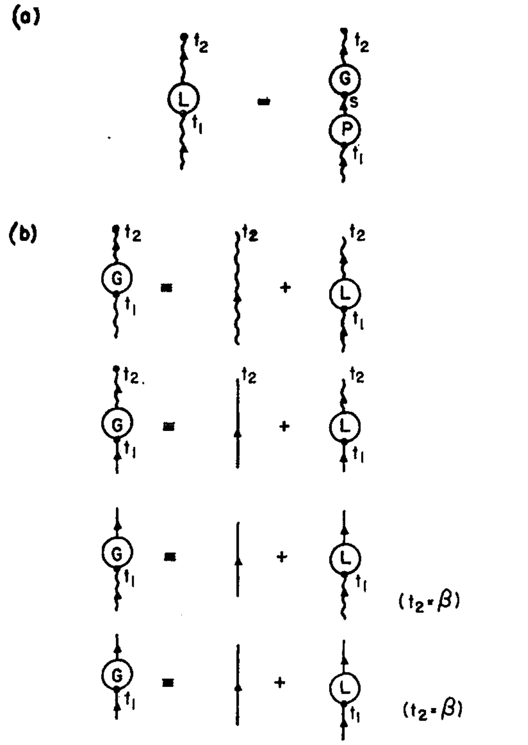

FIG. 2. (a) Diagrammatic representation of Eq. (21) for the case of wiggly external lines; (b) Diagrammatic representation of Eq. (22) for the four possible choices of external lines. For an outgoing solid line, $t_2(=\beta)$ does *not* refer to the temperature variable at the vertex to which the line is directed.

where

$$G(t_2,t_1,\mathbf{k})\equiv\delta(t_2-t_1)+\epsilon L(t_2,t_1,\mathbf{k}). \qquad (22)$$

Equations (21) and (22) are represented graphically in Fig. 2. In particular, the line representing the term $\delta(t_2-t_1)$ is defined to be a solid line unless *both* of the "given" external lines are wiggly lines.

In I we defined a double bond in a graph to be a structure in which two lines connect the same two vertices. We also showed that there are no wiggly-line double bonds in linked-pair ζ graphs, and hence there are none in the quantity $P(t_2,t_1,\mathbf{k})$. We next define the quantity $\mathfrak{P}(t_2,t_1,\mathbf{k})$ to be the sum over all L graphs of the type in Eq. (20), including those with wiggly-line double bonds:

$$\mathfrak{P}(t_2,t_1,\mathbf{k})\equiv\sum(\text{all } L \text{ graphs with given external lines, including those with wiggly line double bonds, which cannot be separated into two } L \text{ graphs by cutting one wiggly line}). \qquad (23)$$

All but two of the one- and two-vertex L graphs in $\mathfrak{P}(t_2,t_1,\mathbf{k}_1)$ are shown in Fig. 3 for the case of external wiggly lines. With the exception of the last L graph, $Q_4^{(2)}(t_2,t_1,\mathbf{k}_1)$, these are also one- and two-vertex L graphs in $P(t_2,t_1,\mathbf{k}_1)$.

The L graphs which contribute in Eq. (23) may be classified as improper or not improper. Figure 3 includes *all* of the one- and two-vertex L graphs in $\mathfrak{P}(t_2,t_1,\mathbf{k}_1)$ which are not improper, whereas, the remaining two 2-vertex L graphs in $\mathfrak{P}(t_2,t_1,\mathbf{k}_1)$ are included among the improper L graphs of Fig. 4.

Definitions

An n-vertex irreducible linked-pair ζ graph $(\zeta=0, 1)$ or an L graph is called *improper* if a group of m vertices $(0<m<n)$ can be isolated so that there is only one incoming and one outgoing line. A ***proper*** graph is an irreducible linked-pair ζ graph $(\zeta=0, 1)$ or an L graph which is not improper, and in which the *internal* lines undergo the replacements prescribed in Fig. 5.

The definition of a proper L graph is such that we may write down the following expression for the quantity $P(t_2,t_1,\mathbf{k})$ of Eq. (20):

$$P(t_2,t_1,\mathbf{k})=\sum(\text{all proper } L \text{ graphs with given external lines, including those with wiggly line double bonds}). \qquad (24)$$

By iterating Eqs. (22) and (21) for the internal line factors of the proper L graphs in Eq. (24), we clearly generate all of the L graphs in $P(t_2,t_1,\mathbf{k})$. Moreover, L graphs with wiggly-line double bonds are explicitly subtracted by the third replacement of Fig. 5. Finally,

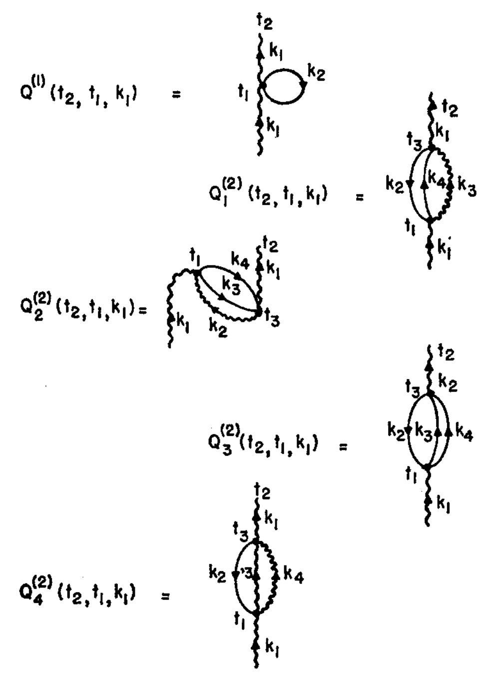

FIG. 3. All but two of the one- and two-vertex L graphs in $\mathfrak{P}(t_2,t_1,\mathbf{k}_1)$, [Eq. (23)], for the case of wiggly external lines. The L graph $Q_4^{(2)}(t_2,t_1,\mathbf{k}_1)$ does not occur in $P(t_2,t_1,\mathbf{k}_1)$, [Eq. (20)]. If the internal line replacements of Fig. 5 are made, then these L graphs become *all* of the one- and two-vertex proper L graphs [see Eq. (24)].

one can verify that the symmetry numbers of the L graphs in Eq. (20) are correctly duplicated when the internal line factors of the proper L graphs in Eq. (24) are iterated. Thus, we see that it is a subset of the L graphs of $\mathfrak{P}(t_2,t_1,\mathbf{k})$, and not $P(t_2,t_1,\mathbf{k})$, that *generates* $P(t_2,t_1,\mathbf{k})$ when we make the internal line replacements of Fig. 5.

Equation (24), in conjunction with Eqs. (21) and (22), constitutes an integral equation, a formal solution of which is rendered by Eq. (20). We shall now show, however, that for a very low-temperature Fermi gas it is incorrect to consider only a finite number of terms of Eq. (20), whereas one may evaluate only the one-vertex proper L graph shown in Fig. 6 to obtain a true first approximation for $L(t_2,t_1,\mathbf{k})$.

In Sec. V of I, it was shown that the pair function (14) can be expressed in terms of reaction matrix, or A matrix, elements. In particular, for an interaction which does not give rise to a two-body bound state, the diagonal elements of the pair function are

$$ {}^{t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_1\mathbf{k}_2\end{bmatrix}_{t_1} = 2\Omega^{-1}\sum_{L=0}^{\infty}(2L+1)[1+\epsilon(-1)^L\delta_{m_1m_2}][C_L(k_{12}k_{12}|t_2t_1)+\delta(t_2-t_1)\langle k_{12}|U_L(t_2,t_2)|k_{12}\rangle], \tag{25}$$

where

$$C_L(kk|t_2t_1)=\left(\frac{2\pi\hbar^2}{m}\right)\Bigg\{(2k)^{-1}\sin 2\delta_L(k)+2\pi^{-1} \times\int_0^\infty dq_0 P\left(\frac{1}{k^2-q_0^2}\right)\cos^2\delta_L(q_0)\langle k|A^{(L)}|q_0\rangle^2\exp[(t_2-t_1)(\omega_k-\omega_{q_0})]\Bigg\}; \tag{26}$$

and

$$\langle k|U_L(t_2,t_2)|k\rangle=\lim_{l\to k}\pi k^{-1}\frac{\partial}{\partial k}\Bigg\{l^{-1}\langle k|A^{(L)}|l\rangle\cos^2\delta_L(l)-k^{-1}\langle l|A^{(L)}|k\rangle\cos^2\delta_L(k) +2\pi^{-1}\int_0^\infty dq_0\cos^2\delta_L(q_0)\langle l|A^{(L)}|q_0\rangle\langle k|A^{(L)}|q_0\rangle\exp[t_2(\omega_k-\omega_l)]\left[P\left(\frac{1}{l^2-q_0^2}\right)-P\left(\frac{1}{k^2-q_0^2}\right)\right]\Bigg\}. \tag{27}$$

We now specialize our considerations to the case of a very low-temperature Fermi gas ($\epsilon=-1$) with strong, short-range interactions. We also make a low-density approximation, in which case the first term of Eq. (26) is proportional to the scattering length a_s. More precisely, the approximations which we shall now make are

$$k|a_s|\sim\rho^{\frac{1}{3}}|a_s|\ll 1 \quad\text{and}\quad |a_s|\lambda_T^{-1}\ll(\rho^{\frac{1}{3}}\lambda_T)^{-1}\ll 1, \tag{28}$$

where λ_T is the thermal wavelength [see (34)].

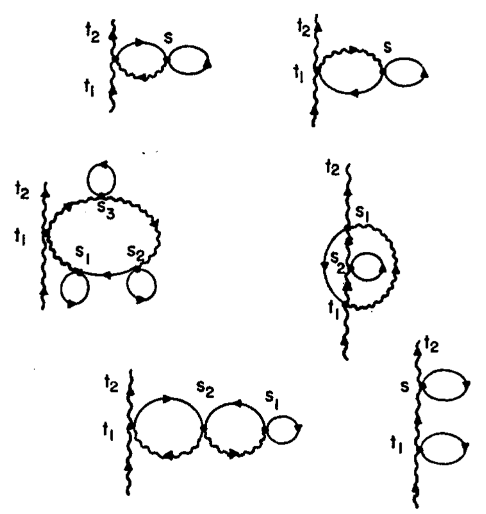

FIG. 4. Some improper L-graphs for the case of wiggly external lines.

If we include only the contribution of the first term of Eq. (26) to the pair function, and if we retain only the one vertex proper L graph in Eq. (24), then in the low-density approximation we obtain the following approximate integral equation for $P^{(1)}(t_2,t_1,\mathbf{k})$:

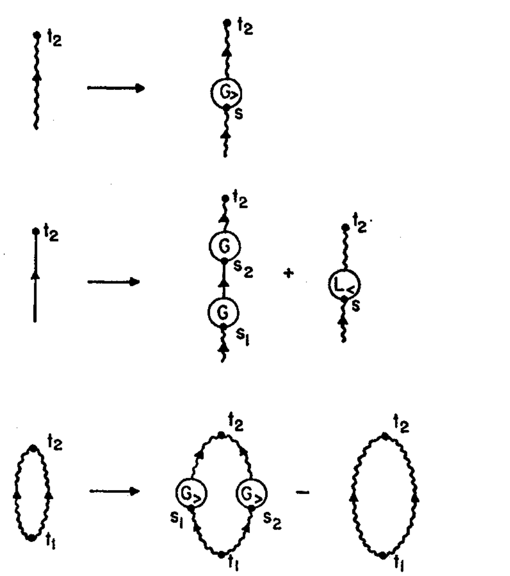

FIG. 5. The internal line replacements which convert a graph which is not improper into a proper graph. The quantities $G_>$ and $L_<$ are: $G_>(t_2,t_1)\equiv G(t_2,t_1)\theta(t_2-t_1)$ and $L_<(t_2,t_1)\equiv L(t_2,t_1)\theta(t_1-t_2)$, where $\theta(x)$ is given by (16).

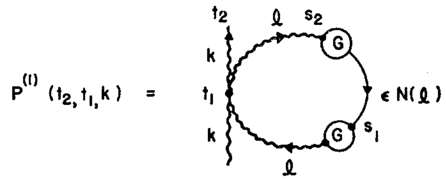

FIG. 6. The one-vertex proper L graph for the case of wiggly external lines.

$$P^{(1)}(t_2,t_1,\mathbf{k})$$
$$=\frac{4J\hbar^2a_s}{(2\pi)^2m}\int d^3l\ N(\mathbf{l})\int_0^\beta ds_1\int_0^\beta ds_2G^{(1)}(t_1,s_1,\mathbf{l})$$
$$\times G^{(1)}(\beta,s_2,\mathbf{l})\theta(t_2-t_1)\theta(s_2-t_1).\quad(29)$$

Equation (29) is diagramatically represented in Fig. 6. The form of the solution for $L(t_2,t_1,\mathbf{k})$ [see Eqs. (21) and (22)] is

$$L^{(1)}(t_2,t_1,\mathbf{k})$$
$$=\Delta^{(1)}(\mathbf{k})\{\exp[-(t_2-t_1)\Delta^{(1)}(\mathbf{k})]\}\theta(t_2-t_1),\quad(30)$$

where

$$\Delta^{(1)}(\mathbf{k})=\frac{4J\hbar^2a_s}{(2\pi)^2m}\int d^3l\ N(\mathbf{l})\ \exp[-\beta\Delta^{(1)}(\mathbf{l})].\quad(31)$$

After substituting Eq. (30) into Eq. (18) and solving for $N(\mathbf{k})$, we find for the lowest order correction to $\langle n(\mathbf{k})\rangle$, Eq. (11),

$$\langle n(\mathbf{k})\rangle\cong\nu^{(1)}(\mathbf{k})\equiv\frac{\exp\{\beta[g-\Delta^{(1)}(\mathbf{k})-\omega_\mathbf{k}]\}}{1+\exp\{\beta[g-\Delta^{(1)}(\mathbf{k})-\omega_\mathbf{k}]\}}.\quad(32)$$

According to Eq. (31), $\Delta^{(1)}(\mathbf{k})$ is actually independent of $\mathbf{k}$. Therefore, to first order the true momentum distribution is the same as the free-particle momentum distribution, and in the low-temperature limit we obtain

$$\Delta^{(1)}=\frac{4J\hbar^2a_s}{(2\pi)^2m}\int d^3l\ \nu^{(1)}(\mathbf{l})$$
$$=8J(3\pi)^{-1}E_F(k_Fa_s)[1+O(k_F\lambda_T)^{-2}],\quad(33)$$

where

$$k_F^3\equiv(2J+1)^{-1}6\pi^2\rho;\quad \lambda_T^2=2\pi\hbar^2\beta m^{-1}.\quad(34)$$

Hence, in the low-temperature, low-density limit, the thermodynamic potential is given by

$$g=E_F[1+8J(3\pi)^{-1}(k_Fa_s)+O(k_Fa_s)^2+O(k_F\lambda_T)^{-2}].\ (35)$$

At $T=0$ Eq. (35) gives the zero of the exponential factors in Eq. (32), which by definition occurs at $k=k_F$. It is clear from this equation that for attractive interactions $g<E_F$ and for repulsive interactions $g>E_F$. This result is consistent with the thermodynamic equation for g at zero temperature, i.e., with

$$g=\left.\frac{\partial\langle E\rangle}{\partial\langle N\rangle}\right|_{\Omega,T}\underset{T\to0}{\longrightarrow}\frac{\langle E\rangle}{\langle N\rangle}+\rho^{-1}\mathcal{P},\quad(36)$$

where $\mathcal{P}$ is the pressure.

We now observe that $\beta\Delta^{(1)}\sim\rho a_s\lambda_T^2$, and therefore we see from Eq. (32) that one cannot calculate the momentum distribution in the low-temperature limit by evaluating a finite number of L graphs in Eq. (17). It was absolutely necessary to consider the integral equation (24) rather than its formal power series solution (20).

The evaluation of the higher order corrections to the momentum distribution is more difficult than the lowest order solution presented in this section, principally because of the complexity of the higher order proper L graphs. It becomes necessary to introduce very systematic procedures for evaluating L graphs, and one is eventually led to introduce the Λ transformation of Sec. IV. We shall also be interested in applying the higher order solutions which we obtain for $L(t_2,t_1,\mathbf{k})$ to the calculation of the thermodynamic quantities of a Fermi gas. For these reasons, a further general study of both ζ graphs and L graphs will form the content of the next section. Then in Sec. IV we shall return to the matter of determining the higher order corrections to Eqs. (30) and (31). We finally remark that the results of the next three sections will substantiate the validity of the lowest order calculation just presented, and it appears to be possible to extend the calculation to all orders.

III. MASTER GRAPHS

In Sec. II we introduced the concept of proper graphs as a certain set of linked-pair graphs with the line replacements of Fig. 5. The important change which we shall make in this section will be to consider the line replacements of Fig. 5 as vertex modifications. This step will then lead to the introduction of master graphs.

We first define a *generalized pair function*,

$$^{t_1t_2}\begin{Bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{Bmatrix}_{t_0}\equiv\int_0^\beta ds_1ds_2\,\Gamma(t_1s_1\mathbf{k}_1;t_2s_2\mathbf{k}_2)\ ^{s_1s_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0},\quad(37)$$

in terms of the pair function (15) and a function $\Gamma(t_1s_1\mathbf{k}_1;t_2s_2\mathbf{k}_2)$, which is in turn defined in terms of $N(k)$ and $G(t_2,t_1,\mathbf{k})$ according to the nature of the outgoing lines at the vertex t_0. If both outgoing lines are *internal* lines, then

$$\Gamma(t_1s_1\mathbf{k}_1;t_2s_2\mathbf{k}_2)$$
$$=\mathcal{G}(t_1,s_1,\mathbf{k}_1)\mathcal{G}(t_2,s_2,\mathbf{k}_2)\quad\text{when}\quad t_1\neq t_2$$
$$=\mathcal{G}(t_2,s_1,\mathbf{k}_1)\mathcal{G}(t_2,s_2,\mathbf{k}_2)-\delta(t_2-s_1)\delta(t_2-s_2)\quad\text{when}\quad t_1=t_2,\quad(38)$$

where

$$\mathcal{G}(t,s,\mathbf{k})\equiv G(t,s,\mathbf{k})+\epsilon\mathfrak{N}(t,\mathbf{k})G(\beta,s,\mathbf{k}),\quad(39)$$

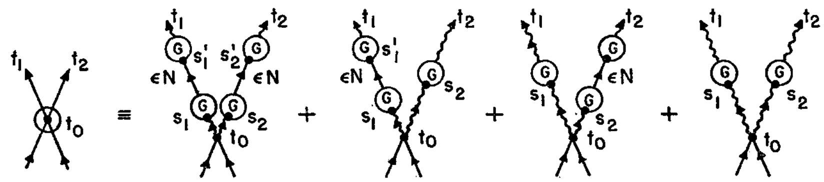

FIG. 7. The definition of the generalized vertex symbol for the case of two outgoing internal lines with $t_1 \neq t_2$; see Eqs. (37), (38), and (39).

and

$$\mathfrak{N}(t,\mathbf{k}) \equiv N(\mathbf{k}) \int_0^\beta ds' G(t,s',\mathbf{k}). \tag{40}$$

The quantity $G(t_2,t_1,\mathbf{k})$ is defined by Eq. (22). If an outgoing line at the vertex t_0 is an *external* line, then in Eq. (38) we must make the replacement

$$\mathcal{G}(t,s,\mathbf{k}) \rightarrow G(t,s,\mathbf{k}) \quad \text{(for an outgoing external line)}, \tag{41}$$

where $\mathbf{k}$ is the momentum variable associated with the external line. In order to indicate in a graph that a vertex represents a generalized pair function we shall use the usual vertex symbol with a circle around it. Such a symbol will be called a *generalized vertex*. The definition (38) is illustrated in Fig. 7 for the case of two outgoing internal lines with $t_1 \neq t_2$.

