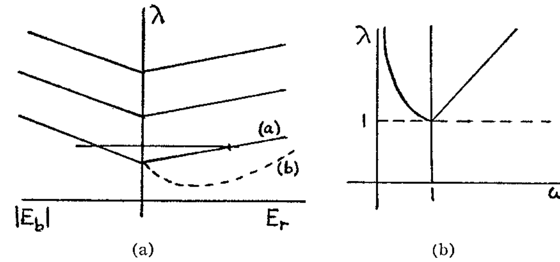


FIG. 2. Resonance and bound state energy as a function of the coupling constant for dynamical resonances. (a) Potential scattering; three bound states have been assumed. (b) Charged scalar meson theory.

for $E\to\infty$ is $\lambda\to\infty$. This gives us the general form of the resonance energy as a function of $\lambda$. If there is a single resonance for fixed angular momentum and fixed $\lambda$, then only curve (a) will appear.

If we increase $\lambda$, for a given potential, there may occur several bound states. One would expect then, for a fixed $\lambda$, $n$ resonances if there are $n$ bound states. For fixed $\lambda$ the phase shift at $E=\infty$ is zero.[6] According to a theorem of Levinson,[7] the phase difference $\delta(0)-\delta(\infty)$ is equal to $n\pi$ if there are $n$ bound states, none of them with binding energy zero; and it is equal to $(n-\frac{1}{2})\pi$ if one of the $n$ bound states has zero binding energy. Therefore the phase difference between two adjacent resonances must be $\pi$ and at $E=0$ the phase shift is discontinuous at the resonance points, as can be seen from Fig. 2.

Exactly the same behavior is found in charged scalar meson theory.[3] The amplitude for the Serber-Lee point-source model of this theory in one-meson approximation can be put in the form

$$e^{i\delta}\sin\delta/k=(\lambda/\omega)/(1-X);\ X=(\lambda/\omega)[1-(1-\omega^2)^{\frac{1}{2}}]. \quad (9)$$

The bound-state curves are given by $\lambda=\omega/[1-(1-\omega^2)^{\frac{1}{2}}]$ and in the physical region ($\omega>1$) the resonance curves are given by $\lambda=\omega$ [Fig. 2(b)].

Thus *the dynamical resonances occur in connection with the bound states* and the resonance energy increases with *increasing* coupling constant. The resonance in the very low-energy neutron-proton scattering is precisely of this type.

## IV

The method of analytic continuation of the coupling constant (i.e., coupling constant as a function of the resonance energy) discussed above has been actually constructed to deal with the solutions of dispersion relations. Given a solution of the dispersion relation, it is not clear whether we are dealing with the true solution or an extra solution. It is important to distinguish between these two types of solutions because one can either eliminate the extra solutions by suitable conditions, or, as the case may be, select the relevant extra solution if the resonance is due to an unstable particle. An important case in this connection is the 33-resonance in pion-nucleon scattering. A study of the analytic continuation of the coupling constant in the dispersion-type equation of Chew and Low[8] for the static $p$-wave meson-nucleon scattering reported separately.[9]

[7] N. Levinson, Kgl. Danske Videnskab. Selskab., Mat.-fys. Medd. 25, No. 9 (1949).

[8] G. F. Chew and F. E. Low, Phys. Rev. 101, 1570 (1956).

[9] A. O. Barut and K. H. Ruei (to be published).

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# Internal State of a Gravitating Gas*

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The significance of a theory of gravitational equilibrium of concentrated masses is discussed in connection with possible general relativistic effects in white dwarf stars. The covariant form of phase space and Liouville's theorem is developed, using the canonical equations for a particle under gravitational and electromagnetic forces. The dynamical isotropy of the ideal fluid is formulated, and the associated equations of state and allowed streaming patterns are found. A covariant kinetic theory yields general relativistic forms for the Maxwell and Fermi distributions in the case of thermal equilibrium, and limits their streaming to rigid motion. Rotating fluids are studied in comoving coordinates, and the problem of determining their gravitational equilibrium is reduced, in most cases of physical interest, to a simple standard form with constant density and vorticity.

## I. INTRODUCTION

THIS paper is one of a series on the theory of great concentrations of gaseous matter capable also of rapid circulation. When a test particle falling freely from infinity towards the center of a resting mass reaches a speed comparable to that of light, the concentration of matter can be called great. When the

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streaming velocity approaches the same speed, the circulation can be called rapid. Thus the absolute limit for all speeds defines the scale of our problem, and sets it within the scope of special relativity.

Whenever internal gravitation acts upon the constituents of a system so that they must move swiftly, effects of general relativity cannot be separated from effects of special relativity. When a single particle can be well described as traversing a periodic orbit in the resultant field of all the others, one can infer from the classical virial theorem that the mean square velocity must be comparable with the mean gravitational potential over the orbit, which, at any point, is twice the square of the speed of free fall from infinity. When that speed is great and the mass concentration is high, the orbital velocity of constituent masses must also be great, and conversely. Examples of this situation are found in the astronomical tests of general relativity where special and general relativistic effects are of the same order.[1]

Where motions are disordered by close collisions involving gravitational or other forces, the virial argument equates the product of pressure and specific volume with the mean gravitational potential. In a homogeneous gas in thermal equilibrium, thermal speeds approach that of light, wherever mass is so concentrated that the gravitational potential requires general relativistic description. Under conditions of Fermi degeneracy, the speed associated with the zero point energy approaches $c$ when general relativistic effects become significant. As first proved by Tolman[2] in certain special systems, regions of lower gravitational potential must have higher (invariant) temperature. (Consider for example, two observers at different potentials attempting to establish thermal equilibrium by exchange of radiation. In order that the color temperature of incoming radiation may match that of outgoing radiation, the effect of the gravitational red shift must be counteracted by a difference of temperature between the two sites.) To this general conclusion we may add a result reached later in this paper, namely that the degeneracy parameter is an invariant that remains constant throughout a gravitating ideal Fermi gas in thermal equilibrium. Thus, though the temperature varies from place to place depending on gravitational potential, the Fermi energy keeps pace with it. As a consequence, both temperature and Fermi energy, together approach the rest energy of a constituent particle where the gravitational potential approaches general relativistic magnitude.

In a system composed of particles of different masses bound to each other by strong forces not of gravitational origin, special and general relativistic effects are not so directly correlated. An extreme example might be a cold, diffuse gas of heavy atoms in which the electrons move swiftly within the atom and thus have a high Fermi energy, while the atoms move slowly in regions of low gravitational potential. The opposite extreme might be illustrated by suddenly removing the electric interactions among all particles. As the system would then progress towards thermal equilibrium the electrons would diffuse to regions of higher gravitational potential, eventually to evaporate into distant field-free regions, leaving behind a homogeneous system of nuclei in which the connection between particle speed and gravitational potential would again be recovered.

