

Scattering of Electrons by Phonons and Impurities in Semiconductors

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A theory is developed for the mobility of an electron in an n -type semiconductor under the combined scattering of phonons and impurities. An attempt is made to combine the two processes in a rigorous manner by treating the effects of impurities as though due to an applied external field. Although in principle a solution for any value of τ , the phonon scattering relaxation time, should be possible, this paper only arrives at a limiting law for the mobility at small values of τ . In the region of applicability satisfactory agreement with experiment is achieved.

I. INTRODUCTION

THE treatment of carrier mobility in semiconductors when both phonon and impurity scattering processes are active has been very approximate. Usually a phonon scattering relaxation time τ and an impurity time τ_I are computed under the assumption that either one of the processes is active alone, and then an effective relaxation time τ_e is estimated by the relation¹

$$1/\tau_e = 1/\tau + 1/\tau_I. \quad (1.1)$$

This relation is only valid when the two scattering processes are independent of one another, that is, when a carrier is scattered by a phonon *or* an impurity, but not by *both* simultaneously. In the case of Coulomb impurities with divergent scattering cross sections it is impossible to achieve this condition. In fact, an examination of the usual method² of treating scattering by Coulomb impurities reveals that the volume of the semiconductor is spanned by a set of close-packed cells, one to each impurity, and a collision is defined as the passage of a carrier through one such cell, the deflection suffered by the carrier in this transit being the scattering angle. Since the cells span the entire volume, a carrier is always being scattered by *some* impurity. Thus, any phonon scattering *must* occur during an impurity collision, and the processes cannot be independent.

This paper will be devoted to a means of avoiding this inconsistency and, thus, to the development of a more rigorous formula for carrier mobility in the presence of both phonon and impurity scattering. There are many other problems which must be solved before an ultra-rigorous formula can be derived, but in this paper attention will be confined to the one problem mentioned above.

Problems such as scattering by bound modes or effects due to electron correlation will not be considered.

II. EQUATION OF CONTINUITY IN NEAREST NEIGHBOR SPACE

Our approach shall be classical. This means that there will be a region in the immediate neighborhood of each

impurity where the theory will not apply directly; quantum considerations being necessary. Then the classical approach will only be acceptable provided suitable boundary conditions can be introduced at the limits of the quantum region whose width should be of the order of the electron wavelength. The precise specification of such boundary conditions represents an additional separate problem, and we shall only comment on it briefly in the present text.

In the classical problem an attempt is usually made to employ the Boltzmann transport equation³:

$$\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{f - f_0}{\tau_e}, \quad (2.1)$$

where $f(\mathbf{r}, \mathbf{v})$ is the density of electrons at the point (\mathbf{r}, \mathbf{v}) in phase space, \mathbf{r} being the vector position and \mathbf{v} the velocity. \mathbf{a} is the acceleration, a function of \mathbf{r} , and τ_e is a relaxation time. f_0 is the value of the distribution function f at equilibrium so that $f - f_0$ is the perturbation in the steady state. For the case we have in mind τ_e is usually taken as given in (1.1), and \mathbf{a} is assumed to arise only in the unidirectional effect of the applied electric field. $\partial/\partial \mathbf{r}$ and $\partial/\partial \mathbf{v}$ are gradient symbols, and in the case where \mathbf{a} is unidirectional, $\partial f/\partial \mathbf{r}$ is set equal to zero. The solution of Eq. (2.1) is then very simple.

In Eq. (2.1), the effects of impurities are to be found on the right in the so-called collision term. These effects enter through Eq. (1.1). The condition under which this treatment is valid has been outlined above. In order to avoid the special problems associated with the use of Eq. (1.1), we shall attempt to construct a scheme wherein the effects of the impurities are manifest on the left of an equation like (2.1), entering specifically in the acceleration \mathbf{a} . Phonon scattering will still be treated by means of a relaxation time through a term such as appears on the right of Eq. (2.1). With this procedure the impurities are actually regarded as the source of an additional applied (although complicated) electric field with which the electrons are always interacting. Phonon scattering can occur simultaneously and the problem of the independence of the two processes does not arise.