A Qth order *master* ζ *graph* ($\zeta = 0$, 1) is defined to be a collection of Q generalized vertices which are connected by $(2Q-\zeta)$ directed lines called *internal* lines, and to which are attached ζ *outgoing external* lines and ζ *incoming external* lines. All of the lines in master ζ graphs are solid lines, and each master ζ graph is *not reducible*. Two master ζ graphs are different if their topological structures are different.

To each master ζ graph we assign a term which is determined by the following procedures:

(i) Associate with each internal line a different integer i ($i=1, 2, \cdots m$), where $m = 2Q-\zeta$, and a corresponding momentum $\mathbf{k}_i$.

(ii) If $\zeta \neq 0$, then associate the external lines with certain pre-given momenta.

(iii) Assign a factor S^{-1} to the entire graph, where

$$S = \text{symmetry number}. \tag{42}$$

The symmetry number S is defined to be the total number of permutations of the m integers associated with the internal lines that leave the graph topologically (including the positions of these numbers relative to the lines) unchanged. For convenience in discussion, we shall in Appendix A refer to the permutations defined here as "internal line permutations" rather than as the "permutations of the numbers associated with the internal lines."

(iv) Associate with each generalized vertex a temperature variable t_i, and assign as a factor a corresponding generalized pair function whose "upper temperature variables" are those at the vertices to which the outgoing lines from the vertex t_i are directed. The upper temperature variable for an outgoing external line is β.

(v) Assign a factor ϵ^{P_B} to the product of the generalized pair functions, where P_B is the total permutation of the $2Q$ bottom-row momenta with respect to the $2Q$ top-row momenta.

(vi) Integrate over all of the temperature variables from 0 to β, and sum over the m internal momentum (and spin) coordinates.

We see that master ζ graphs are structurally the same as irreducible ζ graphs constructed with only T_2-vertex functions (Sec. I). Before demonstrating that the momentum distribution and the grand potential can be expressed in terms of master 1 graphs and 0 graphs, respectively, we introduce master L graphs.

A *master L graph* is defined to be a master 1 graph in which (1) the integration over the temperature variable t_1 at the vertex to which the incoming external line attaches *is not* performed; (2) the last sentence of rule (iv) is changed to read: "The upper temperature variable for an outgoing external line is $t_2 (<\beta)$"; and (3) the external lines may be wiggly or solid, as in the four cases shown in Fig. 2(b).

We can now write down the following equation for the quantity $L(t_2,t_1,\mathbf{k})$ of Eqs. (19) and (21):

$$\begin{aligned} L(t_2,t_1,\mathbf{k}) &= \int_0^\beta ds G(t_2,s,\mathbf{k}) P(s,t_1,\mathbf{k},G) \\ &= \sum (\text{all different master } L \text{ graphs with} \\ &\qquad \text{given external lines}). \end{aligned} \tag{43}$$

The validity of Eq. (43) can be seen in two ways. On the one hand, Eq. (43) follows by comparing the definition of master L graphs with Eq. (24) and the subsequent discussion. On the other hand, one may directly verify that the right-hand side of the second line of Eq. (43) generates all of the L graphs in Eq. (19) with their correct symmetry numbers (see proofs of Theorems 2 and 3 in Appendix A).

According to (41) and the first line of Eq. (43), the function $P(s,t)$ is the sum over all master L graphs with no external line factors. We have explicitly indicated that $P(s,t)$ is a functional of $G(s_2,s_1)$ in the first line of Eq. (43), because we now wish to introduce two new quantities $L_\tau(t_2,t_1)$ and $G_\tau(t_2,t_1)$, which are slight modifications of the functions $L(t_2,t_1)$ and $G(t_2,t_1)$. These new

functions are defined, for $\beta>\tau>(t_2,t_1)$, as follows:

$$L_\tau(t_2,t_1,\mathbf{k})\equiv\int_0^\tau ds G_\tau(t_2,s,\mathbf{k})P(s,t_1,\mathbf{k},G_\beta)\underset{\tau\to\beta}{\longrightarrow} L_\beta(t_2,t_1,\mathbf{k})=L(t_2,t_1,\mathbf{k}),$$
$$G_\tau(t_2,t_1,\mathbf{k})\equiv\delta(t_2-t_1)+\epsilon L_\tau(t_2,t_1,\mathbf{k})\longrightarrow G_\beta(t_2,t_1,\mathbf{k})=G(t_2,t_1,\mathbf{k}). \tag{44}$$

At first sight it would seem that the functions L_τ and G_τ are modifications which are unessential to the present development. However, we shall see below that these functions are encountered when one attempts to express the grand potential in terms of the generalized pair functions (37). We emphasize that the definitions (44) do not entail any modifications of the *internal* line factors of $G(s_2,s_1)$ in $P(s,t)$.

Momentum Distribution

It is clear from the discussion below Eq. (43) and the definition of master L graphs, that Eq. (18) for the quantity $N(\mathbf{k})$ may be written

$$N(\mathbf{k})=\nu(\mathbf{k})[1+N(\mathbf{k})\sum(\text{all different master 1 graphs})]. \tag{45}$$

Thus, by means of Eqs. (11) and (45), the momentum distribution $\langle n(\mathbf{k})\rangle$ is given explicitly in terms of master 1 graphs.

By combining Eqs. (11) and (18), one can also show that the momentum distribution is equal to the function $\mathfrak{N}(\beta,\mathbf{k})$ of Eq. (40):

$$\mathfrak{N}(\beta,\mathbf{k})=\langle n(\mathbf{k})\rangle. \tag{46}$$

Equation (46) suggests, quite apart from any other reasons, that it may be quite useful to express the grand potential in terms of generalized pair functions, even though the quantity which appears in Eq. (39) is $\mathfrak{N}(t,\mathbf{k})$ and not $\mathfrak{N}(\beta,\mathbf{k})$. Thus, one suspects that in the calculation of thermodynamic quantities a formulation in terms of $\mathfrak{N}(t,\mathbf{k})$ will lead more directly to physical results than will a formulation in terms of $N(\mathbf{k})$ or the quite unphysical $\nu(\mathbf{k})$. Indeed, we will find after performing the Λ transformation of the next section, that the calculations of the second and third order corrections to the momentum distribution and ground-state energy of a Fermi gas are quite straightforward in the master-graph formulation.

Grand Potential

In order to demonstrate that the grand potential can be expressed in terms of master 0-graphs we first use Eq. (18) to rewrite Eq. (13) as follows:

$$\begin{aligned}f^{(s)}(N,\Omega,\beta)&=\epsilon\sum_{\mathbf{k}}\ln[1+\epsilon\langle n(\mathbf{k})\rangle]\\&\quad+\sum(\text{all different irreducible 0-graphs})\\&\quad-\epsilon\sum_{\mathbf{k}}N(\mathbf{k})\int_0^\beta dt\, L(\beta,t,\mathbf{k}).\end{aligned} \tag{47}$$

We next write the last term of this equation as

$$\sum_{\mathbf{k}}[\epsilon N(\mathbf{k})]\int_0^\beta dt\, L(\beta,t,\mathbf{k})=\sum_1 S_1^{-1}\sum_{\mathbf{k}}[\epsilon N(\mathbf{k})]L_1(\beta,\mathbf{k}), \tag{48}$$

where $\sum_1$ represents the sum over all irreducible linked pair 1 graphs $S_1^{-1}L_1(\beta,\mathbf{k})$. We have explicitly exhibited the multiplying symmetry number S_1^{-1} for each 1 graph in this sum. Now, when the sum over all momentum states is performed, as in Eq. (48), every irreducible linked pair 1 graph which arises from a given irreducible linked pair 0-graph $S_0^{-1}L_0(\beta)$ has the same expression. That is, if N_0 is the number of solid lines in an irreducible linked-pair 0-graph, then

$$\sum_1 S_1^{-1}\sum_{\mathbf{k}}[\epsilon N(\mathbf{k})]L_1(\beta,\mathbf{k})=\sum_0 N_0S_0^{-1}L_0(\beta), \tag{49}$$

as one can verify by comparing Eqs. (8) and (9), or by referring to LY IV, Eq. (IV.103). Thus, after substituting Eqs. (48) and (49) into Eq. (47), we obtain for the grand potential

$$\Omega f^{(s)}(N,\Omega,\beta)=\epsilon\sum_{\mathbf{k}}\ln[1+\epsilon\langle n(\mathbf{k})\rangle]-\sum_0(N_0-1)S_0^{-1}L_0(\beta). \tag{50}$$

In the last two equations, $\sum_0$ represents the sum over all irreducible linked pair 0-graphs and $L_0(\beta)$ is a linked-pair 0-graph without its symmetry number S_0. Equation (50) is the starting point for the following analysis.

We now distinguish between "open" and "closed" graphs.

Definitions

An improper irreducible linked-pair ζ graph ($\zeta=0,1$) or an improper L graph is an *open graph*, if it includes at least one not improper L graph with an external (with respect to the proper L graph) *solid* line.

A *closed* ζ graph ($\zeta=0,1$) or a *closed L graph* is an irreducible linked-pair ζ graph ($\zeta=0,1$) or an L graph, which is not open, with each internal solid line representing a factor $\epsilon\mathfrak{N}(t,\mathbf{k})G(\beta,s,\mathbf{k})$ instead of $\epsilon N(\mathbf{k})$, and where t and s are the temperature variables at the vertices touched by the head and tail ends, respectively, of the internal solid line. Closed graphs may be either improper or not improper.

In Fig. 8 we have exhibited a number of open and closed 0-graphs together with their associated factors S_0^{-1}, (N_0-1), and S_C^{-1}.

The reason for introducing a distinction between open and closed graphs is that the process of reducing the

grand potential to a sum of terms which are calculated only from master graphs must be made in two steps. In the first step, the grand potential is expressed as a sum over only closed graphs. Then, in the second step the master graphs are introduced. Thus, one must first prove the following result for the grand potential:

$$\Omega f^{(s)}(N,\Omega,\beta)=\epsilon\sum_{\mathbf{k}}\ln[1+\epsilon\langle n(\mathbf{k})\rangle]-\sum_C(N_C-1)S_C^{-1}L_C(\beta). \quad (51)$$

In this equation $\sum_C$ represents the sum over all different closed 0 graphs $S_C^{-1}L_C(\beta)$. The quantities N_C and S_C have the same meanings as the corresponding quantities N_0 and S_0 in Eq. (50).

Equation (51) is proved in Appendix A. Its significance is that the grand potential is expressed in terms of the ordinary pair functions (15) with modified solid line factors. In order to proceed with the introduction of the generalized pair functions (37), we next carry out in reverse the steps which led from Eq. (47) to (50) and obtain

$$\Omega f^{(s)}(N,\Omega,\beta)=\epsilon\sum_{\mathbf{k}}\ln[1+\epsilon\langle n(\mathbf{k})\rangle]+\sum_C S_C^{-1}L_C(\beta)-\epsilon\sum_{\mathbf{k}}\int_0^\beta dtL(\beta,t,\mathbf{k})\mathfrak{N}(t,\mathbf{k}). \quad (52)$$

According to Eq. (43), the last term of (52) can be immediately expressed in terms of master L graphs, a step which it was not convenient to perform earlier in Eq. (47). Therefore, we now only need to consider the second term of Eq. (52), which is the sum over all closed 0-graphs.

As a first guess, one might try to write the second term of Eq. (52) as a sum over all master 0-graphs:

$$\Omega F(N,\Omega,\beta)\equiv\sum(\text{all different master 0-graphs}). \quad (53)$$

When this is attempted, it is found, for example, that

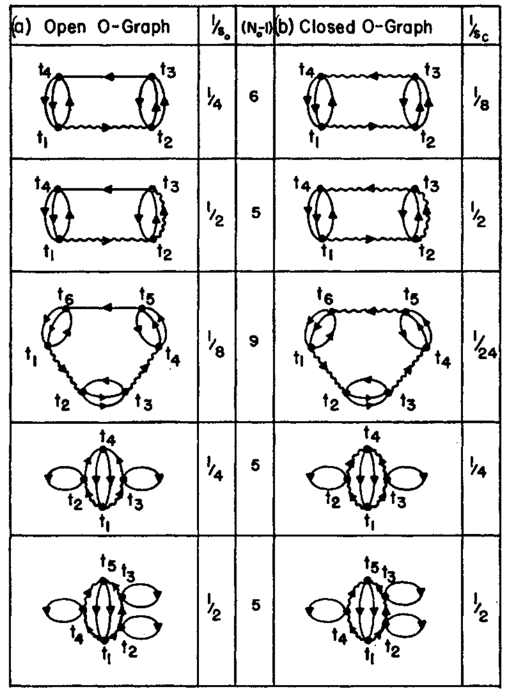

FIG. 8. (a) Some open 0-graphs. For each 0-graph, its inverse symmetry number S_0^{-1} and the number of solid lines minus one, i.e., N_0-1, is given in the second and third column, respectively; (b) Some closed 0-graphs. The corresponding inverse symmetry numbers S_C^{-1} are given in the fifth column. All of the closed 0-graphs shown are improper.

none of the symmetry numbers of Fig. 8(b) are correctly duplicated. The quantity $F(N,\Omega,\beta)$ "overgenerates" the second term of Eq. (52). In spite of the fact that $F(N)$ does not correctly generate the sum over all closed 0-graphs, one may continue in this direction, and try to determine the difference between $F(N)$ and the sum over all closed 0-graphs. This difference can in fact be identified, and one finds the following result.

$$\Omega F(N)-\sum_C S_C^{-1}L_C(\beta)=\epsilon\sum_{\mathbf{k}}\sum_{m=2}^{\infty}\epsilon^m\left(\frac{m-1}{m}\right)\int_0^\beta dt_1dt_2\cdots dt_m\,P(t_1,t_2,\mathbf{k})P(t_2,t_3,\mathbf{k})\cdots P(t_{m-1},t_m,\mathbf{k})P(t_m,t_1,\mathbf{k}), \quad (54)$$

where $P(t_1,t_2,\mathbf{k})$ is the function which appears in Eqs. (20) and (43). Equation (54) is proved in Appendix B.

Before substituting Eq. (54) into Eq. (52), we shall write the right-hand side in a more compact form by including the $m=1$ term (which is zero) and then separating the m/m terms from the $1/m$ terms:

$$\begin{aligned}\Omega F(N)-\sum_C S_C^{-1}L_C(\beta)&=\epsilon\sum_{\mathbf{k}}\sum_{m=1}^{\infty}\epsilon^m\left(\frac{m-1}{m}\right)\int_0^\beta dt_1dt_2\cdots dt_m\,P(t_1,t_2,\mathbf{k})P(t_2,t_3,\mathbf{k})\cdots P(t_m,t_1,\mathbf{k})\\&=\epsilon\sum_{\mathbf{k}}\sum_{m=1}^{\infty}\epsilon^m\int_0^\beta dt_1\cdots dt_m\,P(t_1,t_2,\mathbf{k})\cdots P(t_m,t_1,\mathbf{k})\\&\qquad-\epsilon\sum_{\mathbf{k}}\sum_{m=1}^{\infty}\epsilon^m\int_0^\beta dt_1\int_0^{t_1}dt_2\cdots dt_m\,P(t_1,t_2,\mathbf{k})P(t_2,t_3,\mathbf{k})\cdots P(t_m,t_1,\mathbf{k})\\&=\sum_{\mathbf{k}}\int_0^\beta dt_1dt_2\,G_\beta(t_1,t_2,\mathbf{k})P(t_2,t_1,\mathbf{k})-\sum_{\mathbf{k}}\int_0^\beta dt_1\int_0^{t_1}dt_2\,G_{t_1}(t_1,t_2,\mathbf{k})P(t_2,t_1,\mathbf{k}).\end{aligned} \quad (55)$$

In the last line of this equation, we have used the definition (44) of the function $G_\tau(t_1,t_2,\mathbf{k})$, with $\tau=t_1>t_2$. Equation (55) can now be substituted into (52) to obtain

$$\Omega f^{(s)}(N,\Omega,\beta)=\epsilon\sum_{\mathbf{k}}\ln[1+\epsilon\langle n(\mathbf{k})\rangle]+\Omega F(N,\Omega,\beta)-\epsilon\sum_{\mathbf{k}}\int_0^\beta dtL(\beta,t,\mathbf{k})\mathfrak{N}(t,\mathbf{k})$$
$$-\sum_{\mathbf{k}}\int_0^\beta dt_1\left\{\int_0^\beta dt_2G_\beta(t_1,t_2,\mathbf{k})-\int_0^{t_1}dt_2G_{t_1}(t_1,t_2,\mathbf{k})\right\}P(t_2,t_1,\mathbf{k}).\quad(56)$$

In Eq. (56) all quantities are expressed in terms of the generalized pair functions (37), except the last term which involves a modified external line factor. We thus have achieved our goal of expressing both the momentum distribution and the grand potential in terms of master graphs.