The case of greatest practical interest is intermediate between these extremes, namely the accepted model of the interior of a white dwarf star.[3] The electrons are shared by all nuclei in common, and the Fermi pressure of electrons counters the mutual gravitational attraction of nuclei. Nuclei interact with electrons through electric forces that keep the net charge density practically zero, while the high electron mobility maintains thermal equilibrium. The great disparity between electron and nuclear mass may permit the electron speed to rise well into the relativistic range while the gravitational potential remains within Newtonian bounds. Only if the electron kinetic energy could rise to thousands of times the electron rest energy, would the potential be directly affected by non-Newtonian factors. The obvious mass of that kinetic energy, as well as other, less obvious, general relativistic effects which cannot be unambiguously separated from the gravitation of energy, would then come strongly into play. (Well below this point, however, effects of general relativity may strongly influence the stability of the system.)

Since white dwarf stars have the greatest known mass concentrations, they are important in applications of the theory to be presented here. One may well ask whether the general relativistic effects we have outlined are in any way relevant to the conditions at present imagined to occur in the interior of actual stars. In the study of different stellar models based on the balancing of Fermi pressure against gravitational force, it was observed by Chandrasekhar[4] and Landau[5] that, as the number of particles approaches a certain finite critical value the stable equilibrium radius approaches zero, and the central Fermi energy increases without limit. Above that critical mass no stable configuration is possible. Degenerate systems in general show a tendency to contract as mass is added. It is a specific consequence of the special relativistic limitation of all speeds by the speed of light, that stability becomes impossible above a critical mass. The rapid contraction

[1] W. Pauli, *Relativitätstheorie* (B. G. Teubner, Leipzig, 1922), Vol. 5, Part 2, Sec. 58.

[2] R. C. Tolman, Phys. Rev. **35**, 904 (1930).

[3] S. Chandrasekhar, *An Introduction to the Theory of Stellar Structure* (University of Chicago Press, Chicago, Illinois, 1939), Chap. XI, p. 412.

[4] S. Chandrasekhar, Astrophys. J. **74**, 81 (1931).

[5] L. Landau, Physik. Z. Sowjetunion **1**, 285 (1932).

of stable systems sets in when, for the fermions that create the pressure, the average kinetic energy near the center of the star becomes comparable with their rest energy. Relativistic condensation of white dwarfs is observed as a rapid diminution of luminosity with increasing mass in the neighborhood of a critical value comparable with the mass of the sun. The fate of more massive stars that exhaust their nuclear fuel is at present unknown.

At first sight it might appear that the general relativistic effects we seek can occur only in very condensed stars of negligible luminosity. The effects in question tend to enhance gravitational attraction without increasing pressure, however, and therefore provoke instability not only at a finite mass, but also at a finite critical radius. The first example of this unique general relativistic effect, the existence of a critical *radius*, of some 10 km, was deduced for a hypothetical assembly of neutrons by Oppenheimer and Volkoff.[6]

A dimensional comparison of their arguments with those of references 4 and 5 shows that the critical number of nucleons or electrons is essentially $N \approx (hc/GM^2)^{\frac{3}{2}}$, where $M$ is the nucleon mass, while the critical radius is essentially $N^{\frac{1}{3}}(h/p)$, where $p$ is a mean momentum for the fermions of mass $m$ providing the pressure. A closer analysis[7] of the onset of instability shows that $p/mc$ is certainly less than $(M/m)$, and comparable to $(M/m)^{\frac{1}{3}}$. The general relativistic value of the limiting mass for a white dwarf is practically unchanged from that calculated earlier,[4] while the limiting radius obtained by Oppenheimer and Volkoff must be multiplied by a number between $M/m$ and $(M/m)^{\frac{2}{3}}$, with $m$ the electron mass. This takes into account the difference in mass between the fermions providing the pressure in the neutron assembly and those in the white dwarf. The increased limiting radius is roughly that of the earth—not much smaller than radii observed for the smallest white dwarfs.

Whether general relativistic instability can be distinguished from other unstable factors in thermal or chemical equilibrium to which the critical state is sensitive, or whether a minimum radius is astronomically observable, we cannot say. The minimum radius we have deduced is not obviously negligible, however, and may, perhaps, eventually give rise to a new experimental test of general relativity.[7]

Our main purpose in discussing white dwarfs has been to point to physical applications of the general relativistic kinetic theory of gases set forth in following sections of this paper.

To deal consistently with problems of this kind one needs a generally covariant kinetic theory of gases in close analogy to classical theory, but capable of describing partly ordered states as well as states of complete thermal disorder. Steady states not in equilibrium and the approach to thermal equilibrium should be within its scope. This paper presents such a theory based on a special relativistic description of collisions including transmutations and conversion of mass into energy. Temperature is assumed definable over a region large enough to contain a significant number of particles but small enough so that the average gravitational field does not vary greatly over it. The connection between momentum and space densities in distant regions is established by a generally covariant form of Liouville's theorem, in which the average gravitational forces appear explicitly. Though individual particles are partly described by space-time coordinates as though they were points, the particle fields are not considered singular and the local fluctuations in field as one moves from particle to particle have only a small effect compared to the average field built by numerous distant masses. In similar fashion the rapid time variation of fields during the course of occasional close collisions is considered only as a contribution to thermal disorder, while frequent distant collisions build the average field. Accordingly the differential equations of the field contain an inhomogeneous term given by the average energy and momentum density of the particles. This self-consistent or codetermined model of the motion of field and particles, expressed by the generalized Liouville theorem on the one hand and by Einstein's field equations on the other, is fundamental to the theory.

Naturally, certain singular cases in which the notions of temperature and average field are no longer valid cannot be treated in this way. The preceding discussion of white dwarf stars shows, however, that limiting gravitational instability can occur in the absence of any singularity of statistical or of field quantities. Such instability is furthermore not confined to degenerate systems, but occurs quite generally wherever the particles responsible for the pressure countering gravitation approach the speed of light as a result of the action of the gravitational field. For all stable mass concentrations, including the limiting ones, our average field and statistical description seem to be adequate.

We apply our general theory to the study of massive streaming in a homogeneous ideal Boltzmann or Fermi gas in thermal equilibrium. Just as in nonrelativistic kinetic theory and only streaming compatible with equilibrium is a rigid motion, here defined in a covariant way in the sense of Born[8] and Herglotz.[9] The invariant temperature can furthermore, be related to the gravitational potential $g_{44}$ by exactly the relationship discovered by Tolman[2] for static spherically symmetric fields, namely $T(g_{44})^{\frac{1}{2}}=\text{constant}$, provided that $g_{44}$ is measured in a class of comoving coordinates in which all potentials are independent of comoving time. (Of course one can always construct comoving coordinates that reduce a given time-like velocity field to

[6] J. R. Oppenheimer and G. Volkoff, Phys. Rev. **53**, 374 (1939).
[7] A detailed account of the theory of the critical radius of white dwarf stars is being prepared.
[8] M. Born, Ann. Physik **30**, 1 (1909).
[9] G. Herglotz, Ann. Physik **31**, 393 (1910).

rest, but only in the case of covariant rigid motion can one also make the gravitational potentials static. It must be emphasized that such static potentials are in general incompatible with spherical symmetry and are decidedly different from potentials static in inertial coordinates. In general, $g_{4k}\neq 0$, which results in Coriolis forces upon a freely falling test body. A rigid motion can be steady rotation as well as uniform translation.) We will show that unlike the invariant temperature, the invariant degeneracy parameter, or quotient of chemical potential by absolute temperature is constant throughout a streaming ideal Fermi gas in thermal equilibrium in a gravitational field.