We shall confine our attention to an extrinsic n -type

¹ J. M. Ziman, *Electrons and Phonons* (Clarendon Press, Oxford, England, 1960), p. 286.

² W. Shockley, *Electrons and Holes in Semiconductors* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1950), p. 258.

³ J. M. Ziman, *Electrons and Phonons* (Clarendon Press, Oxford, England, 1960), Chap. VII.

semiconductor in which there are equal concentrations of conduction electrons and ionized (positive) impurities. In the Conwell-Weisskopf theory² it is assumed that the electron interacts only with that impurity in whose cell it happens to be. This is a rough way of saying that every other impurity is shielded by a similar electron momentarily occupying its own cell. In essence, the Conwell-Weisskopf theory assumes that an electron interacts with that impurity nearest to it.

This same assumption will be made in the present development, but it will be introduced in a somewhat more elegant form through the device of *nearest-neighbor space*.^{4,5} This abstract space also provides a natural framework for the treatment of impurities as sources of an external field.

Nearest neighbor space is constructed in the following manner. Consider a given electron and the ion which is nearer to it than any other. Without changing the vector spatial relationship between the electron and ion, rigidly transport the pair and its immediate crystal environment to a new coordinate system, placing the ion at the origin. Do the same thing for another electron and its nearest neighbor, superposing the ion on the same origin. After having performed the same operation for every electron in, say, a unit volume of semiconductor, we are in possession of a space in which many electrons are juxtaposed, but by definition do not interact with one another, only with the positive charge at the origin. This is the nearest-neighbor space.

In constructing nearest-neighbor space a certain error in counting is committed. Some electrons will have the same nearest-neighbor ion in common and so not all the ions will have been moved. However, the fraction of such cases will be rather small, of the order of a few percent, and even so, under the *assumption* of interaction with the nearest neighbor only, this cannot lead to an error.

It is now necessary to derive an equation of continuity in nearest-neighbor space similar to the Boltzmann transport equation in real space. In doing this, it is necessary to recognize the existence of a distributed source-sink function which has no counterpart in real space. This function is however related to the exchange of nearest neighbors which occurs in real space. This is illustrated in Fig. 1 where an electron (circle) traveling with velocity \mathbf{v} is about to relinquish its nearest neighbor (square) with respect to which it is related by the vector distance \mathbf{r} , and to acquire a new nearest neighbor (square) with respect to which it is at \mathbf{r}' . The dashed line is the midplane between the two ions. In nearest-neighbor space, the two ions are superposed at the origin and an electron merely disappears at the point \mathbf{r} in this space to reappear at \mathbf{r}' . This can happen anywhere and accounts for the distributed source-sink.

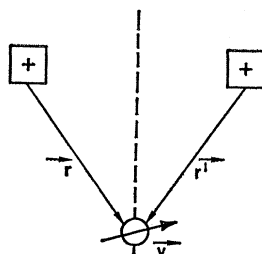


FIG. 1. An electron in the exchange configuration.

The counterpart of Eq. (2.1) in this space will then be

$$\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{f - f_0}{\tau} - \Gamma(\mathbf{r}, \mathbf{v}), \quad (2.2)$$

where $\Gamma(\mathbf{r}, \mathbf{v})$ is the sink term, representing the number of particles lost per unit volume per unit time. The term involving τ is just the standard phonon collision term. The acceleration \mathbf{a} now contains the external field to which must be added the field of the ion at the origin. f is now the distribution function in nearest-neighbor space.

To proceed further, it is necessary to evaluate $\Gamma(\mathbf{r}, \mathbf{v})$ in terms of f . Figure 2 will be of assistance in this connection. It is essentially the same as Fig. 1 except that we elect to center the coordinate system on the electron and view the apparent motion of the ions. In this coordinate system, $\mathbf{R} = -\mathbf{r}$ and $\mathbf{R}' = -\mathbf{r}'$ locate the ions which now move with velocity $\mathbf{V} = -\mathbf{v}$. In place of the density $f(\mathbf{r}, \mathbf{v})$ of electrons in nearest-neighbor space, we can consider the density of ions $F(\mathbf{R}, \mathbf{V})$ such that

$$F(\mathbf{R}, \mathbf{V}) d\mathbf{R} d\mathbf{V} = f(\mathbf{r}, \mathbf{v}) d\mathbf{r} d\mathbf{v}. \quad (2.3)$$