IV. A TRANSFORMATION

Equations (18), (43), and (37) constitute a set A of coupled integral equations, a formal solution of which is given by the linked-pair expansion, Eqs. (18) and (19). As we saw in Sec. II, the latter expansion does not allow one to calculate even the lowest order correction to the momentum distribution of a very low-temperature Fermi gas, while a true first order approximation could be secured by considering the one-vertex master (or proper) L graph, and retaining in the pair function (25) only the term linear in a_s. Similarly, we have found that a true second order approximation can be made by considering both the one-vertex master L graph and the two-vertex master L graph [Fig. 9], retaining in the pair functions (14) of the latter only the terms linear in a_s, but retaining in the pair function of the former terms both linear and quadratic in a_s. Furthermore, the calculation has been successfully extended to third order by an analogous treatment including the three-vertex master L graphs. We have not attempted to prove conclusively the convergence of this treatment.

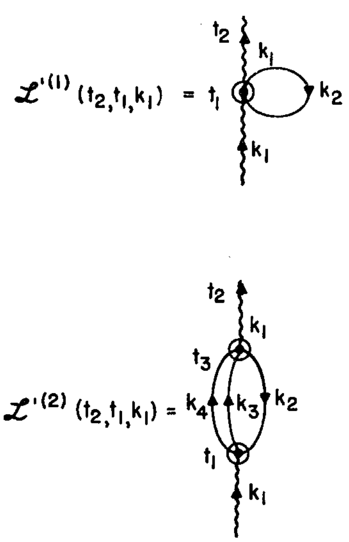

FIG. 9. The one- and two-vertex master L graphs for the case of wiggly external lines. We use the generalized vertex symbol of Fig. 7 for *both* unprimed and primed generalized pair functions, since it is always clear in context which one is meant.

Formal Power Series Solution

The procedure outlined above permits one to write down a set of coupled integral equations $A\{n\}$, which approximates the set A to nth order in the interaction range parameters of the Fermi gas. The solution of the set $A\{n\}$ may be formally obtained as a power series expansion in these parameters by considering all L graphs generated from master L graphs with no more than n vertices. Each L graph, of course, involves factors $N(\mathbf{k})$ which are determined by iterating Eq. (18).

The first step in the power series solution is to replace $L(t_2,t_1,\mathbf{k})$ by zero on the right-hand side of Eqs. (18) and (37). One immediately finds $N(\mathbf{k})=\nu(\mathbf{k})$ and this result is substituted into the right-hand side of Eq. (37). Thus, in the first step we consider only those L graphs generated by Eq. (43) which are *not improper* and we replace $N(\mathbf{k})$ by $\nu(\mathbf{k})$. The calculated $L(t_2,t_1,\mathbf{k})$ is found to possess a "temperature-independent" part, $[A(\mathbf{k})\theta(t_2-t_1)+B(\mathbf{k})\delta(t_2-t_1)]$, as well as a temperature dependent part. For example, for a very low-temperature, low-density Fermi gas of hard spheres of diameter a, (in which the only important momentum values are those for which $k\lesssim\rho^{\frac{1}{3}}$), the former term contributes ϕ to the momentum distribution and the latter contributes χ, where

$$\phi(k)\sim\rho a\lambda_T{}^2[1+b_1(ka)+b_2(ka)^2+\cdots]+d_0(ka)^3[1+d_1(ka)+\cdots],\quad(57)$$
$$\chi(k)\sim C_2(ka)^2+C_3(ka)^3+\cdots.$$

If the calculated $L(t_2,t_1,\mathbf{k})$ of the preceding paragraph is substituted into the right-hand side of Eqs. (18) and (37), a new $N(\mathbf{k})$ and new $L(t_2,t_1,\mathbf{k})$ can be determined. In this second step some of the improper L graphs generated by Eq. (43) are considered. Continuing in this manner one essentially obtains for the dominant low-temperature terms in the momentum distribution successively higher powers of ϕ. The result of formally summing over all L graphs generated from the original master L graphs is that the dominant low-temperature behavior of $L(t_2,t_1,\mathbf{k})$ is an exponential dependence upon ϕ. Thus, the form of the exact solution $L^{(1)}(t_2,t_1,\mathbf{k})$, Eq. (30), for the set of equations $A\{1\}$ is essentially unchanged, except for *small* terms, when higher order corrections are included.

Low-Temperature Approximation Method

It is clear from (57) that the power series solution of the set $\{A\}$ of coupled integral equations is not useful at very low temperatures where $\phi\gg1$. More specifically, in the very low-temperature and low-density region in which $\rho|a_s|\lambda_T{}^2\gg1$, we must attempt to solve the set $\{A\}$ for a Fermi gas using a more direct method. From the preceding analysis we may expect that the most important difference between $L(t_2,t_1,\mathbf{k})$ and its lowest order approximation $L^{(1)}(t_2,t_1,\mathbf{k})$ will be that the function $\Delta^{(1)}$ of Eq. (31) is replaced by a momentum dependent function $\Delta(\mathbf{k})$. We also know that a "small" term of the form $\chi(k)$, Eq. (57), will occur in $L(t_2,t_1,\mathbf{k})$ and $N(\mathbf{k})$. To solve the approximate set of coupled integral equations $A\{n\}$, we therefore try a first solution for $L(t_2,t_1,\mathbf{k})$ and $N(\mathbf{k})$ of the form

$$L^{(1)}(t_2,t_1,\mathbf{k})=\Delta(\mathbf{k})\exp[\epsilon(t_2-t_1)\Delta(\mathbf{k})],$$
$$N^{(1)}(\mathbf{k})=\frac{\exp[\beta(g-\omega_{\mathbf{k}})]}{1-\epsilon\exp[\beta(g-\Delta(\mathbf{k})-\omega_{\mathbf{k}})]},\tag{58}$$

where $N^{(1)}(\mathbf{k})$ is obtained by substituting $L(t_2,t_1)\cong L^{(1)}(t_2,t_1)$ into Eq. (18).

The straightforward procedure which we can now adopt is to substitute the trial solution (58) into the set $A\{2\}$. Replacing $L(t_2,t_1,\mathbf{k})$ and $N(\mathbf{k})$ on the right-hand side of Eq. (37) by the "first solution" (58), we can calculate a "second solution" $L^{(2)}(t_2,t_1,\mathbf{k})$ which can then be substituted into the right-hand side of Eq. (18) to determine $N^{(2)}(\mathbf{k})$. The second solution $L^{(2)}(t_2,t_1,\mathbf{k})$ and $N^{(2)}(\mathbf{k})$, which will include the correct lowest order term of $\chi(k)$ and the second order correction to $\Delta(\mathbf{k})$, can now be substituted into the right-hand sides of the set $A\{3\}$. By successively iterating each of the approximate solutions $L^{(j-1)}(t_2,t_1,\mathbf{k})$ and $N^{(j-1)}(\mathbf{k})$ in the set $A\{j\}$ one can eventually arrive at the nth solution $L^{(n)}(t_2,t_1,\mathbf{k})$ and $N^{(n)}(\mathbf{k})$ to the approximate set of coupled integral equations $A\{n\}$. The nth order approximation to the momentum distribution can then be calculated by substituting $N^{(n)}(\mathbf{k})$ into Eq. (11).

Λ-Transformation

A more elegant method of solving the set $A\{n\}$ consists of first transforming the entire set A so that the dominant low-temperature behavior, i.e., the exponential character, of the solution is taken into account from the start. This requires transforming $N(\mathbf{k})$, $\nu(\mathbf{k})$, the pair function, and $L(t_2,t_1,\mathbf{k})$ in such a way that no temperature independent parts $[A(\mathbf{k})\theta(t_2-t_1)+B(\mathbf{k})\cdot\delta(t_2-t_1)]$ arise in the power series solution of the transformed set A' of integral equations. Therefore, after such a Λ *transformation* the power series treatment yields not just a formal solution, but a true solution valid to any desired order in interaction range parameters. As one might expect, two additional equations must be coupled to the set A' in order to determine the functions $A(\mathbf{k})$ and $B(\mathbf{k})$.

In the preceding discussion, it was somewhat misleading to refer to $A(\mathbf{k})$ and $B(\mathbf{k})$ as temperature-independent functions, since, as we shall see, both $A(\mathbf{k})$ and $B(\mathbf{k})$ include a β dependence, which is negligible for a Fermi gas only in the low-temperature limit. Actually, what is meant by "temperature-independent" parts is an independence of any temperature integration variables, e.g., t_2 and t_1. With these remarks in mind we now introduce the Λ transformation in terms of two momentum dependent functions $A(\mathbf{k},\beta)=A(\mathbf{k})$ and $B(\mathbf{k},\beta)=B(\mathbf{k})$ which will be identified later. We shall also make use of the following derived quantities:

$$\Delta(\mathbf{k},\beta)=\Delta(\mathbf{k})\equiv A(\mathbf{k})[1+\epsilon B(\mathbf{k})]^{-1},\tag{59}$$
$$\Lambda(t_2-t_1,\,\mathbf{k})\equiv A(\mathbf{k})\theta(t_2-t_1)+B(\mathbf{k})\delta(t_2-t_1),\tag{60}$$
$$\Lambda_0(t_2-t_1,\,\mathbf{k})\equiv[1+\epsilon B(\mathbf{k})]^{-1}[B(\mathbf{k})\delta(t_2-t_1)+\Delta(\mathbf{k})\theta(t_2-t_1)],\tag{61}$$
$$G_0(t_2-t_1,\,\mathbf{k})\equiv[1+\epsilon B(\mathbf{k})]\{\delta(t_2-t_1)+\epsilon\Delta(\mathbf{k})\exp[\epsilon(t_2-t_1)\Delta(\mathbf{k})]\theta(t_2-t_1)\},\tag{62}$$
$$\zeta(t_2-t_1,\,\mathbf{k})\equiv[1+\epsilon B(\mathbf{k})]\exp[\epsilon(t_2-t_1)\Delta(\mathbf{k})]\theta(t_2-t_1).\tag{63}$$

The Λ *transformation* is defined to be a transformation of the quantities involved in the set A of the integral equations; namely,

$$N'(\mathbf{k})\equiv N(\mathbf{k})\zeta(\beta,\mathbf{k}),\tag{64}$$
$$\nu'(\mathbf{k})\equiv\frac{\zeta(\beta,\mathbf{k})\nu(\mathbf{k})}{1+\epsilon\nu(\mathbf{k})-\epsilon\zeta(\beta,\mathbf{k})\nu(\mathbf{k})},\tag{65}$$
$$^{t_1t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\equiv\int_0^\beta ds_1\int_0^\beta ds_2\,G_0(t_1-s_1,\,\mathbf{k}_1)G_0(t_2-s_2,\,\mathbf{k}_2)\,{}^{s_1s_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_0}$$
$$\times\exp\{-\epsilon[t_1\Delta(\mathbf{k}_1)+t_2\Delta(\mathbf{k}_2)]\}\exp\{\epsilon t_0[\Delta(\mathbf{k}_3)+\Delta(\mathbf{k}_4)]\},\tag{66}$$
$$L'(t_2,t_1,\mathbf{k})\equiv\exp[-\epsilon(t_2-t_1)\Delta(\mathbf{k})]\left\{L(t_2,t_1,\mathbf{k})-\int_0^\beta dsG(t_2,s,\mathbf{k})\Lambda_0(s-t_1,\,\mathbf{k})\right\}.\tag{67*}$$

What we wish to show is that the set A of integral equations can be expressed in terms of the above four quantities instead of the corresponding unprimed quantities. We shall then show that $A(\mathbf{k})$ and $B(\mathbf{k})$ can be chosen so that $L'(t_2,t_1,\mathbf{k})$ contains no "temperature-independent" parts.

We begin by performing a task which is unessential to the present development, but a necessary preliminary to any application of the method; namely, we simplify Eq. (66) for the primed pair function by inserting the definition (62) of $G_0(s-t,\,\mathbf{k})$. We obtain the following result:

$$\begin{aligned}{}^{t_1t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0} &= \theta(t_1-t_2)\;{}^{t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_2-t_0)+\theta(t_2-t_1)\;{}^{t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_1-t_0)\quad\text{if}\quad t_1\neq t_2\\ &= {}^{t_1}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_0}\theta(t_1-t_0)\quad\text{if}\quad t_1=t_2,\end{aligned}\tag{68}$$

where

$$\begin{aligned}{}^{t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_1} &= [1+\epsilon B(\mathbf{k}_1)][1+\epsilon B(\mathbf{k}_2)]\exp\{\epsilon t_1[\Delta(\mathbf{k}_3)+\Delta(\mathbf{k}_4)]\}\Big\{\exp(-\epsilon t_2[\Delta(\mathbf{k}_1)+\Delta(\mathbf{k}_2)])\;{}^{t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_1}\\ &\quad+\epsilon[\Delta(\mathbf{k}_1)+\Delta(\mathbf{k}_2)]\int_{t_1}^{t_2}ds\,\exp(-\epsilon s[\Delta(\mathbf{k}_1)+\Delta(\mathbf{k}_2)])\;{}^{s}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_1}\Big\}.\end{aligned}\tag{69}$$

In order to perform the last temperature integration in Eq. (69), it is necessary to write out the temperature dependences of the pair function (14) in greater detail. In Sec. V of I, it was shown that the most general form of the pair function is

$$\begin{aligned}{}^{t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}_{t_1} &= \exp[t_1(\omega_1+\omega_2-\omega_3-\omega_4)]f_1(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)+\int d^3k_5d^3k_6\,\exp[t_2(\omega_1+\omega_2-\omega_5-\omega_6)]\\ &\quad\times\exp[t_1(\omega_5+\omega_6-\omega_3-\omega_4)]f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)P\left(\frac{1}{\omega_1+\omega_2-\omega_5-\omega_6}\right)\\ &\quad+\delta(t_2-t_1)\,\exp[t_2(\omega_1+\omega_2-\omega_3-\omega_4)]f_3(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4).\end{aligned}\tag{70}$$

Substitution of Eq. (70) into Eq. (69) yields for the primed pair function the expression

$$\begin{aligned}{}^{t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_1} &= [1+\epsilon B(\mathbf{k}_1)][1+\epsilon B(\mathbf{k}_2)]\Big\{\exp[t_1(\omega_1'+\omega_2'-\omega_3'-\omega_4')]f_1(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\\ &\quad+\int d^3k_5d^3k_6\,\exp[t_2(\omega_1'+\omega_2'-\omega_5-\omega_6)]\,\exp[t_1(\omega_5+\omega_6-\omega_3'-\omega_4')]f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)\\ &\quad\times P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)+\delta(t_2-t_1)\,\exp[t_2(\omega_1'+\omega_2'-\omega_3'-\omega_4')]f_3(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\\ &\quad+\epsilon[\Delta(\mathbf{k}_1)+\Delta(\mathbf{k}_2)]\,\exp[t_1(\omega_1'+\omega_2'-\omega_3'-\omega_4')]\Big[f_3(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4)\\ &\quad-\int d^3k_5d^3k_6f_2(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_5\mathbf{k}_6|\mathbf{k}_3\mathbf{k}_4)P\left(\frac{1}{\omega_1+\omega_2-\omega_5-\omega_6}\right)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_5-\omega_6}\right)\Big]\Big\},\end{aligned}\tag{71}$$

where

$$\omega_{\mathbf{k}}'\equiv\omega_{\mathbf{k}}-\epsilon\Delta(\mathbf{k})=\hbar^2\mathbf{k}^2/2m-\epsilon\Delta(\mathbf{k}).\tag{72}$$

The functions f_1, f_2, and f_3 can be determined from Sec. V of I.

We now turn to the objective of determining the transformed set A' of coupled integral equations. For this purpose it is first necessary to invert Eq. (67), which can be accomplished with the aid of the integral equation satisfied by $G_0(t_2-t_1)$ and $\Lambda_0(t_2-t_1)$; namely,

$$G_0(t_2-t_1,\,\mathbf{k})=\delta(t_2-t_1)+\epsilon\int_0^\beta ds\,G_0(t_2-s,\,\mathbf{k})\Lambda_0(s-t_1,\,\mathbf{k}).\tag{73*}$$

Defining a quantity

$$G'(t_2,t_1,\mathbf{k})\equiv\delta(t_2-t_1)+\epsilon L'(t_2,t_1,\mathbf{k}),\tag{74}$$

one obtains for the inverted form of Eq. (67)

$$G(t_2,t_1,\mathbf{k})=\int_0^\beta ds\,\exp[\epsilon(t_2-s)\Delta(\mathbf{k})]\times G'(t_2,s,\mathbf{k})G_0(s-t_1,\,\mathbf{k}).\quad(75)^*$$

From Eq. (75) it follows that

$$\int_0^\beta dt\,G(\beta,t,\mathbf{k})=\zeta(\beta,\mathbf{k})\int_0^\beta dt\,G'(\beta,t,\mathbf{k}),\quad(76)$$

where $\zeta(\beta,\mathbf{k})=\int_0^\beta dt\,G_0(\beta-t,\,\mathbf{k})$ is given by (63). Using (76) in conjunction with the definitions (64) and (65), one finds that Eq. (18) becomes after the Λ transformation

$$N'(\mathbf{k})=\nu'(\mathbf{k})\left[1+N'(\mathbf{k})\int_0^\beta dt\,L'(\beta,t,\mathbf{k})\right],\quad(77)$$

where from Eqs. (65), (63), (7), and (72), it follows that

$$\nu'(\mathbf{k})=\frac{[1+\epsilon B(\mathbf{k})]\exp[\beta(g-\omega_{\mathbf{k}}')]}{1-\epsilon[1+\epsilon B(\mathbf{k})]\exp[\beta(g-\omega_{\mathbf{k}}')]}.\quad(78)$$

We see from Eq. (77), that the momentum distribution can be expressed in terms of the quantity $L'(t_2,t_1,\mathbf{k})$ defined by the Λ transformation (67), and we now determine the prescription for calculating the latter quantity. From Eqs. (74), (75), (60), and (62), it follows that

$$L'(t_2,t_1,\mathbf{k})=\exp[-\epsilon(t_2-t_1)\Delta(\mathbf{k})]L(t_2,t_1,\mathbf{k})-\int_0^\beta ds\,G'(t_2,s,\mathbf{k})\Lambda(s-t_1,\,\mathbf{k}).\quad(79)^*$$

Moreover, the quantity $\mathcal{G}'(t_2,t_1,\mathbf{k})$ defined by

$$\mathcal{G}'(t_2,t_1,\mathbf{k})\equiv G'(t_2,t_1,\mathbf{k})+\epsilon\mathfrak{N}'(t_2,\mathbf{k})G'(\beta,t,\mathbf{k}),\quad(80)$$

where

$$\mathfrak{N}'(t,\mathbf{k})\equiv N'(\mathbf{k})\int_0^\beta ds\,G'(t,s,\mathbf{k}),\quad(81)$$

satisfies the same transformation law (75) as $G'(t_2,t_1,\mathbf{k})$; namely,

$$\mathcal{G}(t_2,t_1,\mathbf{k})=\int_0^\beta ds\,\exp[\epsilon(t_2-s)\Delta(\mathbf{k})]\times\mathcal{G}'(t_2,s,\mathbf{k})G_0(s-t_1,\,\mathbf{k}).\quad(82)$$

Therefore, after substituting Eqs. (75) and (82) into Eq. (37) and using the definition (66), one finds the following result:

$$\left.{}^{t_1t_2}\begin{Bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{Bmatrix}\right._{t_0}=\exp\{\epsilon[t_1\Delta(\mathbf{k}_1)+t_2\Delta(\mathbf{k}_2)]\}\times\left.{}^{t_1t_2}\begin{Bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{Bmatrix}'\right._{t_0}\exp\{-\epsilon t_0[\Delta(\mathbf{k}_3)+\Delta(\mathbf{k}_4)]\},\quad(83)$$

where the primed generalized pair function is given by

$$\left.{}^{t_1t_2}\begin{Bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{Bmatrix}'\right._{t_0}=\int^\beta ds_1ds_2\,\Gamma'(t_1s_1\mathbf{k}_1;\,t_2s_2\mathbf{k}_2)\left.{}^{s_1s_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'\right._{t_0},\quad(84)$$

and the function $\Gamma'(t_1s_1\mathbf{k}_1;\,t_2s_2\mathbf{k}_2)$ is given in terms of $N'(\mathbf{k})$ and $L'(t_2,t_1,\mathbf{k})$ according to the nature of the outgoing lines at the vertex t_0:

(a) If both outgoing lines are internal lines, then

$$\Gamma'(t_1s_1\mathbf{k}_1;\,t_2s_2\mathbf{k}_2)\left.{}^{s_1s_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'\right._{t_0}=\mathcal{G}'(t_1,s_1,\mathbf{k}_1)\mathcal{G}'(t_2,s_2,\mathbf{k}_2)\left.{}^{s_1s_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'\right._{t_0}\underset{t_1=t_2}{\longrightarrow}\mathcal{G}'(t_1,s_1,\mathbf{k}_1)\mathcal{G}'(t_1,s_2,\mathbf{k}_2)\left.{}^{s_1s_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'\right._{t_0}$$
$$-\delta(t_1-s_1)\delta(t_1-s_2)\exp\{-\epsilon t_1[\Delta(\mathbf{k}_1)+\Delta(\mathbf{k}_2)]\}\left.{}^{s_1s_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}\right._{t_0}\exp\{\epsilon t_0[\Delta(\mathbf{k}_3)+\Delta(\mathbf{k}_4)]\}.\quad(85)$$

(Note that when $t_1=t_2$ the second term involves an *unprimed* pair function.)