In the theory of ordered streaming of a gas developed here, general relativity must be invoked whenever streaming velocities are high. It was predicted long ago that new gravitational forces not envisaged in Newtonian dynamics must couple the momenta of particles in somewhat the same way that electric currents are coupled via magnetic fields, with the interesting difference that parallel momenta repel each other. The gravitational field of a rotating mass ought to act upon another matter current in much the way that a magnet acts upon an electric current.[10] Non-Newtonian forces, though small in astronomical applications developed up to this time, can be directly inferred from the transformation properties of the gravitational potentials which in turn arise from the tensor properties of the energy and momentum that generate the field. (Any theory that ascribes to the current generating gravitation the transformation properties of a tensor of first or higher rank will yield quasi-magnetic forces as a consequence of special relativity alone.)

We escape the complications created by all the different interactions of currents within a streaming gas, by resort to a noninertial system of coordinates comoving with gas at every point. The only current interactions that can survive this transformation are those caused by the counter streaming of all the masses very distant from the system. The non-Newtonian forces that arise from this combine to form the usual Coriolis force introduced by local rotation of comoving coordinates at every point. Thus, although contravariant components of the streaming can be brought to rest, the covariant components survive in the form of $g_{4k}/g_{44}$ which function as components of a vector potential for the space components of the antisymmetric vorticity tensor. These space components of the vorticity tensor cannot be made to vanish by comovement wherever there is any rotation at all. This is part of the tensor property of vorticity and expresses the persistent absolute effects of rotation. The nondiagonal components of energy-momentum tensor cannot be made to vanish; and, in particular, the momentum flux becomes proportional to the vector potential of vorticity. The absolute effects of rotation prevent the system from assuming an equilibrium state of spherical symmetry because of the ineradicability of Coriolis forces caused by local vorticity.

In comoving coordinates, the conservation laws assume a particularly simple form which determines the $g_{4k}$, $g_{44}$ for all time when their spatial distribution is initially known. The field equations for energy and momentum flux then generate further limitations among the gravitational potentials and the pressure and density, corresponding to the classical existence theorem for the centrifugal potential, and for the conservation of angular momentum. In the remaining field equations only the six potentials with spatial indices are effective unknowns.

For certain special problems such as those where pressure and density are functionally related, Synge[11] and Lichnerowicz[12] have proved that a fluid can be organized into a field of vortex filaments of strength constant in time and along their length, just as in classical hydrodynamics. Our device of comoving coordinates not only yields an elementary proof of their results but gives a full account of the further constraints imposed by the field equations, and suggests certain useful generalizations. By exploiting the freedom of coordinate choice still open in comoving systems, it is possible to make the vorticity and circulating mass density equal to a standard unit value, so that different problems are distinguished essentially only by their boundary shapes.

Beyond any formal developments required to bring the field equations into a form comparable with the classical theory of streaming fluids lies the question: Where can one reasonably expect general relativistic effects to be important? When the centrifugal potential, which increases outward from the center of a mass distribution, overtakes the gravitational potential, the assumed streaming pattern becomes unstable. At the limiting boundary of a rotating mass the pressure and density are negligible and the individual particles must then move in nearly Keplerian orbits. The realization of relativistic streaming must therefore be associated with the possibility of correspondingly high speeds in closed orbits, and hence reduces to the problem of obtaining sufficiently high mass concentrations. Since only the most condensed white dwarfs have the necessary property, the first application of the theory will be to improve the description of relativistic degeneracy and to explore the manner in which it is modified by rotational instability.

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[10] J. Lense and H. Thirring, Physik. Z. 19, 156 (1918).

[11] J. L. Synge, Proc. Math. Soc. (London) 43, 376 (1937). Vortex theory in general relativity was independently developed by the authors in the form presented here as a device to reduce the number of dependent variables in the field equations. We are grateful to J. A. Wheeler for pointing out the existence of prior work related to some of our results.

[12] A. Lichnerowicz, Ann. école norm. supér. 58, 285 (1941).

## II. COVARIANT STATISTICAL MECHANICS

A freely falling observer sees his immediate vicinity free of gravitational force. He must therefore be able to construct a local momentum space for passing particles, in which each particle is represented by a point, and a statistical ensemble by a density and current flowing into neighboring space-time regions according to Liouville's theorem. Such a statistical picture ought to be generally covariant, if only it is made Lorentz invariant for the freely falling observer.

To get a Lorentz invariant description, the time coordinate must be adjoined to the dimensions of phase space in order to avoid singling out that particular direction in space-time. An energy dimension must be adjoined to momentum space to preserve the canonical invariance of the theory which underlies Liouville's theorem. Finally, a particle must be represented by a world line rather than a point of the extended phase space. A $\Gamma$ space has been constructed along these lines[13] in a formalism canonically invariant and hence also implicitly generally covariant under the restricted canonical group of space-time coordinate transformations and gauge transformations. Our objective is a $\mu$-space theory substantially equivalent in content, but presenting in explicit form the transformation properties of all quantities of interest under the restricted canonical group.

Our starting point is the equations of motion of a charge $e$ and mass $m$ in a gravitational field described by the metric tensor $g_{\mu\nu}$ and an electromagnetic field described by the field $F_{\mu\nu}=(\partial A_\nu/\partial x^\mu)-(\partial A_\mu/\partial x^\nu)$:

$$(d/ds)(dx_\mu/ds)+\tfrac{1}{2}(\partial g^{\alpha\beta}/dx^\mu)(dx_\alpha/ds)(dx_\beta/ds) = (e/m)F_{\mu\nu}(dx^\nu/ds), \quad (1)$$

where $(dx_\alpha/ds)$ are defined together with $ds$ by

$$g^{\mu\nu}(dx_\mu/ds)(dx_\nu/ds)=g_{\mu\nu}(dx^\mu/ds)(dx^\nu/ds)=1. \quad (2)$$

The quantities set equal to unity in (2) are constants of the motion defined by (1). It has long been known[14] that equations like these can be stated in a form strongly suggestive of Hamilton's canonical equations by introducing a constant mass $M$ and momenta $p_\mu$ defined by

$$dx^\mu/ds=g^{\mu\nu}(p_\nu-eA_\nu)/M. \quad (3)$$

The combination $p_\mu-eA_\mu$ may be denoted $P_\mu$, and Eq. (3) may be substituted into (2) to yield

$$M^2=g^{\mu\nu}(p_\mu-eA_\mu)(p_\nu-eA_\nu)=g^{\mu\nu}P_\mu P_\nu. \quad (4)$$

If we now agree to refer to space-time coordinates only in their contravariant form $x^\mu$, symbolized collectively by $x$, and to momenta only in their covariant form $p_\nu$ symbolized collectively by $p$, we may regard (4) as a unique definition of a function $M(x,p)$ of eight variables such that

$$\begin{aligned}(\partial M/\partial p_\mu)_x&=g^{\mu\nu}(p_\nu-eA_\nu)/M=g^{\mu\nu}P_\nu/M,\\ (\partial M/\partial x^\mu)_p&=(1/2M)(\partial g^{\alpha\beta}/\partial x^\mu)P_\alpha P_\beta\\ &\quad-(e/M)g^{\alpha\beta}P_\alpha(\partial A_\beta/\partial x^\mu).\end{aligned} \quad (5)$$

Substitution of these results into (1) and (3) yields

$$\begin{aligned}&dx^\mu/ds-(\partial M/\partial p_\mu)_x=0,\\ &dp_\nu/ds+(\partial M/\partial x^\nu)_p=(P_\nu/M)(dM/ds),\end{aligned} \quad (6)$$

where, corresponding to the new attitude towards $M$ as a function of $x$ and $p$ defined by (4), we cannot ignore its possible variation over a world line.