From the definition of θ and θ' in Fig. 2, it is apparent that the electron is leaving the neighbor at \mathbf{R} to assume a new neighbor at \mathbf{R}' , only if

$$\theta' < \theta. \quad (2.4)$$

Otherwise, it is relinquishing a neighbor at \mathbf{R}' . Attention will be confined for the moment to the situation to which Eq. (2.4) applies.

Now the density of the situation shown in Fig. 2 is a triplet density. Under the assumption that the electron interacts only with its nearest neighbor which [when (2.4) holds] is at \mathbf{R} , this triplet density is given by

$$NF(\mathbf{R}, \mathbf{V}), \quad (2.5)$$

where N is the bulk average density of ions (or electrons). This is based on the assumption that the chance of the triplet configuration is given by the product of the chance of the nearest-neighbor pair configuration and the chance of finding an ion at \mathbf{R}' . The presence or absence of the electron has no influence on this latter chance provided it does not interact with the ion. Hence, the chance is measured by the bulk average density.

The exchange rate is now determined by the flow of

⁴ H. Reiss, J. Chem. Phys. **25**, 400 (1956).

⁵ H. Reiss, H. L. Frisch, and J. L. Lebowitz, J. Chem. Phys. **31**, 369 (1959).

ions from \mathbf{R}' toward the electron minus the flow from \mathbf{R} , both flows occurring through the equal elements of spherical surface $d\omega$ (see Fig. 2). The flux density at \mathbf{R} and \mathbf{R}' (velocity \times density) is

$$\mathbf{V}NF(\mathbf{R},\mathbf{V}), \quad (2.6)$$

and the flow through $d\omega$ at \mathbf{R} is

$$-\frac{\mathbf{R}}{r} \cdot \mathbf{V}NF(\mathbf{R},\mathbf{V})d\omega, \quad (2.7)$$

while that at \mathbf{R}' is

$$-\frac{\mathbf{R}'}{r} \cdot \mathbf{V}NF(\mathbf{R},\mathbf{V})d\omega. \quad (2.8)$$

As indicated above, the exchange rate through $d\omega$ is the difference between Eqs. (2.8) and (2.7),

$$\frac{(\mathbf{R}-\mathbf{R}')}{r} \cdot \mathbf{V}NF(\mathbf{R},\mathbf{V})d\omega. \quad (2.9)$$

The total exchange rate at (\mathbf{r},\mathbf{v}) in nearest-neighbor space consists of exchanges at \mathbf{r} effected to new neighbors at all values of \mathbf{R}' consistent with $|\mathbf{R}'|=r$. It is therefore a sum of terms like Eq. (2.9), the integral of Eq. (2.9) over all possible orientations of the surface element $d\omega$. If we choose the z axis of our spherical coordinate system to coincide with \mathbf{v} , then

$$d\omega = r^2 \sin\theta' d\theta' d\varphi', \quad (2.10)$$

and for that part of $\Gamma(\mathbf{r},\mathbf{v})$ consistent with Eq. (2.4) we obtain

$$rNF(\mathbf{R},\mathbf{V}) \int_0^{2\pi} d\varphi' \int_0^\theta \mathbf{V} \cdot (\mathbf{R}-\mathbf{R}') \sin\theta' d\theta' \\ = -2\pi vr^2 N f(\mathbf{r},\mathbf{v}) \left[\frac{1}{2} - \cos\theta + \frac{1}{2}(\cos^2\theta) \right], \quad (2.11)$$

since

$$\begin{aligned} \mathbf{V} \cdot \mathbf{R} &= vr \cos\theta, \\ \mathbf{V} \cdot \mathbf{R}' &= vr \cos\theta', \end{aligned} \quad (2.12)$$

and Eq. (2.3) holds.