(b) If an outgoing line at the vertex t_0 is an *external* line, then in Eq. (85) we must make the replacement

$$\mathcal{G}'(t,s,\mathbf{k})\rightarrow G'(t,s,\mathbf{k}).\quad(86)$$

We now define a quantity $\mathfrak{L}'(t_2,t_1,\mathbf{k})$ in analogy with Eq. (43):

$\mathfrak{L}'(t_2,t_1,\mathbf{k})\equiv\sum$(all different master L graphs with given external lines constructed from primed generalized pair functions). (87)

From Eq. (83) it is then clear that

$$\mathfrak{L}'(t_2,t_1,\mathbf{k})=\exp[-\epsilon(t_2-t_1)\Delta(\mathbf{k})]L(t_2,t_1,\mathbf{k}).\quad(88)$$

Equations (84)–(88) provide just the relations which are needed for the identification of $L'(t_2,t_1,\mathbf{k})$. Upon substituting (88) into (79) we finally obtain

$$L'(t_2,t_1,\mathbf{k})=\mathfrak{L}'(t_2,t_1,\mathbf{k})-\int_0^\beta ds\,G'(t_2,s,\mathbf{k})\Lambda(s-t_1,\,\mathbf{k}).\quad(89)^*$$

Equations (77), (89), (87), and (84) constitute the transformed set A' of coupled integral equations.

Referring to the definition (60) of the quantity $\Lambda(t_2-t_1, \mathbf{k})$, we see that the functions $A(\mathbf{k})$ and $B(\mathbf{k})$ should be chosen in such a way that the "temperature-independent" parts are always eliminated from $L'(t_2,t_1,\mathbf{k})$ via Eq. (89). But this may be accomplished by first writing Eq. (87) as follows:

$$\mathcal{L}'(t_2,t_1,\mathbf{k})=\int_0^\beta ds\, G'(t_2,s,\mathbf{k})P'(s,t_1,\mathbf{k}). \qquad (90)^*$$

Then, according to Eqs. (86) and (87), the quantity $P'(s,t_1,\mathbf{k})$ is the sum over all different master L graphs constructed from *primed* generalized pair functions, but with no external line factors. We therefore define $A(\mathbf{k})$ and $B(\mathbf{k})$ by

$$\begin{aligned} A(\mathbf{k}) &\equiv \text{the temperature-independent part of } P'(t_2,t_1,\mathbf{k},G'),\\ B(\mathbf{k})\delta(t_2-t_1) &\equiv [\text{the part of } P'(t_2,t_1,\mathbf{k},G') \text{ which consists of a temperature-independent factor times } \delta(t_2-t_1)]. \end{aligned} \qquad (91)$$

Equation (89) can be written in a more transparent form by substituting Eq. (90).

$$L'(t_2,t_1,\mathbf{k})=\int_0^\beta ds\, G'(t_2,s,\mathbf{k}) \times[P'(s,t_1,\mathbf{k})-\Lambda(s-t_1,\mathbf{k})]. \qquad (92)^*$$

Equation (92) explicitly exhibits the elimination of all "temperature-independent" terms of the form $\Lambda(t_2-t_1, \mathbf{k})$, Eq. (60), from the quantity $L'(t_2,t_1,\mathbf{k})$ which in turn determines the momentum distribution by means of Eq. (77). One may ask what has happened to the temperature-independent terms, since we know that they contribute powers of ϕ, Eq. (57), to the momentum distribution. The answer is that they are explicitly contained in the exponential factors of Eq. (71) and in the function $\nu'(\mathbf{k})$ of Eqs. (77) and (78).

To determine $\nu'(\mathbf{k})$ we must first calculate the quantities $\Delta(\mathbf{k})$ and $B(\mathbf{k})$, which are defined by Eqs. (59) and (91). On the other hand, the primed pair function (71) depends on both the quantities $\Delta(\mathbf{k})$ and $B(\mathbf{k})$. We have therefore introduced *two* new integral equations in performing the Λ transformation from the coupled set A to the coupled set A'. Fortunately, for the problem of a very low-temperature Fermi gas, the quantity $B(\mathbf{k})$ does not enter in the lower orders of the interaction range parameters. Finally, we observe, as remarked previously, that although we have introduced additional integral equations by performing the Λ transformation, it is now possible to solve them for a very low-temperature Fermi gas, using a power series expansion. As pointed out at the beginning of this section, it was not feasible to use this method to solve the untransformed equations.

In Sec. II we calculated the lowest order correction to the momentum distribution, Eq. (32), by introducing an integral equation for $L(t_2,t_1,\mathbf{k})$ in terms of proper graphs. Then, in Sec. III we abandoned the proper graph notation in favor of master graphs. We can now see why this was a necessary step before the Λ transformation could be performed. The reason is simply that the form of Eqs. (75) and (82), which is essential to the Λ transformation, is not maintained when we make a separation into quantities $G_>(t_2,t_1)$ and $L_<(t_2,t_1)$, as in the line replacements of Fig. 5.

Grand Potential

We can also express the grand potential (56) in terms of the quantities $N'(\mathbf{k})$, $L'(t_2,t_1,k)$, $\Delta(\mathbf{k})$, and $B(\mathbf{k})$. To do this we first observe from Eqs. (44), (88), and (90) that

$$\begin{aligned}\mathcal{L}_{t_1}'(t_1,t_1,\mathbf{k}) &\equiv \int_0^{t_1} ds\, G_{t_1}'(t_1,s,\mathbf{k})P'(s,t_1,\mathbf{k},G_\beta')\\ &=\int_0^{t_1} ds\, G_{t_1}(t_1,s,\mathbf{k})P(s,t_1,\mathbf{k},G_\beta), \end{aligned}\qquad (93)$$

where

$$G_\tau'(t_2,t_1,\mathbf{k})\equiv\delta(t_2-t_1)+\epsilon L_\tau'(t_2,t_1,\mathbf{k}),$$

and the quantity $\mathcal{L}_\tau'(t_2,t_1)$ is calculated as discussed below Eq. (44). The quantity $L_{t_1}'(t_1,t_2)$ is determined from the equation

$$L_{t_1}'(t_1,t_2,\mathbf{k})=\int_0^{t_1} ds\, G_{t_1}'(t_1,s,\mathbf{k}) \times[P'(s,t_2,\mathbf{k})-\Lambda(s-t_2,\mathbf{k})], \quad (t_1>t_2), \qquad (94)$$

as can be verified by making the replacement $\beta\dashrightarrow\tau>(t_1,t_2)$ of the upper integration limits in the equations of this section marked with an asterisk. Thus, the quantity $L_{t_1}'(t_1,t_2)$ for $t_1>t_2$ contains no "temperature-independent" parts in the iteration of its outgoing external line factor.

Using Eqs. (88), (40), (75), (64), (81), (93), and (94), one can readily obtain the following result for the grand potential (56):

$$\begin{aligned}\Omega f^{(s)}(N',\Omega,\beta) &=\epsilon\sum_{\mathbf{k}}\ln[1+\epsilon\langle n(\mathbf{k})\rangle]+\Omega F(N',\Omega,\beta)\\ &\quad-\epsilon\sum_{\mathbf{k}}\int_0^\beta dt\, \mathcal{L}'(\beta,t,\mathbf{k})\mathfrak{N}'(t,\mathbf{k})\\ &\quad-\epsilon\sum_{\mathbf{k}}\int_0^\beta dt_1\left\{\int_0^\beta dt_2\, L_\beta'(t_1,t_2,\mathbf{k}) -\int_0^{t_1} dt_2 L_{t_1}'(t_1,t_1,\mathbf{k})\right\}P'(t_2,t_1,\mathbf{k}), \end{aligned}\qquad (95)$$

where from Eqs. (83) and (53), it follows that $F(N)=F(N')$ can be calculated using *either* unprimed *or* primed generalized pair-functions.

We finally observe that the quantity $\mathfrak{N}'(\beta,\mathbf{k})$ of Eq. (81) is equal to $\mathfrak{N}(\beta,\mathbf{k})$, since in analogy with Eq. (46) we can show that

$$\mathfrak{N}'(\beta,\mathbf{k})=\langle n(\mathbf{k})\rangle. \tag{96}$$

V. LOW-TEMPERATURE MOMENTUM DISTRIBUTION IN A FERMI GAS; SECOND ORDER CORRECTIONS

The momentum distribution $\langle n(\mathbf{k})\rangle$ is obtained by substituting Eqs. (77) and (64) into Eq. (11):

$$\langle n(\mathbf{k})\rangle=\nu'(\mathbf{k})+\epsilon\nu'(\mathbf{k})[1+\epsilon\nu'(\mathbf{k})]\times\int_0^\beta dt\, L'(\beta,t,\mathbf{k},N'). \tag{97}$$

In the previous section we have seen how the terms defined by Eq. (91) can be incorporated into the functions $\nu'(\mathbf{k})$, but we have said very little about the quantity $L'(\beta,t)$. In this section we shall show for a Fermi gas with strong, short-range interactions that $L'(\beta,t)$ contributes a small $O(a_s^2)$ term to $\langle n(\mathbf{k})\rangle$, which is of the form of $\chi(k)$, Eq. (57). We shall also calculate the $O(a_s^2)$ term of $\Delta(\mathbf{k})$, Eq. (59), which affects the determination of $\nu'(\mathbf{k})$, Eq. (78).

We begin by writing down the one- and two-vertex master L graphs of Fig. 9. Since the pair-function at each vertex is linear in the scattering length a_s to first approximation, we shall not need to consider any other master L graphs in the second order calculation of this section. As indicated at the beginning of Sec. IV, this seems to be a consistent procedure for calculating the properties of a Fermi gas. One can also verify with the aid of Eqs. (25) and (27) that the leading contribution to $B(\mathbf{k})$, Eq. (91), is third order in the scattering length approximation. Therefore, to second order we have that $\Delta(\mathbf{k})=A(\mathbf{k})$.

Using the notation of Eq. (90), we write the one-vertex master L graph as

$$\begin{aligned}\mathcal{L}'^{(1)}(t_2,t_1,\mathbf{k}_1)&=\int_0^\beta ds_1\, G'(t_2,s_1,\mathbf{k}_1)P'^{(1)}(s_1,t_1,\mathbf{k}_1),\\ P'^{(1)}(s_1,t_1,\mathbf{k}_1)&=\sum_{\mathbf{k}_2}\int_0^\beta ds_2\, \mathcal{G}'(t_1,s_2,\mathbf{k}_2)\,{}^{s_1s_2}\!\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_1\mathbf{k}_2\end{bmatrix}'_{t_1},\end{aligned} \tag{98}$$

where we have used Eqs. (84)–(86) and the temperature and momentum labels of Fig. 9. Similarly, the expression for the two-vertex master L graph is

$$\begin{aligned}\mathcal{L}'^{(2)}(t_2,t_1,\mathbf{k}_1)&=\int_0^\beta ds_1\, G'(t_2,s_1,\mathbf{k}_1)\int_0^\beta dt_3\, P'^{(2)}(s_1t_3t_1|\mathbf{k}_1),\\ P'^{(2)}(s_1t_3t_1|\mathbf{k}_1)&=\tfrac{1}{2}\sum_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\int_0^\beta ds_2\, \mathcal{G}'(t_1,s_2,\mathbf{k}_2)\,{}^{s_1s_2}\!\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{bmatrix}'_{t_3}\Big\{\int_0^\beta ds_3ds_4\mathcal{G}'(t_3,s_3,\mathbf{k}_3)\mathcal{G}'(t_3,s_4,\mathbf{k}_4)\,{}^{s_3s_4}\!\begin{bmatrix}\mathbf{k}_3\mathbf{k}_4\\ \mathbf{k}_1\mathbf{k}_2\end{bmatrix}'_{t_1}\\ &\quad-\exp\{-\epsilon t_3[\Delta(\mathbf{k}_3)+\Delta(\mathbf{k}_4)]\}\,{}^{t_3t_3}\!\begin{bmatrix}\mathbf{k}_3\mathbf{k}_4\\ \mathbf{k}_1\mathbf{k}_2\end{bmatrix}_{t_1}\exp\{\epsilon t_1[\Delta(\mathbf{k}_1)+\Delta(\mathbf{k}_2)]\}\Big\}.\end{aligned} \tag{99}$$

According to Eq. (30), the function $L'(t_2,t_1,\mathbf{k})$ is zero to first order in the scattering length approximation. Therefore, if this function does indeed make only small contributions to the physical quantities of a Fermi gas, then we may evaluate its second order contributions by the power series iteration procedure discussed at the beginning of Sec. IV. Thus, we now assume that the second order terms of $L'(t_2,t_1,\mathbf{k})$ and $\Delta(\mathbf{k})$ can be derived from Eqs. (98) and (99) by the following approximations to $G'(t,s,\mathbf{k})$ and $\mathcal{G}'(t,s,\mathbf{k})$:

$$\begin{aligned}G'(t,s,\mathbf{k})&=\delta(t-s)+O(a_s^2),\\ \mathcal{G}'(t,s,\mathbf{k})&=\delta(t-s)+\epsilon N'(\mathbf{k})\delta(\beta-s)+O(a_s^2).\end{aligned} \tag{100}$$

In order to gain a feeling for the relation between master L graphs (Fig. 9) and ordinary L graphs [Fig. 3], we shall write $\mathcal{L}'(t_2,t_1,\mathbf{k})$ as a sum of terms which correspond to the L graphs of Fig. 3:

$$\begin{aligned}\mathcal{L}'(t_2,t_1,\mathbf{k}_1)&=\mathcal{L}'^{(1)}(t_2,t_1,\mathbf{k}_1)+\mathcal{L}'^{(2)}(t_2,t_1,\mathbf{k}_1)+O(a_s^3)\\ &=\mathcal{L}_>'^{(1)}(t_2,t_1,\mathbf{k}_1)+\mathcal{L}_{>,1}'^{(2)}(t_2,t_1,\mathbf{k}_1)+\mathcal{L}_{>,2}'^{(2)}(t_2,t_1,\mathbf{k}_1)\\ &\qquad+\mathcal{L}_{>,3}'^{(2)}(t_2,t_1,\mathbf{k}_1)+\mathcal{L}_{>,4}'^{(2)}(t_2,t_1,\mathbf{k}_1)+O(a_s^3)\quad(\text{if}\quad t_2>t_1),\\ &=\mathcal{L}_{<,2}'^{(2)}(t_2,t_1,\mathbf{k}_1)+\mathcal{L}_{<,3}'^{(2)}(t_2,t_1,\mathbf{k}_1)+O(a_s^3)\quad(\text{if}\quad t_2<t_1).\end{aligned} \tag{101}$$