The second group of equations in (6) is not canonical in form because of the term in $dM/ds$ which also prevents these equations from leading to any determinate value of $dM/ds$. The constancy of the function $M(x,p)$ must be imposed as a special constraint upon the four momenta with the result that only three can function as independent, dynamical variables. We therefore make a slight but significant change in (6) by striking out the term in $dM/ds$, and asserting instead of (6):

$$\begin{aligned}&dx^\mu/ds-(\partial M/\partial p_\mu)_x=0,\\ &dp_\nu/ds+(\partial M/\partial x^\nu)_p=0.\end{aligned} \quad (7)$$

These equations have exact canonical form, and they determine the change of $M(x,p)$ along a trajectory to be

$$dM/ds=0. \quad (8)$$

Despite the appearance of four independent momenta in (7), the effect of (8) is to make the manifold of trajectories the same as that in (1). The extra degree of freedom is associated with the invariant mass, a constant of the motion. The mass function (4) is therefore a proper Hamiltonian function for canonical equations (7) that do not single out any particular coordinate as the time. Despite their superficial resemblance to ordinary four-vector relations, the canonical equations (7) are not of that kind. They are nevertheless generally covariant in the sense that they are evidently derivable in the same form in every coordinate system and gauge.

The first set of four equations in (7) is gauge invariant. The second set of four equations is not invariant under gauge transformation,

$$\begin{gathered}\bar{A}_\mu=A_\mu+\partial G/\partial x^\mu,\\ x^\mu=\bar{x}^\mu,\quad \bar{p}_\mu=p_\mu+e(\partial G/\partial x^\mu),\quad \bar{M}=M,\\ d\bar{p}_\nu/d\bar{s}=(dp_\nu/ds)+e(\partial^2G/\partial x^\nu\partial x^\alpha)(dx^\alpha/ds),\\ d\bar{x}^\mu/d\bar{s}=dx^\mu/ds,\\ (\partial/\partial\bar{x}^\nu)_{\bar{p}}=(\partial/\partial x^\nu)_p-e(\partial^2G/\partial x^\nu\partial x^\alpha)(\partial/\partial p_\alpha)_x,\\ (\partial/\partial p_\nu)_x=(\partial/\partial\bar{p}_\nu)_{\bar{x}}.\end{gathered} \quad (9)$$

The second set of equations evidently acquires an additive multiple of the first set, so that the entire

[13] P. G. Bergmann, Phys. Rev. **84**, 1026 (1951).

[14] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice Hall, Englewood Cliffs, New Jersey, 1942), p. 97ff. and p. 188ff. In this work, the Hamiltonian form in special relativity is given careful treatment. Our development takes gravitational forces into account according to general relativity.

system of eight quantities in (7) transforms linearly into itself when the gauge is altered.

Under space-time coordinate transformations, the first set of four equations in (7) forms a single contravariant four-vector. The second set is not a space-time tensor of any kind. Under the transformation $x^\mu \to \bar{x}^\mu$, we have $ds = d\bar{s}$, $\bar{M} = M$, and

$$dp_\nu/ds = (d/d\bar{s})(\bar{p}_\alpha \partial\bar{x}^\alpha/dx^\nu) = (\partial\bar{x}^\alpha/dx^\nu)(d\bar{p}_\alpha/d\bar{s}) + (\partial x^\mu/\partial\bar{x}^\beta)(\partial^2\bar{x}^\alpha/\partial x^\mu\partial x^\nu)\bar{p}_\alpha(d\bar{x}^\beta/d\bar{s}). \quad (10)$$

On the other hand, holding constant $\bar{p}_\alpha = (\partial x^\nu/\partial\bar{x}^\alpha)p_\nu$ in forming $x$ derivatives involves an additional $x$-dependent condition:

$$(\partial/\partial x^\nu)_p = (\partial\bar{x}^\alpha/\partial x^\nu)(\partial/\partial\bar{x}^\alpha)_{\bar{p}} - (\partial x^\mu/\partial\bar{x}^\beta)(\partial^2\bar{x}^\alpha/\partial x^\mu\partial x^\nu)\bar{p}_\alpha(\partial/\partial\bar{p}_\beta)_{\bar{x}}. \quad (11)$$

Again the system of eight equations in (7) undergoes linear transformation as a whole.

We therefore regard the quantities in (7) as exemplifying an *eight-vector*, a system of eight quantities $a^\mu$ and $b_\nu$ symbolized by the row matrix $(a,b)$ that transforms according to a linear pattern derived by examination of (9), (10), and (11):

$$(a,b) = (\bar{a},\bar{b})\begin{pmatrix} S & ST \\ 0 & \tilde{S}^{-1}\end{pmatrix}; \quad \tilde{T} = T. \quad (12)$$

Here $S$, $T$ are $4\times 4$ matrices, and $\tilde{S}$ is the transpose of $S$. In this pattern we may interpret $(dx/ds, dp/ds)$ and $(\partial K/\partial p, -\partial K/\partial x)$ (with $K$ any scalar function), as eight-vectors cogredient to $(a,b)$ in (12). For gauge transformations,

$$S = I, \quad T_{\mu\nu} = -e(\partial^2 G/\partial x^\mu\partial x^\nu), \quad (13)$$

while for space-time coordinate transformations,

$$S_\alpha{}^\mu = (\partial x^\mu/\partial\bar{x}^\alpha) = \tilde{S}^\mu{}_\alpha, \quad (\tilde{S}^{-1})^\beta{}_\nu = (\partial\bar{x}^\beta/\partial x^\nu), \quad T_{\mu\nu} = \bar{p}_\alpha(\partial^2\bar{x}^\alpha/\partial x^\mu\partial x^\nu). \quad (14)$$

(In these relations, we adopt the convention of placing the row index nearest to the carrier symbol no matter whether that index is raised or lowered.) Naturally, the transformations of (12) form a group: A transformation indexed by II, following another indexed by I is equivalent to a single transformation defined by

$$S = S_{\mathrm{I}}S_{\mathrm{II}}, \quad T = (S_{\mathrm{II}})^{-1}T_{\mathrm{I}}(\tilde{S}_{\mathrm{II}})^{-1} + T_{\mathrm{II}}.$$

As we have implied by including it in (12), the symmetry of $T$ is a group property: if $T_{\mathrm{I}} = \tilde{T}_{\mathrm{I}}$ and $T_{\mathrm{II}} = \tilde{T}_{\mathrm{II}}$, then $T = \tilde{T}$ also.