When Eq. (2.4) is violated, the density of the configuration in Fig. 2 must be

$$NF(\mathbf{R}',\mathbf{V}), \quad (2.13)$$

rather than Eq. (2.5), since now the ion on the right must be regarded as the nearest neighbor. The contribution to $\Gamma(\mathbf{r},\mathbf{v})$ can still be derived in the manner leading to Eq. (2.11) except that $F(\mathbf{R}',\mathbf{V})$ must now be retained under the integral sign. This leads to the term

$$vr^2 N \int_0^{2\pi} d\varphi' \int_\theta^\pi f(\mathbf{r}',\mathbf{v}) [\cos\theta - \cos\theta'] \sin\theta' d\theta', \quad (2.14)$$

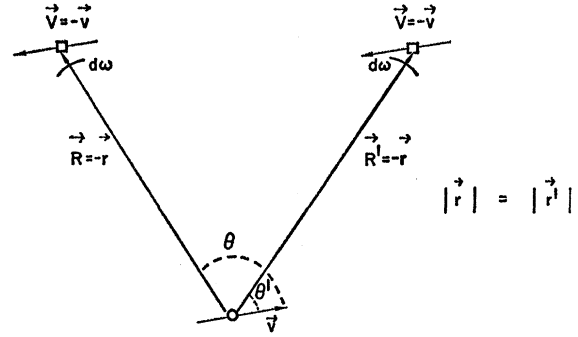


FIG. 2. Diagram to be used in connection with the derivation of $\Gamma(\mathbf{r},\mathbf{v})$. In this figure the electron (circle) is considered stationary and the ions (squares) are regarded as moving with velocity $\mathbf{V} = -\mathbf{v}$.

so that

$$\begin{aligned} \Gamma(\mathbf{r},\mathbf{v}) &= vr^2 N \int_0^{2\pi} d\varphi' \int_\theta^\pi f(\mathbf{r}',\mathbf{v}) \\ &\quad \times [\cos\theta - \cos\theta'] \sin\theta' d\theta' \\ &\quad - 2\pi vr^2 N f(\mathbf{r},\mathbf{v}) \left[\frac{1}{2} - \cos\theta + \frac{1}{2}(\cos^2\theta) \right]. \end{aligned} \quad (2.15)$$

If we deal with the equilibrium situation then $f(\mathbf{r},\mathbf{v})$ is spherical and can be taken outside the integral in Eq. (2.15). For this case

$$\Gamma_0(r,v) = -4\pi vr^2 N f_0(r,v), \quad (2.16)$$

where $f_0(r,v)$ is the equilibrium distribution function in nearest-neighbor space.

Returning to Eq. (2.2) we write

$$\mathbf{a} = -\frac{q}{m} \left(\mathbf{E} - \frac{\partial u}{\partial \mathbf{r}} \right), \quad (2.17)$$

where q is the electronic charge, m the mass of the electron, E the applied field strength, and u the potential due to the ion. At equilibrium $E=0$, and if Eqs. (2.16) and (2.17) are substituted into Eq. (2.2) it is easy to show that the solution of the resulting equation is

$$f_0(r,v) = A \exp(\beta qu - \frac{4}{3}\pi r^3 N) \exp(-\beta mv^2/2), \quad (2.18)$$

where A is a normalizing constant and

$$\beta = 1/kT, \quad (2.19)$$

k being the Boltzmann constant and T the temperature. This can be proved by direct substitution of Eq. (2.18) into the differential equation. Equation (2.18) is just the Hertz distribution,⁶ a well-known nearest-neighbor distribution easily obtained for the equilibrium case by methods not involving the equation of continuity.

⁶ S. Chandrasekhar, Revs. Modern Phys. 15, 1 (1943).