We next substitute the explicit expressions for the pair functions from Eqs. (15), (68), (71), (70) and Sec. V of I, and perform the spin state sums (spin $=J$) for a spin-independent interaction. In the limit of infinite volume, $\Omega\to\infty$, the parts of the functions of Eq. (101) which contribute to second order in the scattering length are as

follows:

$$\begin{aligned}
\mathcal{L}_{>}'^{(1)}(t_2,t_1,\mathbf{k}_1)\cong&\left[\frac{2\epsilon}{(2\pi)^3}\frac{\lambda_T^2}{\beta}\right](2J+1+\epsilon)\int d^3k_2\, N'(\mathbf{k}_2)\bigg\{(2k_{12})^{-1}\sin 2\delta_0(k_{12})+\left[\frac{2}{(2\pi)^3}\frac{\lambda_T^2}{\beta}\right]\\
&\times\int d^3k_3d^3k_4\, P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3-\omega_4}\right)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\exp[(t_2-t_1)(\omega_1'+\omega_2'-\omega_3-\omega_4)]\\
&\times k_{34}^{-2}\langle k_{12}|A^{(0)}|k_{34}\rangle^2\cos^2\delta_0(k_{34})-[\Delta(\mathbf{k}_1)+\Delta(\mathbf{k}_2)]\left[\frac{2\epsilon}{(2\pi)^3}\frac{\lambda_T^2}{\beta}\right]\int d^3k_3d^3k_4\, P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3-\omega_4}\right)\\
&\times P\left(\frac{1}{\omega_1+\omega_2-\omega_3-\omega_4}\right)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)k_{34}^{-2}\langle k_{12}|A^{(0)}|k_{34}\rangle^2\cos^2\delta_0(k_{34})\bigg\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{>,1}'^{(2)}(t_2,t_1,\mathbf{k}_1)\cong&\,2\left[\frac{2}{(2\pi)^3}\frac{\lambda_T^2}{\beta}\right]^2(2J+1+\epsilon)\int d^3k_2d^3k_3d^3k_4\, N'(\mathbf{k}_2)N'(\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\\
&\times(k_{12}|k_{34})(k_{34}|k_{12})\int_{t_1}^{t_2}dt_3\exp[(t_3-t_1)(\omega_1'+\omega_2'-\omega_3'-\omega_4')],
\end{aligned}\tag{102}$$

$$\begin{aligned}
\mathcal{L}_{>,2}'^{(2)}(t_2,t_1,\mathbf{k}_1)\cong&\left[\frac{2}{(2\pi)^3}\frac{\lambda_T^2}{\beta}\right]^2(2J+1+\epsilon)\int d^3k_2d^3k_3d^3k_4\, N'(\mathbf{k}_3)N'(\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\\
&\times(k_{12}|k_{34})(k_{34}|k_{12})\int_0^{t_1}dt_3\exp[(t_3-t_1)(\omega_1'+\omega_2'-\omega_3'-\omega_4')],
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{>,3}'^{(2)}(t_2,t_1,\mathbf{k}_1)\cong&\,\epsilon\left[\frac{2}{(2\pi)^3}\frac{\lambda_T^2}{\beta}\right]^2(2J+1+\epsilon)\int d^3k_2d^3k_3d^3k_4\, N'(\mathbf{k}_2)N'(\mathbf{k}_3)N'(\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\\
&\times(k_{12}|k_{34})(k_{34}|k_{12})\int_0^{t_2}dt_3\exp[(t_3-t_1)(\omega_1'+\omega_2'-\omega_3'-\omega_4')],
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{>,4}'^{(2)}(t_2,t_1,\mathbf{k}_1)\cong&\,\epsilon\left[\frac{2}{(2\pi)^3}\frac{\lambda_T^2}{\beta}\right]^2(2J+1+\epsilon)\int d^3k_2d^3k_3d^3k_4\, N'(\mathbf{k}_2)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)(k_{12}|k_{34})(k_{34}|k_{12})\\
&\times\int_{t_1}^{t_2}dt_3\{\exp[(t_3-t_1)(\omega_1'+\omega_2'-\omega_3'-\omega_4')]-\exp[(t_3-t_1)(\omega_1'+\omega_2'-\omega_3-\omega_4)]\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{<,2}'^{(2)}(t_2,t_1,\mathbf{k}_1)\cong&\left[\frac{2}{(2\pi)^3}\frac{\lambda_T^2}{\beta}\right]^2(2J+1+\epsilon)\int d^3k_2d^3k_3d^3k_4\, N'(\mathbf{k}_3)N'(\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\\
&\times(k_{12}|k_{34})(k_{34}|k_{12})\int_0^{t_2}dt_3\exp[(t_3-t_1)(\omega_1'+\omega_2'-\omega_3'-\omega_4')],
\end{aligned}\tag{103}$$

$$\begin{aligned}
\mathcal{L}_{<,3}'^{(2)}(t_2,t_1,\mathbf{k}_1)\cong&\,\epsilon\left[\frac{2}{(2\pi)^3}\frac{\lambda_T^2}{\beta}\right]^2(2J+1+\epsilon)\int d^3k_2d^3k_3d^3k_4\, N'(\mathbf{k}_2)N'(\mathbf{k}_3)N'(\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\\
&\times(k_{12}|k_{34})(k_{34}|k_{12})\int_0^{t_2}dt_3\exp[(t_3-t_1)(\omega_1'+\omega_2'-\omega_3'-\omega_4')],
\end{aligned}$$

where

$$(k|l)\equiv l^{-1}\langle k|A^{(0)}|l\rangle\cos^2\delta_0(l)=-a_s[1+O(ka_s)^2].\tag{104}$$

In Sec. II, Eq. (33), we showed that $\Delta(\mathbf{k})\sim E_F(k_Fa_s)$ in the low-density, low-temperature approximations (28). Then the third term of $\mathcal{L}_{>}'^{(1)}(t_2,t_1,\mathbf{k}_1)$, above, is $\sim a_s^3$ and can be neglected, as can a similar term which arises from the lower t_3-integration limit in $\mathcal{L}_{>,4}'^{(2)}(t_2,t_1,\mathbf{k}_1)$. We now set $\epsilon=-1$ in the above expressions and keep only terms to second order in the scattering length. We then perform the t_3 temperature integrations and combine all of the resulting terms into a temperature-dependent and temperature-independent part according to the prescription of Eq. (89):

$$\begin{aligned}
\mathcal{L}'(t_2,t_1,\mathbf{k}_1)&=L_{>}'(t_2,t_1,\mathbf{k}_1)+A(\mathbf{k})+O(a_s^3)\quad\text{if}\quad t_2>t_1,\\
&=L_{<}'(t_2,t_1,\mathbf{k}_1)+O(a_s^3),\quad\text{if}\quad t_2<t_1.
\end{aligned}\tag{105}$$

The quantity $\Delta(\mathbf{k})\cong A(\mathbf{k})$ acquires contributions from $\mathcal{L}_>'^{(1)}$, $\mathcal{L}_{>,1}'^{(2)}$, and $\mathcal{L}_{>,2}'^{(2)}$. It can be written as

$$\Delta(\mathbf{k})=A(\mathbf{k})+O(a_s^3)$$
$$=2J\pi^{-2}E_F(k_Fa_s)\left\{k_F^{-3}\int d^3k_2\, N'(\mathbf{k}_2)+\frac{4}{15}(k_Fa_s)[P_a(\mathbf{k}_1)+P_b(\mathbf{k}_1)]+O(k_Fa_s)^2\right\}; \tag{106}$$

where k_F is defined in terms of the density by Eq. (34). The quantities $P_a(\mathbf{k}_1)$ and $P_b(\mathbf{k}_1)$ are given by the integrals

$$P_a(\mathbf{k}_1)=15(2\pi)^{-2}E_Fk_F^{-6}\int d^3k_2d^3k_3d^3k_4\, N'(\mathbf{k}_3)N'(\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right),$$
$$P_b(\mathbf{k}_1)=-30(2\pi)^{-2}E_Fk_F^{-6}\int d^3k_2d^3k_3d^3k_4\, N'(\mathbf{k}_2)N'(\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right). \tag{107}$$

Finally, the functions $L_>'(t_2,t_1)$ and $L_<'(t_2,t_1)$ can be written in the following manner:

$$L_>'(t_2,t_1,\mathbf{k}_1)=-2J\left[\frac{E_F(k_Fa_s)}{\pi^2k_F^3}\right]^2\int d^3k_2d^3k_3d^3k_4\,\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)$$
$$\times\{[1-N'(\mathbf{k}_2)]N'(\mathbf{k}_3)N'(\mathbf{k}_4)\exp[t_1(\omega_3'+\omega_4'-\omega_1'-\omega_2')]$$
$$+N'(\mathbf{k}_2)[1-N'(\mathbf{k}_3)][1-N'(\mathbf{k}_4)]\exp[(t_2-t_1)(\omega_1'+\omega_2'-\omega_3'-\omega_4')]\},$$
$$L_<'(t_2,t_1,\mathbf{k}_1)=2J\left[\frac{E_F(k_Fa_s)}{\pi^2k_F^3}\right]^2\int d^3k_2d^3k_3d^3k_4[1-N'(\mathbf{k}_2)]N'(\mathbf{k}_3)N'(\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)$$
$$\times P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)\{\exp[(t_2-t_1)(\omega_1'+\omega_2'-\omega_3'-\omega_4')]-\exp[t_1(\omega_3'+\omega_4'-\omega_1'-\omega_2')]\}. \tag{108}$$

If the integrals of Eqs. (108) are cast into dimensionless form by the substitutions $\mathbf{k}_i=k_F\mathbf{l}_i$, then one finds that there is an exponential dependence in the integrands upon the parameter $(k_F\lambda_T)^2$. At first sight, this would seem to be very distressing, for it would indicate that a further investigation of higher order master L graphs should be made, in order to derive a correct description of the temperature-density region $k_F\lambda_T\gg1$. Fortunately, such an investigation does not seem to be necessary for a Fermi gas, and we have investigated this matter to third order in the scattering-length approximation. The reason is that the contribution of $L'(t_2,t_1,\mathbf{k})$ to physical quantities is small, i.e., it is of the form of the function $\chi(k)$, Eq. (57). To see how this comes about, we first observe that to the order which we are calculating we may replace $N'(\mathbf{k})$ in Eqs. (108) by $\nu'(\mathbf{k})$. We then substitute the first of Eqs. (108) into Eq. (97) and perform the temperature integration. After using the identity

$$\nu'(\mathbf{k})=[1+\epsilon B'(\mathbf{k})][1+\epsilon\nu'(\mathbf{k})]\exp[\beta(g-\omega_{\mathbf{k}}')]$$
$$\cong[1+\epsilon\nu'(\mathbf{k})]\exp[\beta(g-\omega_{\mathbf{k}}')], \tag{109}$$

one can verify that the $O(a_s^2)$ contribution of $L'(\beta,t,\mathbf{k})$ to the momentum distribution is

$$\langle n(\mathbf{k})\rangle=\nu'(\mathbf{k})+2J\pi^{-2}(k_Fa_s)^2\{[1-\nu'(\mathbf{k})]M_a(\mathbf{k})-\nu'(\mathbf{k})M_b(\mathbf{k})\}+O(k_Fa_s)^3, \tag{110}$$

where

$$M_a(\mathbf{k}_1)=\pi^{-2}E_F^2k_F^{-6}\int d^3k_2d^3k_3d^3k_4[1-\nu'(\mathbf{k}_2)]\nu'(\mathbf{k}_3)\nu'(\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)^2$$
$$\times\underset{k_1\gg k_F}{\longrightarrow}\frac{4}{9}(k_F/k_1)^4[1+O(k_F/k_1)^2], \tag{111}$$
$$M_b(\mathbf{k}_1)=\pi^{-2}E_F^2k_F^{-6}\int d^3k_2d^3k_3d^3k_4\,\nu'(\mathbf{k}_2)[1-\nu'(\mathbf{k}_3)][1-\nu'(\mathbf{k}_4)]\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)^2.$$

In these last two integrals we may replace $\omega_{\mathbf{k}}'=\omega_{\mathbf{k}}+\Delta(\mathbf{k})$ by $\omega_{\mathbf{k}}=(2m)^{-1}\hbar^2\mathbf{k}^2$, because the integrals are convergent, and we are only interested in calculating the momentum distribution to second order. Similarly, we may set $\omega_{\mathbf{k}}'=\omega_{\mathbf{k}}$ in P_a and P_b, Eqs. (107).

In the very low-temperature limit, the function

$\nu_{\mathbf{k}}'(1-\nu_{\mathbf{k}}')$ of Eq. (97) becomes a δ function, i.e.,

$$\nu_{\mathbf{k}}'(1-\nu_{\mathbf{k}}')\rightarrow\beta^{-1}\delta(\omega_{\mathbf{k}}'-g)\quad\text{when}\quad\beta\rightarrow\infty.$$

It is because of the singular nature of this function and the identity (109) that it was difficult to determine beforehand whether or not the exponential factors in the integrands of Eqs. (108) would give "large" low-temperature contributions to the momentum distribution. In fact, the terms M_a and M_b of Eq. (110) give *no* contribution to the density $\rho=(2J+1)(2\pi)^{-3}\int d^3k\langle n(\mathbf{k})\rangle$, as one can readily verify.

The contributions of M_a and M_b to the zero-temperature momentum distribution can be easily understood, because of the factors $(1-\nu')$ and ν' with which they are associated. Since these integrals are both positive definite, we see that the $M_a(\mathbf{k})$ term has the effect of adding a tail to the momentum distribution, whereas the $M_b(\mathbf{k})$ term lowers the momentum distribution for $k<k_F$. These terms can also be derived using the pseudopotential method,[10] and are therefore meaningful in the language of ordinary perturbation theory. Finally, we see that the leading contribution of $L'(t_2,t_1,\mathbf{k})$ to the momentum distribution is $\sim(k_Fa_s)^2$, thereby justifying the approximations (100).

We now turn our attention to the calculation of $\Delta(\mathbf{k})$ to $O(a_s^2)$. At the end of Sec. II we showed that the momentum distribution including the first order contribution to $\Delta(\mathbf{k})$, Eq. (32), is the same as the free particle momentum distribution. Therefore, the iteration of the integral equation (77) in Eqs. (107) has, to second order, only the effect of replacing $N'(\mathbf{k})\cong\nu'(\mathbf{k})$ by the free particle momentum distribution, and we can write $\Delta(\mathbf{k})$ as

$$\Delta(\mathbf{k})=8J(3\pi)^{-1}E_F(k_Fa_s)\{1+(5\pi)^{-1}(k_Fa_s)\times[P_a(\mathbf{k}_1)+P_b(\mathbf{k}_1)]+O(k_Fa_s)^2\}.\quad(112)$$

The integrated expression for $[P_a(\mathbf{k}_1)+P_b(\mathbf{k}_1)]$, Eqs. (107), is

$$[P_a(\mathbf{k}_1)+P_b(\mathbf{k}_1)]=\tfrac{1}{2}[P(k_1/k_F)+P(-k_1/k_F)],$$

$$P(x)=11-\tfrac{1}{2}x^4\ln(1-x^2)^2+2x^4\ln x^2+x^{-1}(10-10x^2-x^5)\ln(1+x)^2$$
$$+\begin{Bmatrix}(2x)^{-1}(2-x^2)^{\frac{5}{2}}\left(\ln\left[\dfrac{2+x+(2-x^2)^{\frac{1}{2}}}{2+x-(2-x^2)^{\frac{1}{2}}}\right]^2-\ln\left[\dfrac{x+(2-x^2)^{\frac{1}{2}}}{x-(2-x^2)^{\frac{1}{2}}}\right]^2\right) & \text{if}\quad x^2<2\\[2ex] 2x^{-1}(x^2-2)^{\frac{5}{2}}\left(\tan^{-1}\left[\dfrac{2+x}{(x^2-2)^{\frac{1}{2}}}\right]-\tan^{-1}\left[\dfrac{x}{(x^2-2)^{\frac{1}{2}}}\right]\right) & \text{if}\quad x^2>2\end{Bmatrix}.\quad(113)$$

At the end of Sec. II we also determined the thermodynamic potential g at zero temperature in terms of the Fermi momentum k_F, i.e., in terms of the density, by setting the arguments of the exponential factors of $\nu^{(1)}(\mathbf{k})$, Eq. (32), equal to zero at $k=k_F$ [see Eq. (35)]. Thus, in the limit $\beta\rightarrow\infty$, we have $\nu^{(1)}(\mathbf{k})=1$ when $\omega_{\mathbf{k}}<g-\Delta^{(1)}$ and $\nu^{(1)}(\mathbf{k})=0$ when $\omega_{\mathbf{k}}>g-\Delta^{(1)}$. The generalization of this procedure for a Fermi gas is to set the argument of the exponential factors of $\nu'(\mathbf{k})$, Eq. (78), equal to zero at $k=k_F$ (assuming that there is only one zero), in order to determine g at $T=0$. We therefore obtain

$$\lim_{T\rightarrow0}(g)=E_F+\Delta(\mathbf{k}_F).\quad(114)$$

Equation (114) is valid as long as the only contribution to the density ρ is $\nu'(\mathbf{k})$ and provided that the argument of the exponential factors of $\nu'(\mathbf{k})$ has only one zero, i.e., provided there is no "gap" in the zero temperature momentum distribution.

The result of substituting Eqs. (112) and (113) into Eq. (114) is an expression for the zero-temperature thermodynamic potential which is correct to second order in the scattering-length approximation [see Eq. (35)]:

$$\lim_{T\rightarrow0}g=E_F\{1+8J(3\pi)^{-1}(k_Fa_s)[1+(5\pi)^{-1}(k_Fa_s)\times(11-2\ln2)+O(k_Fa_s)^2]\}.\quad(115)$$

Equation (114) is an important result. Its significance is that we may calculate the ground state, i.e., the $T=0$, properties of a Fermi gas by using for $\nu'(\mathbf{k})$ the free particle momentum distribution. The momentum and temperature dependence of $\Delta(\mathbf{k})$ only affects the momentum distribution at nonzero temperatures. This result therefore suggests that perhaps the present application of the methods of quantum statistics may be used to justify the many-body perturbation theories and to extend their range of applicability to nonzero temperatures. It must be stressed, however, that the validity of Eq. (114) depends on the two qualifying remarks below the equation, which must always be verified before the use of Eq. (114) is justified.

VI. GROUND-STATE ENERGY AND MOMENTUM DISTRIBUTION OF A FERMI GAS; THIRD ORDER CORRECTIONS

The method of Sec. IV together with the results of Sec. V can readily be applied to the calculation of the ground-state energy and momentum distribution of a Fermi gas to third order in its scattering parameters. As before, we assume that the two-body interaction is strong and of short range, such as for hard core repulsions.

[10] F. Mohling, thesis, University of Washington, 1958 (unpublished).

In Sec. V we calculated the second order contributions to the momentum distribution by making the approximations (100) in the one- and two-vertex master L graphs of Fig. 9. Similarly, since to leading order $L'(t_2,t_1,\mathbf{k})$ is second order in the scattering length, one may calculate the third order contributions to the momentum distribution by making the approximations (100) in the two- and three-vertex master L graphs. For the one-vertex master L graph it may be verified after some lengthy manipulations, following the substitution of Eqs. (80), (81), (77), (74), and (108) into Eq. (98) for $P'^{(1)}(s,t,\mathbf{k}_1)$, that the quantity $\mathfrak{L}'^{(1)}(t_2,t_1,\mathbf{k}_1)$ of Eq. (98) can be written as follows:

$$\mathfrak{L}'^{(1)}(t_2,t_1,\mathbf{k}_1)=\sum_{\mathbf{k}_2}[\epsilon\nu'(\mathbf{k}_2)]\int_0^\beta ds\, G'(t_2,s,\mathbf{k}_1)\times{}^{s\,t_2}\begin{bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_1\mathbf{k}_2\end{bmatrix}'_{t_1}+O(a_s^4). \quad (116)$$

We next observe that since the linear scattering length term in the diagonal pair function is a "temperature-independent" term [see Eqs. (25) and (26)], we may also make the approximation $G'(t_2,t_1)\cong\delta(t_2-t_1)$ in Eq. (116). This follows from the prescription (89) for calculating $L'(t_2,t_1)$. Thus, the momentum distribution in a Fermi gas may be calculated to third order by making the approximations (100) together with $N'(\mathbf{k})\cong\nu'(\mathbf{k})$ in *all* of the primed generalized pair functions of the one-, two-, and three-vertex master L graphs.