The transformations of tensors in space-time depend only on the quantities $S$ in (14). Eight-vector transformations depend also on $T$, and hence involve explicit momentum values and derivatives of $S$. An infinitesimal canonical transformation, with generating function $\delta U$, creates an eight-vector field $(\delta x, \delta p) = (\partial/\partial p, -\partial/\partial x) \times (\delta U)$ if $\delta U$ is an invariant. Consequently, infinitesimal coordinate transformations of space-time $\bar{x}^\mu = x^\mu + \delta x^\mu$ and infinitesimal gauge transformations $\delta G$ are generated by invariant transformation functions $\delta U = p_\mu \delta x^\mu(x)$ and $\delta U = e\delta G(x)$, respectively. These remarks suggest that $(\delta x, \delta p)$, *any* infinitesimal displacement in our eight-dimensional phase space, transforms like an eight-vector. This is not immediately evident only because $p_\nu$ is a four-vector at $x^\mu$ while $p_\nu + \delta p_\nu$ is a four-vector located at $x^\mu + \delta x^\mu$, so that $\delta p_\nu$ has transformation properties characteristic of both points. This situation is relieved as usual by parallel displacement of $p_\nu + \delta p_\nu$ back to $x^\mu$ to form

$$(p_\nu + \delta p_\nu) - \Gamma_{\nu\tau}{}^\sigma(p_\sigma + \delta p_\sigma)\delta x^\tau = p_\nu + (\delta p_\nu - \Gamma_{\nu\tau}{}^\sigma p_\sigma \delta x^\tau),$$

apart from quantities of higher order, with $\Gamma_{\nu\tau}{}^\sigma$ the usual connection coefficients of space-time. The displaced vector is a four-vector at $x^\mu$, and its difference from $p_\nu$ is $\delta p_\nu - \Gamma_{\nu\tau}{}^\sigma p_\sigma \delta x^\tau$, which is also such a four-vector. This reduces to the familiar covariant differential when $p_\nu$ is taken equal to some special vector function of $x$. From the well-known transformation property of connection coefficients

$$\Gamma_{\tau\mu}{}^\sigma = \bar{\Gamma}_{\alpha\beta}{}^\gamma(\partial x^\sigma/\partial\bar{x}^\gamma)(\partial\bar{x}^\beta/\partial x^\tau)(\partial\bar{x}^\alpha/\partial x^\mu) + (\partial x^\sigma/\partial\bar{x}^\alpha)(\partial^2\bar{x}^\alpha/\partial x^\mu\partial x^\tau),$$

and from the four-vector transformation of $\delta p_\nu - \Gamma_{\nu\tau}{}^\sigma p_\sigma \delta x^\tau$, we find

$$\delta p_\nu = (\partial\bar{x}^\alpha/\partial x^\nu)\delta\bar{p}_\alpha + (\partial x^\mu/\partial\bar{x}^\beta)(\partial^2\bar{x}^\alpha/\partial x^\nu\partial x^\mu)\bar{p}_\alpha\delta\bar{x}^\beta,$$

which is cogredient to (10). As for gauge transformations,

$$\delta\bar{p}_\nu = \delta p_\nu + e(\partial^2 G/\partial x^\nu\partial x^\mu)\delta x^\mu,$$

which, by (9), completes the proof that $(\delta x, \delta p)$ is an eight-vector. In this sense, phase space is an eight-vector space. One may partly characterize the eight-vector property by saying that $(a,b)$ is an eight-vector under space-time coordinate transformations if both $a^\mu$ and $b_\nu - \Gamma_{\tau\nu}{}^\sigma a^\tau p_\sigma$ are four-vectors.

The components of an eight-vector $(a,b)$ may be rearranged to form a column matrix $\begin{pmatrix} -\tilde{b} \\ \tilde{a}\end{pmatrix}$ that transforms according to

$$\begin{pmatrix} -\tilde{b} \\ \tilde{a}\end{pmatrix} = \begin{pmatrix} S^{-1} & -T\tilde{S} \\ 0 & \tilde{S}\end{pmatrix}\begin{pmatrix} -b' \\ a'\end{pmatrix} \quad (15)$$

where $a'$ and $b'$ denote the transposed forms of $\bar{a}$ and $\bar{b}$. It is evident that the matrix in (15) is precisely the reciprocal of that in (12). The elements of $\begin{pmatrix} -\tilde{b} \\ \tilde{a}\end{pmatrix}$ may therefore be said to form an eight-vector that transforms in a manner contragredient to $(a,b)$, with the aid of which we may construct invariants by matrix multiplication of a cogredient into a contragredient eight-vector:

$$(a^*, b^*)\begin{pmatrix} -\tilde{b} \\ \tilde{a}\end{pmatrix} = b_\mu{}^* a^\mu - b_\mu a^{\mu *}. \quad (16)$$

The matrix that transforms the columns of $(a,b)$ into the rows of $\begin{pmatrix}-\tilde{b}\\ \tilde{a}\end{pmatrix}$ namely $\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$, must function as a kind of "metric tensor" of phase space. When treated as a contragredient eight-tensor of the second rank, it always transforms into the same numerical matrix independent of $x$ and $p$. The "metric" resembles in this respect the metric of a Euclidean space in Cartesian coordinates transforming under the rotation group. The analogy must be treated carefully, however, for the metric tensor is antisymmetric, and the "length" of every vector vanishes. Consider, for example, the field of directions defined by the world lines, one through every point $(x,p)$. These can be viewed as the field of normals to the surfaces $M(x,p)=\text{constant}$, because the gradient of a scalar field ought to be given by the contragredient eight-vector $\begin{pmatrix}\partial M/\partial x\\ \partial M/\partial p\end{pmatrix}$, which is evidently parallel to the contragredient forms of the eight-vector $(dx/dx,dp/ds)$ tangent to the world line through $(x,p)$. The fact that $dM/ds=0$, i.e., that the normals to $M$ are both perpendicular and parallel to world lines, is an example of the limitations of ordinary geometrical concepts applied to phase space.

In spite of difficulties with the idea of "length," scalar products of different eight-vectors, and the idea of volume density can be usefully applied. For example, the antisymmetric scalar product of two eight-vector gradients

$$(\partial K/\partial p,\ -\partial K/\partial x)\quad\text{and}\quad(\partial L/\partial p,\ -\partial L/\partial x),$$

yields, according to (16), the invariant Poisson bracket of scalars $K$, $L$:

$$[K,L]=(\partial K/\partial x^{\mu})_{p}(\partial L/\partial p_{\mu})_{x}-(\partial K/\partial p_{\mu})_{x}(\partial L/\partial x^{\mu})_{p}. \quad (17)$$

An important special case evaluated by means of the canonical equations (7) is

$$[K,M]=(\partial K/\partial x^{\mu})_{p}dx^{\mu}/ds+(\partial K/\partial p_{\mu})_{x}dp_{\mu}/ds=dK/ds. \quad (18)$$

It must be emphasized that the Poisson bracket does not, in general, preserve the eight-tensor character of quantities entering into it. Consider the important relations $[x^{\mu},K]=(\partial K/\partial p_{\mu})_{x}$ and $[p_{\nu},K]=-(\partial K/\partial x^{\nu})_{p}$, where $K$ and $(\partial K/\partial p,\ -\partial K/\partial x)$ are eight-tensors while $(x,p)$ is not.