III. CALCULATION OF THE MOBILITY

The combination of Eqs. (2.2), (2.15), and (2.17) yields

$$\begin{aligned} & \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{q}{m} \left\{ \mathbf{E} - \frac{\partial u}{\partial \mathbf{r}} \right\} \cdot \frac{\partial f}{\partial \mathbf{v}} \\ &= \frac{f - f_0}{\tau} - v r^2 N \int_0^{2\pi} d\varphi' \int_0^\pi f(\mathbf{r}', \mathbf{v}) \\ & \quad \times [\cos\theta - \cos\theta'] \sin\theta' d\theta' \\ & \quad + 2\pi v r^2 N f(\mathbf{r}, \mathbf{v}) \left[\frac{1}{2} - \cos\theta + \frac{1}{2}(\cos^2\theta) \right]. \quad (3.1) \end{aligned}$$

Attention will be limited to the case of ohmic conduction so that it is reasonable to attempt to solve Eq. (3.1) by recourse to a power series in E with retention of the linear term. Thus, we write f in the form

$$f(\mathbf{r}, \mathbf{v}) = f_0(r, v) [1 + E\psi(\mathbf{r}, \mathbf{v})]. \quad (3.2)$$

Employing Eq. (2.18) and substituting Eq. (3.2) into Eq. (3.1) yields an equation for ψ by collection of the coefficient of E .

$$\begin{aligned} & \tau \left\{ \mathbf{v} \cdot \frac{\partial \psi}{\partial \mathbf{r}} + \frac{q}{m} \frac{\partial u}{\partial \mathbf{r}} \cdot \frac{\partial \psi}{\partial \mathbf{v}} + q\beta(\mathbf{v} \cdot \hat{e}) \right. \\ & \quad + v r^2 N \int_0^{2\pi} d\varphi' \int_0^\pi \psi(\mathbf{r}', \mathbf{v}) [\cos\theta - \cos\theta'] \sin\theta' d\theta' \\ & \quad \left. - 2\pi v r^2 N \psi \left[\frac{1}{2} + \cos\theta + \frac{1}{2}(\cos^2\theta) \right] \right\} = \psi, \quad (3.3) \end{aligned}$$

where \hat{e} is the unit vector in the direction of \mathbf{E} .

For phonon scattering, the relaxation time τ is velocity dependent.⁷ In fact,

$$\tau = l/v, \quad (3.4)$$

where l is the mean free path for phonon scattering and depends upon temperature but not on velocity. If τ were not velocity dependent, Eq. (3.3) could be solved by a power series in τ , at least for a certain range of τ . However, the radius of convergence of this solution is not yet known. It is evident from Eq. (3.3) that such a solution fails for $\tau \rightarrow \infty$. This corresponds to no-phonon scattering (the pure impurity case) and we shall leave this important aspect of the problem to a later paper where we hope to provide a more complete solution.

For the moment, however, the problem of the velocity dependence of τ must be dealt with. If Eq. (3.4) is substituted into Eq. (3.3) and an attempt is made to solve the resulting equation by a power series in l , no success is achieved. This follows from the fact that the series diverges for $v=0$. One might have expected this in view of the fact that $v=0$ corresponds to $\tau=\infty$. Thus,

⁷ W. Shockley, *Electrons and Holes in Semiconductors* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1950), p. 277.

another method of solution besides the series method is required. To avoid this problem we shall assume that τ is temperature dependent but not velocity dependent by choosing a suitable average τ at each temperature. In essence this will amount to

$$\tau = l/\bar{v}, \quad (3.5)$$

where \bar{v} is a mean thermal velocity.

With the series solution, the case of very small τ (dominant phonon scattering), for which our solution should be valuable, will therefore provide a limiting law, useful at higher temperatures.

By direct substitution of

$$\psi = \psi_0 + \tau\psi_1 + \tau^2\psi_2 + \tau^3\psi_3 + \dots \quad (3.6)$$

into Eq. (3.3) recursion formulas are established for ψ_0 , ψ_1 , ψ_2 , ψ_3 , etc., and it is a matter of algebra and vector notation to show that

$$\psi_0 = 0, \quad (3.7)$$

$$\psi_1 = q\beta(\mathbf{v} \cdot \hat{e}), \quad (3.8)$$

$$\psi_2 = \frac{q^2\beta}{m} (\hat{r} \cdot \hat{e}) \frac{du}{dr}, \quad (3.9)$$