A similar result can be verified for the calculation of the grand potential (95) to third order in scattering parameters. We define the quantity $F(\nu')$ [see Eq. (53)]:

$$\Omega F(\nu')\equiv\sum[\text{all different master 0-graphs constructed from primed generalized pair functions (84) and calculated by making the approximations (100) and } N'(\mathbf{k})\cong\nu'(\mathbf{k})]. \quad (117)$$

After some more lengthy manipulations, similar to those which led to Eq. (116), one can show that the difference $[F(N')-F(\nu')]$ occurs in the sum of the third and fourth terms of the grand potential (95). In particular, Eq. (95) may be written in the following convenient form.

$$\Omega f^{(s)}(N',\Omega,\beta)=\epsilon\sum_{\mathbf{k}}\ln[1+\epsilon\nu'(\mathbf{k})]+\Omega F(\nu')-\sum_{\mathbf{k}}[\epsilon\nu'(\mathbf{k})]\int_0^\beta dt\Lambda(\beta-t,k)+O(a_s^4). \quad (118)$$

We are interested in calculating the energy per particle of a Fermi system, which is derived from the grand potential according to the relation

$$\langle E\rangle/\langle N\rangle=g-\rho^{-1}\partial f/\partial\beta, \quad (119)$$

where g is the thermodynamic potential per particle. After substituting the preceding equation for f into this relation and using Eqs. (78), (72), (59), and (60), we obtain the following equation for the energy per particle

$$\langle E\rangle/\langle N\rangle=(\rho\Omega)^{-1}\sum_{\mathbf{k}}\nu'(\mathbf{k})\omega(\mathbf{k})-(\rho\beta)^{-1}F(\nu')+\rho^{-1}T(\beta)+O(a_s^4), \quad (120)$$

where

$$T(\beta)\equiv\beta\frac{\partial}{\partial\beta}\{\Omega^{-1}\sum_{\mathbf{k}}[\epsilon\nu'(\mathbf{k})]A(\mathbf{k})-\beta^{-1}F(\nu')\}-\beta\Omega^{-1}\sum_{\mathbf{k}}[\epsilon\nu'(\mathbf{k})]\frac{\partial}{\partial\beta}A(\mathbf{k})+\frac{\partial}{\partial\beta}\sum_{\mathbf{k}}[\epsilon\nu'(\mathbf{k})]B(\mathbf{k})\times\underset{\beta\to\infty}{\longrightarrow}0,\quad\text{for}\quad\epsilon=-1. \quad (121)$$

In the limit of infinite volume and zero temperature, Eq. (120) therefore becomes

$$\lim_{\substack{T\to 0\\ \Omega\to\infty}}\frac{\langle E\rangle}{\langle N\rangle}=\tfrac{3}{5}E_F-(\rho\beta)^{-1}F(\nu')+O[E_F(k_Fa_s)^4], \quad (122)$$

where we have used Eq. (114) to calculate the first term. Thus, to third order the corrections to the free-particle ground-state energy of a Fermi gas are determined entirely by the function $F(\nu')$ of Eq. (117). We

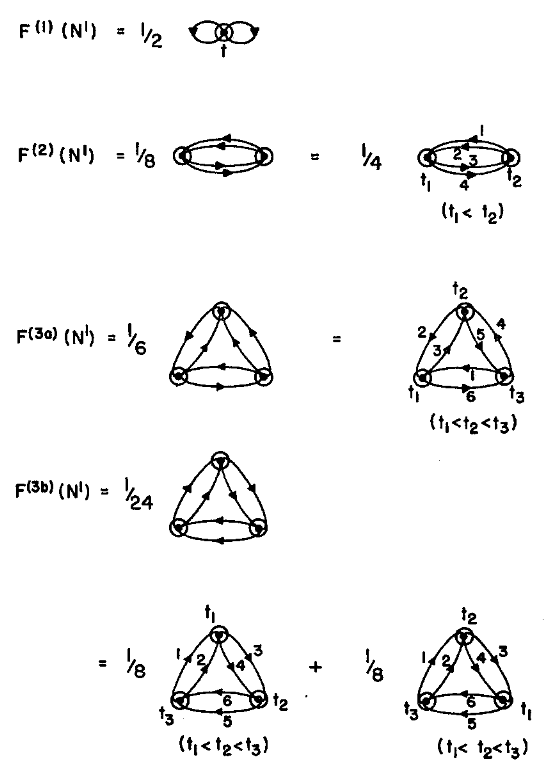

FIG. 10. The one-, two-, and three-vertex master 0-graphs. The inverse symmetry number S^{-1} has been included with each 0-graph.

observe that the quantity $B(\mathbf{k})$ of Eq. (91), although of third order in interaction range parameters, does not contribute in the third order calculation of the ground-state energy and momentum distribution.

We now turn our attention to the calculation of $F(\nu')$. Since the pair function at each vertex of a master 0-graph is linear in the scattering length to first approximation, we need only consider the four master 0-graphs shown in Fig. 10 for our present third-order calculation. We have utilized the symmetry of these 0-graphs to order the temperature variables at the different vertices. Thus, for the function $F(N')$ we obtain

$$F(N')=F^{(1)}(N')+F^{(2)}(N')+F^{(3a)}(N')+F^{(3b)}(N')+O(a_s^4). \quad (123)$$

The explicit expressions for the first four terms of Eq. (123), using the temperature and momentum labels of Fig. 10, are as follows:

$$\begin{aligned}
\Omega F^{(1)}(N')&=\frac{1}{2}\int_0^\beta dt \sum_{\mathbf{k}_1\mathbf{k}_2} {}^{tt}\begin{Bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_1\mathbf{k}_2\end{Bmatrix}'_t,\\
\Omega F^{(2)}(N')&=\frac{1}{4}\int_0^\beta dt_2\int_0^{t_2} dt_1 \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} {}^{t_2t_2}\begin{Bmatrix}\mathbf{k}_3\mathbf{k}_4\\ \mathbf{k}_1\mathbf{k}_2\end{Bmatrix}'_{t_1} {}^{t_1t_1}\begin{Bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_3\mathbf{k}_4\end{Bmatrix}'_{t_2},\\
\Omega F^{(3a)}(N')&=\epsilon\int_0^\beta dt_3\int_0^{t_3} dt_2\int_0^{t_2} dt_1 \sum_{\mathbf{k}_1\cdots\mathbf{k}_6} {}^{t_2t_3}\begin{Bmatrix}\mathbf{k}_3\mathbf{k}_6\\ \mathbf{k}_1\mathbf{k}_2\end{Bmatrix}'_{t_1} {}^{t_3t_1}\begin{Bmatrix}\mathbf{k}_2\mathbf{k}_5\\ \mathbf{k}_3\mathbf{k}_4\end{Bmatrix}'_{t_2} {}^{t_1t_2}\begin{Bmatrix}\mathbf{k}_1\mathbf{k}_4\\ \mathbf{k}_5\mathbf{k}_6\end{Bmatrix}'_{t_3},\\
\Omega F^{(3b)}(N')&=\frac{1}{8}\int_0^\beta dt_3\int_0^{t_3} dt_2\int_0^{t_2} dt_1 \sum_{\mathbf{k}_1\cdots\mathbf{k}_6}\Bigg\{ {}^{t_1t_2}\begin{Bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_5\mathbf{k}_6\end{Bmatrix}'_{t_3} {}^{t_2t_2}\begin{Bmatrix}\mathbf{k}_3\mathbf{k}_4\\ \mathbf{k}_1\mathbf{k}_2\end{Bmatrix}'_{t_1} {}^{t_3t_3}\begin{Bmatrix}\mathbf{k}_5\mathbf{k}_6\\ \mathbf{k}_3\mathbf{k}_4\end{Bmatrix}'_{t_2}\\
&\qquad + {}^{t_2t_2}\begin{Bmatrix}\mathbf{k}_1\mathbf{k}_2\\ \mathbf{k}_5\mathbf{k}_6\end{Bmatrix}'_{t_3} {}^{t_1t_1}\begin{Bmatrix}\mathbf{k}_3\mathbf{k}_4\\ \mathbf{k}_1\mathbf{k}_2\end{Bmatrix}'_{t_2} {}^{t_3t_3}\begin{Bmatrix}\mathbf{k}_5\mathbf{k}_6\\ \mathbf{k}_3\mathbf{k}_4\end{Bmatrix}'_{t_1}\Bigg\}.
\end{aligned} \quad (124)$$

In order to obtain the function $F(\nu')$ we must now make the approximations (100) and $N'(\mathbf{k})\cong\nu'(\mathbf{k})$ in each of the above terms. After using the explicit expressions for the pair functions [see Eqs. (70) and (71) and Sec. V of I] and performing many tedious manipulations, one can regroup these terms into six other quantities, convenient for calculational purposes, whose sum gives $F(\nu')$ to third order in scattering parameters:

$$(\rho\beta)^{-1}F(\nu')=[T_1+T_2+T_3+T_4+T_5+T_6]+O(a_s^4). \quad (125)$$

We give below only the final expressions which are obtained for the T_i:

$$\begin{aligned}
T_1&=\frac{(2J+1)}{(2\pi)^3\rho}\left(\frac{E_F}{\pi^2k_F^2}\right)\int d^3k_1d^3k_2\,\nu'(\mathbf{k}_1)\nu'(\mathbf{k}_2)[J(2k_{12})^{-1}\sin 2\delta_0(k_{12})+3(J+1)(2k_{12})^{-1}\sin 2\delta_1(k_{12})]\\
&=-4J(3\pi)^{-1}E_F(k_Fa_s)\left\{1+\tfrac{3}{10}(k_Fa_s)^2\left[\left(\frac{r_1^3-a_s^3}{a_s^3}\right)+3\left(\frac{J+1}{J}\right)\left(\frac{a_p}{a_s}\right)^3\right]\right\}[1+O(k_Fa_s)^2+O(l/\lambda_T)^2],
\end{aligned}$$

where

$$\rho\equiv l^{-3}=\left(\frac{2J+1}{6\pi^2}\right)k_F^3,$$

$$\begin{aligned}
T_2&=-\frac{(2J+1)}{(2\pi)^3\rho}\left(\frac{E_F}{\pi^2k_F^2}\right)^2\int d^3k_1d^3k_2d^3k_3d^3k_4(\nu_1'+\nu_2')\nu_3'\nu_4'\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\\
&\qquad\times(k_{12}|k_{34})(k_{34}|k_{12})P\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right),\\
&=-\frac{8J}{35\pi^2}E_F(k_Fa_s)^2(11-2\ln 2)[1+O(k_Fa_s)^2+O(l/\lambda_T)^2]
\end{aligned}$$

using the notation of Eq. (104),

$$\begin{aligned}
T_3&=\frac{4J(2J+1)}{(2\pi)^3\rho}\left(\frac{E_F}{\pi^2k_F^2}\right)^3\int d^3k_1\cdots d^3k_6\,\nu_1'\nu_2'\nu_3'\nu_5'\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6)\\
&\qquad\times(k_{12}|k_{34})(k_{34}|k_{56})(k_{56}|k_{12})P\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)P\left(\frac{1}{\omega_5'+\omega_6'-\omega_1'-\omega_2'}\right)\\
&=-\tfrac{1}{3}(2J)E_F(k_Fa_s)^3(0.258\pm0.002)[1+O(k_Fa_s)^2+O(l/\lambda_T)^2],
\end{aligned}$$

$$T_4=\frac{J(2J+1)}{(2\pi)^3\rho\beta}\left(\frac{E_F}{\pi^2k_F{}^2}\right)^3\int d^3k_1\cdots d^3k_6\,\nu_1'\nu_2'\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6) \tag{126}$$

$$\times\int_0^\beta dt\,\exp[t(\omega_1'+\omega_2'-\omega_3-\omega_4)](k_{12}|k_{34})(k_{34}|k_{56})(k_{56}|k_{12})P\left(\frac{1}{\omega_5+\omega_6-\omega_3-\omega_4}\right)P\left(\frac{1}{\omega_5+\omega_6-\omega_1'-\omega_2'}\right)$$

$$=-\frac{\pi^2J(2J+1)}{(2\pi)^3\rho}\left(\frac{E_Fa_s}{\pi^2k_F{}^2}\right)^3[1+O(k_Fa_s)]\int d^3k_1\cdots d^3k_6\nu_1'\nu_2'\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6)$$
$$\times\delta(\omega_5+\omega_6-\omega_3-\omega_4)\delta(\omega_5+\omega_6-\omega_1'-\omega_2')$$

$$=-4J(3\pi)^{-1}E_F(k_Fa_s)^3\times\tfrac{3}{10}[1+O(k_Fa_s)+O(l/\lambda_T)^2],$$

$$T_5=-\frac{J(2J+1)}{(2\pi)^3\rho}\left(\frac{E_F}{\pi^2k_F{}^2}\right)^3\int d^3k_1\cdots d^3k_6\,\nu_1'\nu_2'\nu_3'\nu_4'\nu_5'\nu_6'\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6)$$
$$\times(k_{12}|k_{34})(k_{34}|k_{56})(k_{56}|k_{12})P\left(\frac{1}{\omega_5'+\omega_6'-\omega_1'-\omega_2'}\right)P\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right)$$

$$=-\frac{\pi^2J(2J+1)}{3(2\pi)^3\rho}\left(\frac{E_F}{\pi^2k_F{}^2}\right)^3\int d^3k_1\cdots d^3k_6\,\nu_1'\nu_2'\nu_3'\nu_4'\nu_5'\nu_6'\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6)$$
$$\times(k_{12}|k_{34})(k_{34}|k_{56})(k_{56}|k_{12})\delta(\omega_5'+\omega_6'-\omega_1'-\omega_2')\delta(\omega_3'+\omega_4'-\omega_1'-\omega_2')$$

$$=J(15\pi)^{-1}E_F(k_Fa_s)^3[1+O(k_Fa_s)^2+O(l/\lambda_T)^2],$$

$$T_6=\frac{4J(J-1)(2J+1)}{(2\pi)^3\rho}\left(\frac{E_F}{\pi^2k_F{}^2}\right)^3\int d^3k_1\cdots d^3k_6\,\nu_1'\nu_2'\nu_3'(1-\nu_4')(1-\nu_5')(1-\nu_6')\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_4-\mathbf{k}_6)$$
$$\times\delta^{(3)}(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_2-\mathbf{k}_5)(k_{12}|k_{46})(k_{34}|k_{25})(k_{56}|k_{13})P\left(\frac{1}{\omega_1'+\omega_2'-\omega_4'-\omega_6'}\right)P\left(\frac{1}{\omega_1'+\omega_3'-\omega_5'-\omega_6'}\right)$$

$$=-\tfrac{1}{3}4J(J-1)E_F(k_Fa_s)^3(0.170\pm0.001)[1+O(k_Fa_s)+O(l/\lambda_T)^2].$$

In each of the above six terms the first line always serves to indicate the origin of a term. Thus, T_1 is a term in $F^{(1)}(\nu')$, whereas T_6 gives the entire third order contribution of $F^{(3a)}(\nu')$. The expansion of each T_i to third order in the scattering parameters a_s, r_1, and a_p is also explicitly given, where a_s is the S-wave scattering length, a_p is the P-wave scattering length and $2a_s{}^{-2}r_1{}^3$ is the effective range. We observe that the expansions of the integrands in powers of ($k\times$length) is permissible in all but two of the T_i, because of the "momentum-cutoffs" provided by the factors $\nu'(\mathbf{k})$. The leading terms of T_4 and T_6, however, are those of asymptotic expansions, and one cannot obtain higher order terms from these quantities by simple expansions of their integrands. The leading terms of T_3 and T_6 have been calculated by approximation methods which are estimated to be reliable to within the quoted errors. Finally, we have written $(2m)^{-1}\hbar^2$ as $k_F{}^{-2}E_F$, because the factors $\nu'(\mathbf{k})$ always introduce the cutoff-momentum k_F. Therefore, since k_F and E_F are the only parameters which occur in any of the third order integrals which are performed, the dependences upon these parameters can be removed as scaling factors and cancelled by the multiplying factors of E_F and $k_F{}^{-1}$ outside.

We now substitute Eqs. (125) and (126) into Eq. (122) to obtain the ground-state energy of a Fermi gas to third order in scattering parameters.

$$\lim_{\substack{T\to0\\ \Omega\to\infty}}\frac{\langle E\rangle}{\langle N\rangle}=E_F\Big\{\tfrac{3}{5}+\frac{4J}{3\pi}(k_Fa_s)+\frac{8J}{35\pi^2}(11-2\ln2)(k_Fa_s)^2+\frac{2J}{5\pi}(k_Fr_1)^3$$
$$+\tfrac{1}{3}(2J)[(0.258)+2(J-1)(0.170)-(10\pi)^{-1}](k_Fa_s)^3+6(5\pi)^{-1}(J+1)(k_Fa_p)^3+O(k_Fa_s)^4\Big\}. \tag{127}$$

This result, without the third order terms, has appeared in a number of papers[10,11] and was first written down by Huang and Yang using the pseudopotential method.[12] The third order terms have been previously calculated by Martin and DeDominicis,[13] also using the pseudopotential method. As is well-known, the many-body pseudopotential method is *only* valid for the calculation

[11] T. D. Lee and C. N. Yang, Phys. Rev. **117**, 12 (1960).

[12] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957). See also Elliot Lieb, Proc. Natl. Acad. Sci. U. S. **46**, 1000 (1960).

[13] P. Martin and C. DeDominicis, Phys. Rev. **105**, 1417 (1957).

of thermodynamic quantities which depend upon two-body scattering parameters and not upon the "close-in" behavior of the two-body wave function. One sees from the above result that this is true for the ground-state energy of a Fermi gas to third order, and therefore it is not surprising that the result (127) should have been correctly derived using the pseudopotential method. It is very likely, however, that a calculation of fourth order terms would reveal a dependence of the ground-state energy upon the close-in behavior of the two-particle wave function. For these terms the pseudopotential method would then no longer be valid.