The density-in-phase, or distribution function, $N(x,p)$, must measure the invariant number, $dn$, of world lines traversing a differential volume element of phase space. The product of eight coordinate and momentum differentials symbolized by $dxdp$ is an invariant, because $dx\sqrt{(-g)}$ and $dp/\sqrt{(-g)}$ are invariants under space-time coordinate transformation, with $g$ denoting the determinant of the space-time metric tensor. That $dxdp$ is the invariant volume element of phase space is confirmed by the unit value of the determinant of the pseudo-Euclidean metric eight-tensor. Consider now the unique world line passing through $(x,p)$. Let $ds_0$ be the interval length of the segment of that world line included within the space-time volume element $dx$. Then the relation between number of world lines $dn$ and density in phase $N$ must be

$$dnds_0=Ndxdp. \quad (19)$$

This defines phase space density $N$ as an invariant. To interpret $N$ under usual conditions we can perform a sequence of transformations to a coordinate system $\bar{x}^{\mu}$: rotate coordinates at $x$ to make $g_{\mu\nu}$ diagonal there, change coordinate scales to make the diagonal values Minkowskian, and then Lorentz transform so that $d\bar{x}^4=ds_0$, $d\bar{x}^k/ds=0$, and $d\bar{p}_4=(M/p_4)dM$. Then make a gauge transformation to $M=\bar{p}_4$ so that $dn=Nd^3\bar{x}d^3\bar{p}dM$, a product of familiar three-dimensional volume elements taken in the locally Euclidean system comoving with the unique world line through $(x,p)$. The invariant $N$ reduces to the ordinary nonrelativistic density in phase, except for the extra mass dimension which can be suppressed at will by assuming $N$ to be a sum of invariant delta functions in the mass.

The current density of representative points or particles in $\mu$ phase space is the eight-vector $N(dx/ds,dp/ds)$, which must obey the differential conservation rule: The eightfold divergence of the current must vanish. The latter has a unique meaning that rests only on the definition of a scalar product (16) and an invariant volume element $dxdp$ in which the square root of the metric determinant is precisely unity:

$$\mathrm{Div}(a,b)\equiv-(\partial/\partial p,\ -\partial/\partial x)\begin{pmatrix}-\tilde{b}\\ \tilde{a}\end{pmatrix}=\left(\frac{\partial a^{\mu}}{\partial x^{\mu}}\right)_{p}+\left(\frac{\partial b_{\nu}}{\partial p_{\nu}}\right)_{x}. \quad (20)$$

In applying the transformation law (12) to the operator $(\partial/\partial p,\ -\partial/\partial x)$ one might think, at first, that the matrices of transformation coefficients ought to stand to the left of the transformed differential operators as they do in (9) and (11). They may as well stand to the right of the transformed operators, however, in the following way:

$$\left(\frac{\partial}{\partial p},\frac{\partial}{\partial x}\right)=\left(\frac{\partial}{\partial \bar{p}},\frac{\partial}{\partial \bar{x}}\right)\begin{pmatrix}S, & ST\\ 0, & \tilde{S}^{-1}\end{pmatrix}=\left(\frac{\partial}{\partial \bar{p}}S,\ \frac{\partial}{\partial \bar{p}}ST-\frac{\partial}{\partial \bar{x}}\tilde{S}^{-1}\right). \quad (21)$$

Applying the transformed differential operators to the transformation matrices as well as to whatever else may stand on the right, makes no difference: with $S$ and $T$ defined by (14), $\partial/\partial\bar{p}$ applied to $S$ gives zero;

and also

$$[(\partial/\partial\bar{p})ST-(\partial/\partial\bar{x})S^{-1}]_\nu = (\partial/\partial\bar{p}_\alpha)[\bar{p}_\tau(\partial x^\mu/\partial\bar{x}^\alpha)(\partial^2\bar{x}^\tau/\partial x^\mu\partial x^\nu)] - (\partial/\partial\bar{x}^\alpha)(\partial\bar{x}^\alpha/\partial x^\nu)=0.$$

For gauge transformations, with $S$, $T$ defined by (13), we reach the same conclusion. Hence, (21) is justified. It follows immediately from the formal scalar product construction of (20), that $\mathrm{Div}(a,b)$ is an invariant. In fact, the operation of Div applied to any eight-tensor must yield an eight-tensor of rank reduced by one.

One special case of this theorem is the identical vanishing of the invariant eightfold Laplacian of any scalar. Another special case can be expressed with the aid of (17)

$$[K,L]=\mathrm{Div}(K\partial L/\partial p,\ -K\partial L/\partial x). \qquad (22)$$

The conservation law for representative points in eight-space is of this type. With the aid of (7) and (22) it takes the form

$$0=\mathrm{Div}(Ndx/ds,Ndp/ds)=[N,M], \qquad (23)$$

which, by (18), is precisely Liouville's theorem in invariant form,

$$dN/ds=0. \qquad (24)$$

The most directly useful form of Liouville's theorem is obtained by substitution of (7) into (24) and introduction of the gauge-invariant derivative

$$\begin{aligned}(\partial/\partial x^\mu)_P&=(\partial/\partial x^\mu)_{p-eA}\\&=(\partial/\partial x^\mu)_p+e(\partial A_\nu/\partial x^\mu)(\partial/\partial p_\nu)_x,\\0&=P^\mu(\partial N/\partial x^\mu)_P+\tfrac{1}{2}(\partial g_{\mu\nu}/\partial x^\sigma)P^\mu P^\nu(\partial N/\partial P_\sigma)_x\\&\quad-eF_{\mu\nu}P^\mu(\partial N/\partial P_\nu)_x.\end{aligned} \qquad (25)$$

A scalar $K(x,p)$ gives rise to a world-scalar density $\mathfrak{K}(x)$ by integration over all momentum-energy space since $dp/\sqrt{(-g)}$ is an invariant volume element: $\mathfrak{K}(x)=\int K(x,p)dp$. The four-vector density constructed of two scalars $K$, $L$ according to $\mathfrak{K}^\mu=\int K(\partial L/\partial p_\mu)_x dp$ satisfies the interesting relation derived with the aid of (22),

$$\partial\mathfrak{K}^\mu/\partial x^\mu=\int[K,L]dp+\int(\partial/\partial p_\mu)[K(\partial L/\partial x^\mu)_p]dp.$$

With reasonable limitations on the behavior of $K$ and $L$ at large values of $p_\mu$, $\partial\mathfrak{K}^\mu/\partial x^\mu=\int[K,L]dp$. One main application of this theorem will be to the case $L=M$, $K=K(N,M)$, for which, as a consequence of Liouville's theorem (23),

$$(\partial/\partial x^\mu)\int K(N,M)(p^\mu-eA^\mu)dp/M=0. \qquad (26)$$

The mass current density formed from $K=NM$ and the electric current density formed from $K=Ne$ satisfy differential conservation laws as a consequence of Liouville's theorem.