$$\begin{aligned} \psi_3 = \frac{q^2\beta}{m} \left\{ (\hat{r} \cdot \hat{e})(\mathbf{v} \cdot \hat{r}) \frac{d^2u}{dr^2} + (\mathbf{v} \cdot \hat{e}) \frac{1}{r} \frac{du}{dr} \right. \\ \left. - (\mathbf{v} \cdot \hat{r})(\hat{r} \cdot \hat{e}) \frac{1}{r} \frac{du}{dr} - 2\pi v r^2 N (\hat{r} \cdot \hat{e}) \frac{du}{dr} \right. \\ \left. \times \left[\frac{1}{2} + (\hat{v} \cdot \hat{r}) + \frac{1}{2}(\hat{v} \cdot \hat{r})^2 \right] - 2\pi v r^2 N (\hat{v} \cdot \hat{e}) \frac{du}{dr} \right. \\ \left. \times \left[\frac{1}{3} + \frac{1}{2}(\hat{v} \cdot \hat{r}) - \frac{1}{6}(\hat{v} \cdot \hat{r})^2 \right] \right\}, \quad (3.10) \end{aligned}$$

where \hat{r} , \hat{e} , and \hat{v} are unit vectors.

If A in (2.18) is chosen so that

$$\begin{aligned} A = N \int_{\infty}^{\infty} \exp(-\beta m v^2/2) dv \\ \times \int_{\infty}^{\infty} \exp(\beta q u - \frac{4}{3}\pi r^3 N) d\mathbf{r}, \quad (3.11) \end{aligned}$$

where ∞ in the limits indicates that the integrals are to be taken over all of configuration space and all of velocity space, nearest-neighbor space will have been composed out of a unit volume of real space. Then the current will be

$$I = -q \int_{\infty}^{\infty} d\mathbf{v} \int_{\infty}^{\infty} \mathbf{v} f(\mathbf{r}, \mathbf{v}) d\mathbf{r}, \quad (3.12)$$

which upon substitution of Eq. (3.2) yields

$$I = -qE \int_{\infty}^{\infty} d\mathbf{v} \int_{\infty}^{\infty} \mathbf{v} f_0(r, v) \psi(\mathbf{r}, \mathbf{v}) d\mathbf{r}, \quad (3.13)$$

since f_0 itself yields no current. Now Eq. (3.6) together with Eqs. (3.7), (3.8), (3.9), and (3.10) provides an expression for ψ out to order τ^3 . For the moment, the expression is general and not restricted to Coulomb scatterers since it is given in terms of u . It may have applications, for example, to the scattering of electrons by defects in metals. However, it is our intent to examine the Coulomb case.

At this stage it is necessary to examine the consequences of the departure from classical behavior to be expected at small values of r . Obviously, the series of formulas developed above cannot be employed in this region. In fact, if it is assumed that there are no quantum effects and the formulas are applied down to $r=0$, the integrals in Eqs. (3.11) and (3.12) diverge for the Coulomb potential

$$u = q/\kappa r, \quad (3.14)$$

where κ is a dielectric constant. It is the quantum effects which save the situation. In effect the uncertainty principle allows the electron to leak out of the infinitely deep potential well, thus preventing the catastrophe responsible for the divergence in Eq. (3.11).

A possible means of handling the situation is to assume that the classical equations apply down to $r=0$, but that the potential $u(r)$ becomes constant at some point $r=a$, thereby reproducing the effect of the uncertainty principle and averting pile-up of electrons. Only derivatives of u up to the second appear in Eq. (3.10) so that it is only necessary to assure the continuity of u and its first two derivatives at $r=a$. We thus define

$$\begin{aligned} u(r) &= q/\kappa r, & r > a, \\ du/dr &= 0, & u(r) = u(a), & r < a, \end{aligned} \quad (3.15)$$

where u and its first two derivatives are defined to be continuous at $r=a$. That this procedure is permissible can be seen by allowing the transition from pure Coulomb behavior to occur in some finite interval λ about $r=a$, introducing this function into Eq. (3.7) through Eq. (3.12), and then allowing λ to approach zero. The result is the same as when the limit formula Eq. (3.15) is employed directly. In this connection it is important to notice that the second derivative d^2u/dr^2 in Eq. (3.10) can be eliminated immediately upon insertion in Eq. (3.12) through an integration by parts. Thus Eq. (3.15) is sufficient for all boundary conditions.