We now return to the discussion of the momentum distribution. In order to calculate $L'(t_2,t_1,\mathbf{k})$ to third order, one must, of course, carry out the program of Sec. V by further including the three-vertex master L graphs. However, to determine the momentum distribution to third order, such a program is unnecessary. Instead, one may utilize the result of Eq. (116) and, in particular, the statements below this equation. Since both the momentum distribution and $F(\nu')$, Eq. (117), are calculated by making the approximations (100) and $N'(\mathbf{k})\cong\nu'(\mathbf{k})$, the quantity $\int_0^\beta dt\, L'(\beta,t,\mathbf{k})$ of Eq. (97) can be determined (in the limit of infinite volume) with the aid of the following identity

$$\int_0^\beta dt\, \mathcal{L}'(\beta,t,\mathbf{k}) =-\frac{\delta}{\delta\nu'(\mathbf{k})}[(2\pi)^3(2J+1)^{-1}F(\nu')]_{R'}+O(k_Fa_s)^4. \quad (128)$$

Equation (128) is derived from Eqs. (117) and (87) in analogy with the derivation of Eq. (9) from Eq. (8). The subscript R', means that the primed pair functions must be held constant during the functional differentiation.

Upon substituting Eqs. (125) and (126) into Eq. (128), we immediately see that *only* the quantity $\int_0^\beta dt\, \Lambda(\beta-t,\mathbf{k})\cong\int_0^\beta dt\, A(\mathbf{k})$ [see Eq. (89)] is obtained, whereas one knows from Eq. (110) that there must be other terms in the momentum distribution. The difficulty lies in the fact that Eq. (125) has been derived by using the identity (109), which, since the pair function must be held constant in (128), cannot be used until after the functional differentiation with respect to $\nu'(\mathbf{k})$ has been performed. Therefore, in order to calculate the third order corrections to the momentum distribution using Eq. (128), we must identify those terms in $F(\nu')$ in which the identity (109) has been used, and perform the functional differentiation first. For example, the only second order term in this category is

$$F_1^{(2)}(\nu')\equiv\frac{J(2J+1)}{(2\pi)^3}\left(\frac{E_F}{\pi^2k_F^2}\right)^2\int d^3k_1d^3k_2d^3k_3d^3k_4(1-\nu_1')(1-\nu_2')\nu_3'\nu_4'\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)$$
$$\times(k_{12}|k_{34})(k_{34}|k_{12})\left(\frac{1}{\omega_1'+\omega_2'-\omega_3'-\omega_4'}\right)^2\{\exp[\beta(\omega_3'+\omega_4'-\omega_1'-\omega_2')]-1\}. \quad (129)$$

With the aid of the identity (109), it is readily verified that $F_1^{(2)}(\nu')$ is zero.

In order to specify precisely the manner in which the contribution of $F_1^{(2)}(\nu')$ to the momentum distribution is calculated, we first define a quantity $F_1(\nu')$:

$$F_1(\nu')\equiv F(\nu')-\rho\beta[T_1+T_2+T_3+T_4+T_5+T_6]+O(a_s^4)$$
$$=F_1^{(2)}(\nu')+O(a_s^3). \quad (130)$$

Then the quantity $\int_0^\beta dt\, L'(\beta,t,\mathbf{k})$ is obtained by replacing $F(\nu')$ by $F_1(\nu')$ on the right-hand side of (128) [see Eq. (89)]. Combining the result of this substitution with Eq. (97), we obtain

$$\langle n(\mathbf{k})\rangle=\nu_\mathbf{k}'+\nu_\mathbf{k}'(1-\nu_\mathbf{k}')\frac{\delta}{\delta\nu_\mathbf{k}'}$$
$$\times[(2\pi)^3(2J+1)^{-1}F_1(\nu')]_{R'}+O(k_Fa_s)^4. \quad (131)$$

Thus, upon substituting Eq. (129) into Eq. (131), we obtain the function $M_2(\mathbf{k})$, where

$$M_2(\mathbf{k})\equiv\nu_\mathbf{k}'(1-\nu_\mathbf{k}')\frac{\delta}{\delta\nu_\mathbf{k}'}[(2\pi)^3(2J+1)^{-1}F_1^{(2)}(\nu')]_{R'}$$
$$=2J\pi^{-2}(k_Fa_s)^2[(1-\nu_\mathbf{k}')M_a(\mathbf{k})-\nu_\mathbf{k}'M_b(\mathbf{k})]. \quad (132)$$

The quantities $M_a(\mathbf{k})$ and $M_b(\mathbf{k})$ are given by Eqs. (111).

We now write the momentum distribution as

$$\langle n(\mathbf{k})\rangle=\nu'(\mathbf{k})+M_2(\mathbf{k})+M_3(\mathbf{k})+O(k_Fa_s)^4. \quad (133)$$

The third order contribution $M_3(\mathbf{k})$ is determined in complete analogy with the determination of $M_2(\mathbf{k})$, and one finds the result:

$$M_3(\mathbf{k}_0)=2J\left(\frac{E_Fa_s}{\pi^2k_F^2}\right)^3\int d^3k_1d^3k_2d^3k_3d^3k_4d^3k_5(\nu_1'+\nu_2')[(1-\nu_0')(1-\nu_5')\nu_3'\nu_4'-\nu_0'\nu_5'(1-\nu_3')(1-\nu_4')]$$
$$\times\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\delta^{(3)}(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_0)\left(\frac{1}{\omega_5'+\omega_0'-\omega_3'-\omega_4'}\right)^2P\left(\frac{1}{\omega_3'+\omega_4'-\omega_1'-\omega_2'}\right). \quad (134)$$

We observe that neither $M_2(\mathbf{k})$ nor $M_3(\mathbf{k})$ makes a contribution to the density $\rho=(2J+1)(2\pi)^{-3}\int d^3k\langle n(\mathbf{k})\rangle$.

We finally evaluate the third order correction to the thermodynamic potential g, [see Eqs. (114) and (115)]. At first sight it would seem to be necessary to calculate the quantity $\Delta(\mathbf{k})\cong A(\mathbf{k})$ by directly using Eq. (128). This would indeed be a tedious procedure. For the ground-state problem, however, we note that $\nu'(\mathbf{k})$ is a step function with its discontinuity at $k=k_F$. Moreover, according to Eq. (114) the quantity which we need in order to determine g is $\Delta(\mathbf{k}_F)$. Therefore, since we only require the value of $\Delta(\mathbf{k})$ at $k=k_F$, we may use the following relation derived from Eq. (128):

$$\Delta(\mathbf{k}_F)=-\frac{k_F}{3\rho\beta}\frac{\partial}{\partial k_F}[F(\nu')-F_1(\nu')]_{R'}+O[E_F(k_Fa_s)^4]. \quad (135)$$

Since the identity (109) was not used in the derivation of any of the terms of Eqs. (126), we may substitute Eqs. (126) directly into Eq. (135) to obtain

$$\begin{aligned}\lim_{T\to 0}g&=E_F+\Delta(k_F)\\&=E_F\Big\{1+\frac{8J}{3\pi}(k_Fa_s)+\frac{8J}{15\pi^2}(11-2\ln 2)(k_Fa_s)^2\\&\quad+\frac{16J}{15\pi}(k_Fr_1)^3+\tfrac{1}{9}(16J)[(0.258)\\&\quad+2(J-1)(0.170)-(10\pi)^{-1}](k_Fa_s)^3\\&\quad+16(5\pi)^{-1}(J+1)(k_Fa_p)^3+O(k_Fa_s)^4\Big\}. \quad (136)\end{aligned}$$

One might have anticipated that the calculation of the thermodynamic potential would be given by a relation such as (135); since it is an intensive quantity and should not depend upon the details of a distribution function, i.e., upon the explicit calculation of $\Delta(\mathbf{k})$ for all values of $\mathbf{k}$.

VII. GROUND-STATE ENERGY OF A FERMI GAS OF HARD SPHERES WITH AND WITHOUT WEAK S-WAVE ATTRACTIONS

It is of considerable interest to consider a Fermi gas composed of two different kinds of Fermions with the same mass, e.g., neutrons and protons. These Fermions can be characterized by the different projections of an isotopic spin quantum number $I=\frac{1}{2}$. For interactions which are independent of both isotopic and ordinary spin projections, the matrix elements of $R(t_i,t_j)$, Eq. (14), include factors of $\delta_{m_1m_2}\delta_{q_1q_2}$, where m_i is the magnetic spin quantum number and q_i is a corresponding projection of the isotopic spin $\mathbf{I}$.

The spin factors, which correspond to the extra degrees of freedom of the isotopic spin, can readily be calculated for each of the T_i of Eqs. (126). One then finds that the only replacement which is necessary in any of these equations is to write

$$2J\to(2J+1)(2I+1)-1. \quad (137)$$

For example, the Fermi momentum k_F [see Eq. (34)] is now defined by the relation

$$\rho=l^{-3}=(6\pi^2)^{-1}(2J+1)(2I+1)k_F^3. \quad (138)$$

Thus, for a Fermi gas with two kinds of Fermions which interact with strong, short-range, charge-independent and spin-independent forces, Eq. (127) for the ground-state energy per particle becomes

$$\begin{aligned}\langle E\rangle/\langle N\rangle=E_F\{&\tfrac{3}{5}+\tfrac{1}{3}[(2J+1)(2I+1)-1][2\pi^{-1}(k_Fa_s)+12(35\pi^2)^{-1}(11-2\ln 2)(k_Fa_s)^2+C_{J,I}(k_Fa_s)^3\\&+(5\pi)^{-1}k_F^3(3r_1^3-a_s^3)]+3(5\pi)^{-1}[(2J+1)(2I+1)+1](k_Fa_p)^3+O(k_Fa_s)^4+O(l/\lambda_T)^2\}, \quad (139)\end{aligned}$$

where

$$C_{J,I}=(0.258)+[(2J+1)(2I+1)-3]\times(0.170)+(10\pi)^{-1}, \quad (140)$$

$$C_{\frac{1}{2},\frac{1}{2}}=(0.460\pm 0.003).$$

We now examine Eq. (139) for two particular two-body interactions; namely, hard core repulsions with and without weak S-wave attractions. In a gas of hard spheres there is only one interaction length, which is the diameter a of a single particle. According to Table I of I, the scattering parameters a_s, a_p, and r_1 are related to the core diameter a as follows:

$$a_s=a;\quad a_p^3=r_1^3=\tfrac{1}{3}a^3\ \text{(repulsive core only)}. \quad (141)$$

For these values of the scattering parameters, Eq. (139) becomes (with $I=J=\frac{1}{2}$)

$$\frac{\langle E\rangle}{\langle\langle N}=E_F\{\tfrac{3}{5}+2\pi^{-1}(k_Fa)+12(35\pi^2)^{-1}(11-2\ln 2)(k_Fa)^2+(0.778)(k_Fa)^3+O(k_Fa)^4+O(l/\lambda_T)^2\}. \quad (142)$$

In Fig. 11, the cumulative contributions of the four terms of Eq. (142) have been plotted as a function of the ratio (l/a) in order to determine the relative importance of the various terms. The value of l/a for hexagonal close-packing is $l_H/a=2^{-1/6}=0.89$, a number quite outside the range of the curves of Fig. 11.

In their treatment of the nuclear many-body problem, Weisskopf, Walecka, and Gomes[14] first calculate the energy per nucleon due to the S-wave part of the hard core repulsion by the Brueckner method, and then calculate the energy due to nuclear attraction by per-

[14] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. 3, 241 (1958).

turbation theory. In such a calculation it has meaning to refer to the repulsive energy per nucleon in nuclear matter. The separation of particles in nuclear matter, $l_N \cong 1.66\times 10^{-13}$ cm, and the scale of the Fermi momentum has been indicated in Fig. 11 for a hard core diameter, $a=0.4\times 10^{-13}$ cm. To $O(k_F a)^3$ and for $l/a \gtrsim 4$, Eq. (142) closely approximates the result calculated by the Brueckner method.

A Fermi gas of weakly attracting hard spheres is characterized by two interaction lengths; namely, the diameter a of a single particle and the negative part b of the scattering length due to the weak attraction alone. The A matrix for this problem can be derived by considering the problem of an S wave δ function attraction outside of a hard core repulsion. According to Table I of I, the scattering parameters a_s, a_p, and r_1, which one obtains in this case are

$$a_s=a+b;\quad a_p{}^3=\tfrac{1}{3}a^3;\quad r_1{}^3=\tfrac{1}{3}[(a+b)^3-b^3]$$
$$[\text{repulsive core+zero-range } (L=0) \text{ attraction}]. \quad (143)$$

For these values of the scattering parameters, Eq. (139) with $(I=J=\frac{1}{2})$ becomes:

$$\begin{aligned}\langle E\rangle/\langle N\rangle = E_F\{&\tfrac{3}{5}+2\pi^{-1}(k_Fa_s)+12(35\pi^2)^{-1}\\ &\times(11-2\ln 2)(k_Fa_s)^2+(0.460)(k_Fa_s)^3\\ &+\pi^{-1}(k_Fa)^3-(5\pi^{-1})(k_Fb)^3\\ &+O(k_Fa_s)^4+O(l/\lambda_T)^2\}. \quad (144)\end{aligned}$$

The effect of the $-(1/5\pi)(k_Fb)^3$ term is both small and positive for weak attractions. The energy per particle, as determined by Eq. (144), has been plotted in Fig. 12 for various values of the ratio b/a. The free particle curve $(3/5)E_F$ has also been dashed in for comparison.

For $-2\lesssim b/a<0$, the energy curves depart from the pure repulsive core curve in a way which might be expected. That is, at a fixed density an increase in the attractive force causes the energy per particle to decrease and the slope of the curve remains negative as it should for any physical system. The difference between the $b/a=-1$ $(a_s=0)$ curve and the dashed curve in Fig. 12 explicitly gives the effect of the P wave and b^3 terms upon the energy. In this case the P-wave contribution is exactly five times the contribution of the b^3 term.

At first sight, the curves for $b/a\lesssim -2$ in Fig. 12 seem to imply that for large and negative scattering lengths a Fermi system of attracting hard spheres condenses. The conclusion is here not justified, because in the region $l/a\sim 4$ and $b/a<-2$ Eq. (144) would be very sensitive to $O(a_s/l)^4$ terms. Moreover, the curves for $b/a<-2$ are clearly incorrect since they imply that the ground-state pressure, $\mathcal{P}=-[d/d(1/\rho)](E/N)$, of the physical system is negative. However, the curves may still be (asymptotically) valid in the low-density region $l/a>5$ where fourth order effects are expected to be unimportant.

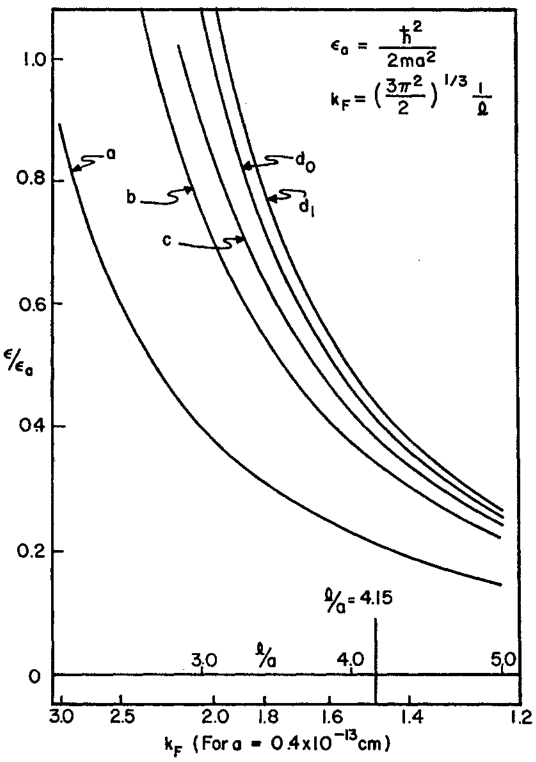

FIG. 11. The ground-state energy per particle for a Fermi gas of hard spheres as a function of interparticle separation $[I=J=\frac{1}{2}]$. Curve a—the free particle energy; Curve b—the energy of a gas of hard spheres to $O(a/l)$; Curve c—the energy of a gas of hard spheres to $O(a/l)^2$; Curve d_0—the energy of a gas of hard spheres to $O(a/l)^3$, $L=0$ terms only; Curve d_1—the energy of a gas of hard spheres to $O(a/l)^3$, both $L=0$ and $L=1$ terms.

CONCLUSIONS

To be of value a theory must be fruitful in the new results which it can calculate. Therefore, in a certain sense the present report serves as a stepping stone for further theoretical investigations. We here indicate a number of directions for further study, which are suggested by the results of this paper.

The contribution of the fourth order (in k_Fa_s) terms to the ground-state energy of a Fermi gas would be a result of great interest. In the first place, there is the question as to whether there is any dependence in the fourth order terms on the close-in behavior of the two-body wave function. This in turn has a bearing on the limit of validity of the pseudopotential method.[12] In the second place, there is the matter of the convergence of the low-density expansion of the energy in terms of interaction range parameters (see Fig. 11). It may well be true, even at very low densities, that this expansion is only asymptotically correct.[15]

A second question of interest is the explicit temperature dependence of the thermodynamic quantities of low-density Fermi gases. The results for quantities such as the specific heat and the density of states would

[15] In this connection see L. Van Hove, Physica 25, 849 (1959).

probably have implications for corresponding real Fermi systems such as liquid He^3. There is also the problem of determining the excitation spectrum in the ground state of a Fermi gas. Hugenholtz and Van Hove[16] have shown that it has meaning to talk about single-particle excitations only when $k \sim k_F$. They also show that the limiting value of the excitation energy as $k \to k_F$ is the thermodynamic potential g [as one can verify with the aid of Eq. (36)]. It would be interesting to understand the results of Hugenholtz and Van Hove in the present context and to generalize their studies to nonzero temperatures.

A problem of continuing interest is the nuclear many-body problem; namely, the determination of the equilibrium density and binding energy of infinite nuclear matter. As mentioned in Sec. VII, the Weisskopf, Walecka, and Gomes[14] treatment of the nuclear many-body problem is to use the Brueckner method to calculate the energy due to hard core repulsions alone and then to add the attractive energy by using perturbation theory. Although the mathematical analysis by these authors is somewhat unsatisfactory, the physical justifications presented by them seem quite sound. This suggests, therefore, that the methods introduced in this paper might be capable of giving additional insight into the nuclear many-body problem by using the investigations of Weisskopf *et al.* as a model.