If only gravitational forces are present, it is possible to relate the covariant divergence of the symmetric tensor density,

$$\mathfrak{K}^{\mu\nu\cdots}=\int K(dx^\mu/ds)(dx^\nu/ds)\cdots dp,$$

to the change of $K$ along a trajectory. The covariant divergence is given by

$$\begin{aligned}\partial_\mu\mathfrak{K}^{\mu\nu\sigma\cdots}&+\Gamma_{\mu\alpha}{}^\nu\mathfrak{K}^{\mu\alpha\sigma\cdots}+\cdots\\&=\int(\partial/\partial x^\mu)_p(Kdx^\mu/ds)(dx^\nu/ds)\cdots dp\\&\quad+\int K[(dx^\mu/ds)(\partial/\partial x^\mu)(dx^\nu/ds)\\&\qquad+\Gamma_{\mu\alpha}{}^\nu(dx^\mu/ds)(dx^\alpha/ds)]\\&=\int(dK/ds)(dx^\nu/ds)\cdots dp\\&\quad-\int K(dp_\mu/ds)(\partial/\partial p_\mu)_x[(dx^\nu/ds)\cdots]dp\\&\quad-\int[(\partial/\partial p_\mu)_x(Kdp_\mu/ds)](dx^\nu/ds)\cdots dp,\end{aligned}$$

on applying (22), and the geodesic equation (1) in the form

$$[(dx^\mu/ds)(\partial/\partial x^\mu)_p+(dp_\mu/ds)(\partial/\partial p_\mu)_x](dx^\nu/ds)+\Gamma_{\mu\alpha}{}^\nu(dx^\mu/ds)(dx^\alpha/ds)=0.$$

Since all terms of the equation in $K$ except the first sum to

$$-\int(\partial/\partial p_\mu)_x[K(dp_\mu/ds)(dx^\nu/ds)(dx^\sigma/ds)\cdots]dp,$$

we may transform them into surface integrals over momentum space and thence into zero, leaving

$$K^{\mu\nu\sigma\cdots}{}_{;\mu}=\int\frac{dK}{ds}\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds}\cdots\frac{dp}{\sqrt{(-g)}}, \qquad (27)$$

when only gravitational fields act on the particle. In particular, when $K=K(N,M)$, $K^{\mu\nu\sigma\cdots}{}_{;\mu}=0$ by Liouville's theorem. In the presence of electromagnetic fields, similar theorems cannot be recovered because of the intervention of the field strengths $F_{\mu\nu}$ in the equations of motion. One would have, in fact,

$$K^{\mu\nu\sigma\cdots}{}_{;\mu}=\int\frac{dK}{ds}\frac{dx_\nu}{ds}\frac{dx^\sigma}{ds}\cdots\frac{dp}{\sqrt{(-g)}}-e\int\frac{K}{M}F_\mu{}^\nu\frac{dx^\mu}{ds}\frac{dx^\sigma}{ds}+\cdots\frac{dp}{\sqrt{(-g)}}.$$

In one special case, however, the added terms can be evaluated, i.e., when $K=NM$ and, the tensor is of second rank. Then

$$e\int NF_{\mu}{}^{\nu}(dx^{\mu}/ds)dp/\surd(-g)=F_{\mu}{}^{\nu}j^{\mu}(x), \qquad (28)$$

with $j^{\mu}$ the electric current density four-vector, the whole representing the divergence of the energy-momentum tensor of the electromagnetic field. The quantity

$$T^{\mu\nu}=\int(N/M)(p^{\mu}-eA^{\mu})(p^{\nu}-eA^{\nu})dp/\surd(-g) \qquad (29)$$

is thus the energy-momentum tensor of matter and satisfies the appropriate conservation law as a consequence of Liouville's theorem when the corresponding tensor of the electromagnetic field is adjoined to it.

## III. EQUATIONS OF STATE

We turn now to the self-consistent solution of the field equations for the fields and their generating currents. In a continuous picture of the currents of matter, electricity, etc., it is usual to meet the claims of mechanics by enforcing the conservation rules. In statistical description, those claims are more severe. They are met by the Liouville differential equation for the distribution function $N(x,p)$, rather than by differential equations of conservation which follow from it, and which merely restrict the space-time variation of certain integrals of $N$ over all momentum space.

Some of the limitations that continuous description ignores are essential. Einstein[15] has shown that when the constituent particles are not limited to speeds less than light, a Schwarzschild singularity of the field may occur, and cause the gravitational red shift factor to vanish. An example is the incompressible liquid,[16] which necessarily transmits sound signals with unlimited speed. Other difficulties arise from failing to maintain the implied particle density positive in every region of momentum space as well as in space-time. None of these inconsistencies will be necessarily evident in the continuous picture. Though some particular fluid models that obviously violate relativistic mechanics and basic statistical laws might be rejected at once, others might pass undetected. The only secure basis for an equation of state must be a consistent relativistic statistical mechanics.

The contrast between statistical and fluid descriptions may be illustrated by the familiar "ideal" fluid, for which the tensor of matter is determined from two scalar fields, pressure $\wp$ and density $\rho$, and a time-like vector field $v^{\mu}$ representing the massive streaming at every point:

$$T_{\mu\nu}=\rho v_{\mu}v_{\nu}-\wp(g_{\mu\nu}-v_{\mu}v_{\nu}). \qquad (30)$$

The velocity field is subject to

$$v^{\mu}v_{\mu}=1,$$

and the covariant derivative of that relation

$$v^{\mu}v_{\mu;\nu}=0. \qquad (31)$$

The pressure and density obey conservation rules that take the form $T^{\mu}{}_{\nu;\mu}=0$ in the case of special interest with particles interacting only through gravitation. By constructing components parallel and normal to $v^{\mu}$, those appear in the form of Euler equations

$$v^{\alpha}{}_{;\alpha}=-v^{\alpha}\rho_{,\alpha}/(\wp+\rho), \qquad (32)$$

$$v^{\alpha}v_{\mu;\alpha}=(g_{\mu}{}^{\alpha}-v^{\alpha}v_{\mu})\wp_{,\alpha}/(\wp+\rho). \qquad (33)$$

It is generally assumed that the hypothesis of ideal fluid behavior can be made to represent a large class of phenomena by adjoining to it a more or less plausible relation between pressure and density, termed in this connection, the "equation of state." It is less generally noted that the ideal fluid hypothesis is itself a drastic limitation on the motion of a fluid, for it limits the transport of energy to simple massive streaming, and the transport of momentum to isotropic pressure. Transport processes of much greater generality, are, however, a necessary consequence of any interaction among particles, and can be suppressed, or reduced to this special form, only if the system happens to be in a very special state, either of order or disorder. Such a state might be of the type usually assumed in cosmological discussions. Another example would be complete thermal equilibrium. In any case, assumption of the ideal fluid condition (30) itself conditions the equations of state in a manner not apparent in the continuous picture.