The physical validity of Eq. (3.15) can only be examined through a careful wave-mechanical analysis. For example, it is conceivable that the most meaningful choice of a will have a depend upon v , since the wavelength of the electron ought to be involved. One might then be forced to choose an average a dependent on temperature. It may even happen that a will depend upon τ , since the time the electron spends in a given state will affect the precision with which its energy and therefore its position in the potential well can be determined. In the absence of a detailed analysis, however,

it is probably sufficient to assume a to be approximately specified by the radius of the first Bohr orbit. We shall follow this practice below.

For reasonably large values of τ to which the usual treatments of combined impurity and lattice scattering address themselves, the electron is not considered to be in thermal equilibrium during "collision" with an impurity. The trajectory of the passage by the impurity is simply computed upon the assumption that the chance of a passage at distance r is proportional to the volume $4\pi r^2 dr$ at r and to nothing else. Thus, there is no singularity as r goes to zero and in fact the contribution of near passes to the total cross section is negligible. Because of this, quantum considerations are not necessary.

On the other hand, for small values of τ such that our perturbation solution is applicable, the electron is quickly thermalized in the presence of the impurity. This means that the Boltzmann distribution must apply in configuration space so that the density of electrons near the ion is very high. As a result, the precise value chosen for a exercises an important influence on the mobility. It will therefore be necessary to have a reliable estimate for a .

Returning to the development and substituting Eq. (3.15) into Eq. (3.7) through Eq. (3.12) in the manner outlined above, we obtain the following result:

$$\begin{aligned} I = -NEq \left(\frac{q\tau}{m} \right) & \left\{ 1 - \frac{4\pi q^4 \tau^2 A}{3mN\kappa^2 kT} \left(\frac{2\pi}{\beta m} \right)^{\frac{3}{2}} \right. \\ & \left. \times \int_a^\infty \frac{1}{r^2} \exp \left(\frac{q^2}{\kappa kT r} - \frac{4}{3} \pi r^3 N \right) dr \right\}. \end{aligned} \quad (3.16)$$

The process leading to Eq. (3.16) is algebraically tedious but straightforward. Most of the definite integrals can be found in tables. A great deal of cancellation occurs. The quantity A can be evaluated by performing the integration in Eq. (3.11) by parts taking due account of Eq. (3.15). The result is

$$\begin{aligned} A = & \frac{N^2 (\beta m / 2\pi)^{\frac{3}{2}}}{\frac{q^2 \beta}{\kappa} \int_a^\infty \frac{1}{r^2} \exp \left(\frac{q^2}{\kappa kT r} - \frac{4}{3} \pi r^3 N \right) dr - \exp(q^2 \beta / \kappa a)} \end{aligned} \quad (3.17)$$

The integrals in both Eqs. (3.16) and (3.17) are rapidly convergent, because of the $1/r^2$ factor, even in the absence of $\exp(-\frac{4}{3}\pi r^3 N)$, the convergence property of nearest-neighbor space. In fact, at the densities with which we shall deal, very little error is committed by ignoring $\exp(-\frac{4}{3}\pi r^3 N)$ in these integrals. Under the latter condition the integrals can be evaluated analytically and Eq. (3.16) with Eq. (3.17) substituted becomes

$$-\frac{I}{NEq} = \mu = \mu_0 \left\{ 1 - \frac{4\pi q^4 \tau^2 N}{3m\kappa} \left[\exp \left(\frac{q^2}{\kappa kT a} \right) - 1 \right] \right\}, \quad (3.18)$$