A physical system which very closely resembles a low-density Fermi gas is the system of free electrons in a conductor. It would seem, then, that a detailed study of the low-temperature electron gas, using the present methods, could lead to explicit predictions for the momentum distribution (and other thermodynamic quantities) which could be compared with experiment. For example, we have seen in Eq. (133) for a low-density Fermi gas with strong, short-range interactions that the ground-state momentum distribution has a sharp discontinuity. In a recent paper,[17] Luttinger has indicated that this may also be true for an electron gas.

There is finally another system to which the present methods can be applied; namely, the Bose gas. Thus, Eq. (97) for $\langle n(\mathbf{k})\rangle$ can readily be applied to a Bose gas by setting $\epsilon = +1$. In this case it is not the region $\rho|a_s|\lambda_T^2 \gg 1$ which necessitates expressing the momentum distribution as a functional of $\nu'(k)$, but the temperature region $T \gtrsim T_C$, where T_C is the critical or transition temperature. As Lee and Yang have shown,[5,18] the transition temperature for a Bose gas is characterized by momentum space condensation into the zero-momentum state, i.e., by the fact that $\langle n(\mathbf{k})\rangle \to \infty$ for $\mathbf{k}=0$ and $T \to T_C +$. Moreover, it is quite likely that the function $\nu'(\mathbf{k})$ contains the essential features of this singularlty, whereas $\nu(\mathbf{k})$, Eq. (7), certainly does not. It would thus be of value to compare the present method for this problem with the variational method used by Lee and Yang.

[16] N. M. Hugenholtz and L. Van Hove, Physica **24**, 363 (1958).
[17] J. M. Luttinger, Phys. Rev. **119**, 1153 (1960). See also W. Kohn and S. H. Vosco, Phys. Rev. **119**, 912 (1960).
[18] T. D. Lee and C. N. Yang, Phys. Rev. **117**, 897 (1960).

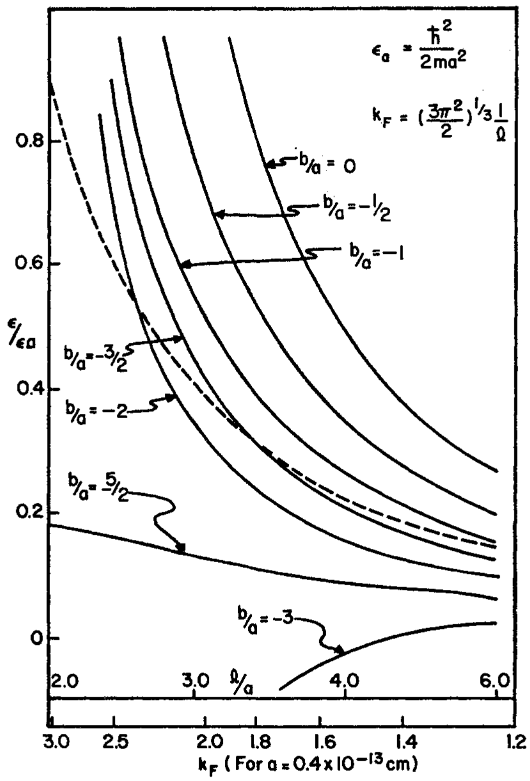

FIG. 12. The ground-state energy per particle for a Fermi gas of weakly attracting hard spheres as a function of interparticle separation. The parameter (b/a) is the ratio of the attractive part of the scattering length $(a_s = a+b)$ to the hard-core diameter. The energy has been calculated to third order in the scattering length approximation for the case $I=J=\frac{1}{2}$ and an $L=0$ attraction. The dashed curve is the energy per particle for free Fermions.

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APPENDIX A

In this Appendix we prove that Eqs. (50) and (51) are equivalent. That is, we prove the following identity:

$$\sum_0 \binom{N_0-1}{S_0} L_0(\beta) = \sum_C \binom{N_C-1}{S_C} L_C(\beta). \qquad \text{(A.1)}$$

We begin by defining several terms necessary to the proof of (A.1).

Definitions

The contraction of an open 0-graph into two linked-pair 0-graphs is a procedure in which an open 0-graph is first separated into two linked-pair 1 graphs, by cutting both a wiggly line and a solid line and then replacing both external wiggly lines by external solid lines. The procedure then consists of forming two linked-pair 0-graphs by separately joining the ends of the two linked-pair 1 graphs.

The *contraction of an open 0-graph into closed 0-graphs* consists of successively contracting all open 0-graphs formed in the above contraction into two linked-pair 0-graphs, until there remain only 0-graphs which are not-open.

A *closed subgraph* in an open 0-graph is a set of vertices together with all of the lines which *touch* these vertices, which becomes a 0-graph which is not-open by performing the above contraction procedure. *Each closed subgraph corresponds uniquely to a closed 0-graph.*

$n_C \equiv$ (the number of closed 0-graphs of a given kind associated with an open 0-graph by the preceding correspondence). (A.2)

Examples

If the contraction procedure is applied to the open 0-graphs of Fig. 8(a), then both of the new 0-graphs formed in the first two cases are not-open. For the remaining three cases, however, one of the new 0-graphs formed after the first contraction is open and the other is not-open.

Theorem 1

$$(N_0-1)=\sum_C (N_C-1)n_C. \tag{A.3}$$

Proof. If N_1 and N_2 are the numbers of solid lines in the two linked-pair 0-graphs formed by the first contraction of an open 0-graph, then we have the sum rule

$$(N_0-1)=(N_1+N_2-1)-1=(N_1-1)+(N_2-1).$$

Iteration of this sum rule in the successive contractions of an open 0-graph yields Eq. (A.3).

We next identify each of the closed 0-graphs of Eq. (A.3) with corresponding closed 0-graphs on the right-hand side of (A.1). The definitions of the solid-line factors $\epsilon\mathfrak{N}(t,\mathbf{k})G(\beta,s,\mathbf{k})$ of closed graphs are then such that each of these closed 0-graphs generates the open 0-graph $S_0^{-1}L_0(\beta)$ when we iterate the functions $G(t,s)$. According to Theorem 1, therefore, we have now only to show that the "symmetry number" $(n_C S_0^{-1})$ is correctly "generated" by *each* of the closed 0-graphs on the right-hand side of (A.1).

We assign a label α to each of the closed subgraphs of an open 0-graph $L_0(\beta)$, and study the symmetry of $L_0(\beta)$ in terms of its closed subgraphs. For this purpose, it is necessary to define five quantities.

Definitions

$m_{\alpha 0}\equiv$[the number of line permutations [see below (42)] in $L_0(\beta)$ which involve the lines of the closed subgraph α_0, but which consist only of permutations of entire closed subgraphs, and which leave the 0-graph $L_0(\beta)$ topologically unchanged]. (A.4)

This number is unity for all of the closed subgraphs of Fig. 8(a), except in the fourth open 0-graph where $m_{\alpha 0}=2$ for the one-vertex closed subgraphs.

$S_{\alpha 0}\equiv$[the symmetry number of the closed 0-graph $L_{\alpha 0}(\beta)$, which corresponds to the closed subgraph α_0], (A.5)

$S_{\alpha 0}'\equiv$[the number of line permutations in $L_0(\beta)$ which involve the lines of the closed subgraph α_0, and which leave the 0-graph $L_0(\beta)$ topologically unchanged]. (A.6)

Clearly $S_{\alpha 0}'$ includes the number $m_{\alpha 0}$ as a factor. Now, if we consider a given closed 0-graph $L_{\alpha 0}(\beta)$, which corresponds to the closed subgraph α_0, then corresponding to all of the other closed subgraphs we may associate closed L graphs. These L graphs are generated "along the lines" of the closed 0-graph $L_{\alpha 0}(\beta)$ in the process of generating $L_0(\beta)$. For each such closed L graph we define.

$\sigma_{\alpha,\alpha 0}\equiv$[the symmetry number of the closed L graph $L_\alpha(t_2,t_1,\mathbf{k})$, which corresponds to the closed subgraph α, with respect to the closed subgraph α_0], (A.7)

$\sigma_{\alpha,\alpha 0}'\equiv$[the number of line permutations in $L_0(\beta)$ which involve the *internal* lines, with respect to the closed subgraph α_0, of the closed subgraph α, and which leave the 0-graph $L_0(\beta)$ topologically unchanged]. (A.8)

We now prove two theorems in terms of the five quantities (A.4)–(A.8).

Theorem 2

The number of ways of generating an open 0-graph $L_0(\beta)$ from a given closed 0-graph $L_C(\beta)$ is

$$\sum_{\alpha 0\in C}\left(\frac{S_{\alpha 0}}{S_{\alpha 0}'}\right)\prod_{\alpha\neq\alpha 0}\left(\frac{\sigma_{\alpha,\alpha 0}}{\sigma_{\alpha,\alpha 0}'}\right);\quad \text{with}\quad \sum_{\alpha 0\in C}1=n_C, \tag{A.9}$$

where the summation is over all the closed sub graphs α_0 in $L_0(\beta)$ which correspond to the same closed 0-graph $L_C(\beta)$.

Proof. Each number in the sum of (A.9) is equal to the number of ways of destroying the symmetry of the

closed 0-graph $L_{\alpha 0}(\beta)$ and the various L graphs $L_\alpha(t_2,t_1,\mathbf{k})$ when the subgraph α_0 is chosen as the basic 0-graph of $L_0(\beta)$. We may therefore say that each of these numbers is also the number of ways of generating $L_0(\beta)$ from $L_{\alpha 0}(\beta)$.

Referring to the definition (A.4), we note that $m_{\alpha 0}$ is the number of closed subgraphs in $L_0(\beta)$ which appear symmetrically with the subgraph α_0. From this symmetry, we conclude that only one of the $m_{\alpha 0}$ closed subgraphs should be considered as the basic structure for generating $L_0(\beta)$, in which case one must include a factor of $m_{\alpha 0}$ for each term in the sums of (A.9) [see comment below (A.6)]. Alternatively, (A.9) is correct if we consider *each* of the $m_{\alpha 0}$ closed subgraphs as the basic structure for generating $L_0(\beta)$, as we have done.

Theorem 3

The symmetry number S_0 of an open 0-graph $L_0(\beta)$ can be computed as the product

$$S_0 = S_{\alpha 0}' \prod_{\alpha\neq\alpha_0} \sigma_{\alpha,\,\alpha 0}', \tag{A.10}$$

where α_0 refers to any closed subgraph of $L_0(\beta)$.

Proof. Theorem 3 follows from the definition of the symmetry number of a linked-pair 0-graph (I.58) and the definitions (A.6) and (A.8).

We now observe that a factor $S_{\alpha 0}^{-1}(\prod_{\alpha\neq\alpha 0}\sigma_{\alpha,\,\alpha 0}^{-1})$ occurs with each term in the iteration of the right-hand side of (A.1). From this fact and Theorems 2 and 3 we may conclude that each of the different closed 0-graphs corresponding to the closed subgraphs of an open 0-graph $L_0(\beta)$ generates $L_0(\beta)$ with the "symmetry number" $(n_C S_0^{-1})$. The identity (A.1) follows by combining this result with Theorem 1.

APPENDIX B

We here prove Eq. (54):

$$\Omega F(N) - \sum_C S_C^{-1} L_C(\beta) = \epsilon \sum_{\mathbf{k}} \sum_{m=2}^{\infty} \epsilon^m \left(\frac{m-1}{m}\right) \int_0^\beta dt_1 dt_2 \cdots dt_m P(t_1,t_2,\mathbf{k}) \times P(t_2,t_3,\mathbf{k}) \cdots P(t_m,t_1,\mathbf{k}). \tag{B.1}$$

The first part of this proof proceeds in analogy with the proof of Eq. (A.1), Appendix A.

Definitions

The *contraction of a closed 0-graph into master 0-graphs* consists of the following procedures.

(1) Replace all wiggly lines in the closed 0-graph by solid lines.

(2) If the resulting 0-graph is reducible, cut two lines to form two 1 graphs.

(3) Separately join the ends of each 1 graph created in step (2) to form two new 0-graphs.

(4) Repeat (2) and (3) for each new reducible 0-graph which is formed.

A *master subgraph* in a closed 0-graph is a set of vertices together with the lines which *touch* these vertices, which becomes a not-reducible 0-graph as a result of the above contraction procedure. *Each master subgraph corresponds uniquely to a master 0-graph.*

$$n_M \equiv (\text{the number of master subgraphs in a closed 0-graph}). \tag{B.2}$$

Theorem 4

$$\Omega F(N) = \textstyle\sum_C n_M S_C^{-1} L_C(\beta). \tag{B.3}$$

Proof. From the last paragraph of Appendix A we may conclude that the *sum* $(\sum_C)$ over all different closed 0-graphs, corresponding to the closed subgraphs of an open 0-graph $L_0(\beta)$, generates $L_0(\beta)$ with the "symmetry number" $(\sum_C n_C)S_0^{-1}$, where n_C is the number of closed subgraphs which correspond to the same closed 0-graph $L_C(\beta)$. Similarly, after making slight changes in the terminology of Theorems 2 and 3 and the preceding definitions, one may conclude that Theorem 4 is correct, i.e., that the sum over all different master 0-graphs, corresponding to the master subgraphs of a closed 0-graph $L_C(\beta)$, generates $L_C(\beta)$ with the "symmetry number" $(n_M S_C^{-1})$. [Theorem 4 should be checked for each of the closed 0 graphs of Fig. 8(b).]

We substitute Eq. (B.3) into (B.1) to obtain

$$\sum_C \left(\frac{n_M - 1}{S_C}\right) L_C(\beta) = \epsilon \sum_{\mathbf{k}} \sum_{m=2}^{\infty} \epsilon^m \left(\frac{m-1}{m}\right) \int_0^\beta dt_1 dt_2 \cdots dt_m\, P(t_1,t_2,\mathbf{k}) \times P(t_2,t_3,\mathbf{k}) \cdots P(t_m,t_1,\mathbf{k}). \tag{B.4}$$

In this form of Eq. (B.1), the function $P(t_i,t_j)$ may be considered to be the sum over all closed L graphs with no external line factors (41), as one can prove using a modification, for L graphs, of Theorems 2 and 3. Then, all of the terms of (B.4) are *improper* closed graphs, which we shall now examine from a different viewpoint.

Definitions

It is possible to cut two (wiggly) lines in an improper closed 0-graph so as to separate the 0-graph into two parts. The lines for which this is possible are called *improper lines.*

An *improper loop* in a closed 0-graph is a collection of improper lines, any two of which can be cut so as to separate the 0-graph into two parts. All pairs of improper lines with this property belong to the same improper

loop, and *different* improper loops never have an improper line in common.

Examples

In each of the first three closed 0-graphs of Fig. 8(b) there is only one improper loop, whereas in the last two 0-graphs there are two improper loops.

Definition

$$\begin{aligned} n_l \equiv & [\text{number of improper lines in the } l\text{th improper loop of an improper closed 0-graph } L_C(\beta)], \\ = & [\text{number of master subgraphs connected by the improper lines of the } l\text{th improper loop of an improper closed 0-graph } L_C(\beta)]. \end{aligned} \tag{B.5}$$

For a closed 0-graph which is not improper we define $n_l = 1$. We also observe that whereas two improper loops can never have improper lines in common, they may have a master subgraph in common.

Theorem 5

For a given closed 0-graph $L_C(\beta)$,

$$(n_M - 1) = \sum_l (n_l - 1), \tag{B.6}$$

where $\sum_l$ indicates the sum over all different improper loops in $L_C(\beta)$.

Proof. Theorem 5 is obviously true for a closed 0-graph with only one improper loop. It is also true for a closed 0-graph with two improper loops, since in this case the number of improper lines is (n_M+1), i.e., the two improper loops must have one master subgraph in common. An induction proof serves to establish the general case.

We substitute Eq. (B.6) into (B.4) to obtain

$$\sum_C \sum_{l \epsilon C} \left(\frac{n_l - 1}{S_C}\right) L_C(\beta) = \epsilon \sum_{\mathbf{k}} \sum_{m=2}^{\infty} \epsilon^m \left(\frac{m-1}{m}\right) \int_0^\beta dt_1 dt_2 \cdots dt_m \, P(t_1,t_2,\mathbf{k}) \times P(t_2,t_3,\mathbf{k}) \cdots P(t_m,t_1,\mathbf{k}). \tag{B.7}$$

But now we use the second form of the definition (B.5), and also interpret each of the factors $P(t_i,t_j)$ on the right-hand side of (B.7) as the sum over all possible master subgraphs (each of which may or may not include other master subgraphs along its internal lines). If, on the left-hand side of (B.7), we next interchange the sum over all closed 0-graphs with the sum over all improper loops for each 0-graph, then a one-to-one correspondence between the terms on the two sides of this equation can be established. We therefore identify the numbers n_l with the numbers m and write Eq. (B.7) as

$$\sum_{m=2}^{\infty} \sum_{C \epsilon m} N_m^{(C)} \left(\frac{m-1}{S_C}\right) L_C(\beta) = \epsilon \sum_{\mathbf{k}} \sum_{m=2}^{\infty} \epsilon^m \left(\frac{m-1}{m}\right) \times \int_0^\beta dt_1 dt_2 \cdots dt_m P(t_1,t_2,\mathbf{k}) \cdots P(t_m,t_1,\mathbf{k}), \tag{B.8}$$

where

$$N_m^{(C)} \equiv [\text{number of improper loops in } L_C(\beta) \text{ with } m \text{ improper lines}]. \tag{B.9}$$

Equation (B.8) is our final form of Eq. (B.1). In order to verify this equation, it must be shown that $1/m$ times the integrated product of $(m)P(t_i,t_j,\mathbf{k})$'s "generates" the correct "symmetry number" $(N_m^{(C)} S_C^{-1})$ for the closed 0-graph $L_C(\beta)$. To prove this one must first verify that Theorem 3, together with appropriate changes in the definitions (A.4)–(A.8), also applies to a closed 0-graph $L_C(\beta)$. It is also necessary to observe that for each of the master subgraphs α_0 connected by the improper lines of a given improper loop l, the corresponding factor S_{α_0}' of (A.6) always includes a factor $\sigma_{\alpha_0,\,l}'$ where

$$\sigma_{\alpha_0,\,l}' \equiv [\text{the number of line permutations in } L_C(\beta) \text{ which involve the } \textit{internal} \text{ lines with respect to the } l\text{th improper loop of the master subgraph } \alpha_0, \text{ and which leave the 0-graph } L_C(\beta) \text{ topologically unchanged}]. \tag{B.10}$$

Equation (B.8), and therefore also (B.1), can finally be verified by combining a study of the symmetries of improper loops in closed 0-graphs with the above discussion.