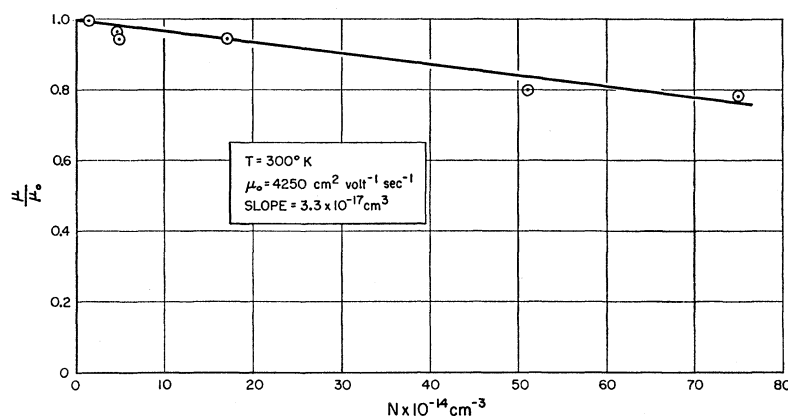


FIG. 3. Plot of μ/μ_0 versus N for arsenic-doped germanium at 300°K. The data points are due to Debye and Conwell.¹⁰ The straight line is not theoretical but is merely the best curve through the points.

where μ is the mobility and

$$\mu_0 = q\tau/m \quad (3.19)$$

is the mobility in the absence of impurity scattering, a standard result.⁸

If a is chosen to be the Bohr radius, then

$$a \approx \kappa h^2 / 4\pi^2 m q^2, \quad (3.20)$$

in which h is Planck's constant. In this expression the appearance of κ in the numerator is a direct result of the application of an effective mass approximation to a hydrogen-like atom embedded in a dielectric. For germanium a reasonable value of m taken from cyclotron resonance measurements⁹ is $0.12m_0$, where m_0 is the normal electron mass. The dielectric constant is 16. Substituting these values into Eq. (3.20) yields

$$a = 7.05 \times 10^{-7} \text{ cm}. \quad (3.21)$$

IV. COMPARISON WITH EXPERIMENT

By substitution of (3.19) into (3.18), one sees that the slope of a plot of μ/μ_0 against N is given by

$$-\frac{4\pi m \mu_0^2}{3\kappa} \left[\exp\left(\frac{q^2}{\kappa k T a}\right) - 1 \right]. \quad (4.1)$$

Of course, this expression must only be valid when τ (or μ_0) is small enough so that the expansion retaining terms of order τ^3 is sufficient. Physically, this ought to be the case when the energy imparted to an electron by an impurity due to the acceleration in one mean free path is less (on the average) than kT . The maximum energy which can be acquired by an electron in this

manner is, of course,

$$q^2/\kappa a. \quad (4.2)$$

For long free paths ($>10^{-5}$ cm) of the kind involved in germanium or silicon we may take Eq. (4.2) as average. Then the criterion for applicability of the expansion is something like

$$q^2/\kappa a k T \ll 1. \quad (4.3)$$

The only data on mobility versus impurity concentration which seem adequate for testing Eq. (4.1) are those of Debye and Conwell¹⁰ on n -type germanium doped with arsenic. Those authors quote data at 78°, 150°, and 300°K and obtain values of μ_0 equal to 1.2×10^7 , 3.8×10^6 , and 1.28×10^6 cm² statvolt⁻¹ sec⁻¹, respectively. Using Eq. (3.21) for a , we find $q^2/\kappa a k T$ for these three temperatures to be 1.92, 1.00, and 0.50, respectively, at 78°, 150°, and 300°K. Thus, only the 300°K case can be used for testing Eq. (4.1).

In Fig. 3, μ/μ_0 for this case has been plotted against N . As expected, a straight line results. The slope is

$$-3.3 \times 10^{-17} \text{ cm}^3. \quad (4.4)$$

This is to be compared with the value calculated from Eq. (4.1), namely

$$-3.04 \times 10^{-17} \text{ cm}^3. \quad (4.5)$$

Agreement is satisfactory.

Comparison with the slopes at 150° and 78°K yields poor agreement; the situation becoming progressively worse the lower the temperature. According to Eq. (4.3) this is to be expected.

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⁸ W. Shockley, *Electrons and Holes in Semiconductors* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1950), p. 198.

⁹ R. N. Dexter, H. J. Zeiger, and Benjamin Lax, *Phys. Rev.* **104**, 637 (1956).

¹⁰ P. P. Debye and E. M. Conwell, *Phys. Rev.* **93**, 693 (1954).