At every point in space and time the velocity field of streaming points out a local time direction and a local energy direction in phase space. In the three dimensional momentum space normal to the energy direction, one can distinguish a momentum magnitude determined essentially by mass and energy, and a spatial momentum direction represented by a latitude and longitude. If the distribution function at a given space-time point is expanded in spherical harmonics of degree $l$ and multiplicity $(2l+1)$ in these angular coordinates, the form of (29) suggests that only harmonics up to the second degree are effective in determining the energy momentum tensor. The form of (30) suggests further that only the zeroth order harmonic is effective in an ideal fluid, which, in turn, points to a kind of local dynamical isotropy.

To bring a spherical harmonic analysis of $N$ into general relativistic form, observe first that at a particular point in space-time, one may align the time-axis with $v^{\mu}$ and then transform the metric into the Lorentz-

[15] A. Einstein, Ann. Math. 40, 922 (1939).
[16] K. Schwarzschild, Berlin. Ber. (1916), p. 424.

Minkowski form of special relativity with a local Euclidean affine space geometry. The momentum variables at that point can then be made orthogonal to the energy in Euclidean normal form. As is well known, spherical harmonics of degree $l$ can be rearranged to form the components of an Euclidean space tensor of rank $l$, symmetric and traceless on all pairs of space-vector indices. This leads to the construction of a spherical harmonic of degree $l$ by means of a world tensor of rank $l$, symmetric and traceless on all index pairs, e.g., $\mathfrak{N}^{\mu_1\mu_2\cdots\mu_l}p_{\mu_r}p_{\mu_s}\cdots p_{\mu_l}$ with

$$g_{\mu_r\mu_s}\mathfrak{N}^{\mu_1\mu_2\cdots\mu_l}=0 \quad (1\leqq r,\, s\leqq l). \tag{34}$$

The coefficients $\mathfrak{N}$ depend only on the energy $v^\mu p_\mu$, the mass $M=(g^{\mu\nu}p_\mu p_\nu)^{\frac{1}{2}}$, and the space-time coordinates. To prevent components of the $\mathfrak{N}$ tensors parallel to $v^\mu$ from generating a spherical harmonics of lower degree we must also maintain the limitation

$$v_{\mu_r}\mathfrak{N}^{\mu_1\mu_2\cdots\mu_l}=0 \quad (1\leqq r\leqq l). \tag{35}$$

By examining the result in its local Euclidean form, it is clear that we have constructed the harmonic analysis of a world scalar according to Eqs. (34), (35) and

$$N(p,x)=\textstyle\sum_l p_{\mu_1}p_{\mu_2}\cdots p_{\mu_l}\mathfrak{N}^{\mu_1\mu_2\cdots\mu_l}(v^\mu p_\mu,M,x). \tag{36}$$

Because of Eq. (35), one can replace $p_\mu$ in (36) by $p_\mu{}^*$, the component of $p_\mu$ "normal" to $v^\mu$ obtained by means of the projection operator $g_\nu{}^{\mu*}$ that already has entered significantly into (30) and (33)

$$g_\nu{}^{\mu*}=g_\nu{}^\mu-v^\mu v_\nu, \quad g_\nu{}^{\mu*}p_\mu\equiv p_\nu{}^*, \tag{37}$$

with

$$g_\alpha{}^{\mu*}g_\nu{}^{\alpha*}=g_\nu{}^{\mu*}, \quad g_{\mu\nu}{}^*v^\nu=0, \quad g_\mu{}^{\mu*}=3.$$

Upon introducing, for brevity components "parallel" to $v^\mu$ defined as

$$p_0=v^\mu p_\mu \quad\text{and}\quad g_{0\nu}{}^*=0, \quad g_{00}=1, \tag{38}$$

Eq. (36) becomes

$$N(p,x)=\textstyle\sum_l p_{\mu_1}{}^*p_{\mu_2}{}^*\cdots p_{\mu_l}{}^*\mathfrak{N}^{\mu_1\mu_2\cdots\mu_l}(p_0,M,x). \tag{39}$$

Integrals over momentum space may now be reduced according to

$$\int d^4p/M(-g)^{\frac{1}{2}}\cdots = \int dM\int_M^\infty dp_0(p_0{}^2-M^2)^{\frac{1}{2}}\int d\Omega\cdots, \tag{40}$$

where the volume element has been separated in local normal coordinates, and the integral over all momenta takes into account that $p^\mu p_\mu{}^*=p_\mu{}^*p^{\mu*}=M^2-p_0{}^2$, and $\int d\Omega$ represents the integral over the element of solid angle in local polar coordinates of momentum space. It is now evident that integration over $d\Omega$ of $N(p,x)\times\mathfrak{L}^{\mu_1\mu_2\cdots\mu_l}(p_0,M,x)p_{\mu_1}p_{\mu_2}\cdots p_{\mu_l}$, where $\mathfrak{L}^{\mu_1\mu_2\cdots\mu_l}$ is a tensor, symmetric, traceless, normal to $v^\mu$ in the sense of (34) and (35), and dependent only on $p_0$, $M$, $x$, will yield zero except for the term of degree $l$ in (36), because of the orthogonality of spherical harmonics of different degree:

$$\int N(p,x)\mathfrak{L}^{\mu_1\mu_2\cdots\mu_l}p_{\mu_1}{}^*p_{\mu_2}{}^*\cdots p_{\mu_l}{}^*d\Omega =4\pi c_l(M^2-p_0{}^2)^l\mathfrak{N}^{\mu_1\mu_2\cdots\mu_l}\mathfrak{L}_{\mu_1\mu_2\cdots\mu_l}, \tag{41}$$

with $c_l$ a number characteristic of the normalization of spherical harmonics in this representation, i.e.,

$$c_0=1, \quad c_1=1/3, \quad c_2=2/15, \quad \text{etc.}$$

A prime application of (41) is the evaluation of the energy momentum tensor of matter defined in (30). Consider the projection of $T_{\mu\nu}$ on two arbitrary directions $K^\mu$, $L^\nu$. The evaluation then proceeds by resolving the coefficients of $p_\mu p_\nu$ in $K^\mu p_\mu L^\nu p_\nu$ into their irreducible parts—symmetric, traceless tensors normal to $v^\mu$:

$$\begin{aligned}K^\mu p_\mu L^\nu p_\nu=&[v_\alpha v_\beta p_0{}^2+\tfrac{1}{3}(M^2-p_0{}^2)g_{\alpha\beta}{}^*]K^\alpha L^\beta\\ &+[(K^{\mu*}L_0+L^{\mu*}K_0)p_0p_\mu{}^*]\\ &+[K^{\mu*}L^{\nu*}-\tfrac{1}{3}g^{\mu\nu*}K^{\lambda*}L_\lambda]p_\mu{}^*p_\nu{}^*.\end{aligned} \tag{42}$$

The three successive lines evidently represent the spherical harmonics of degrees zero, one, and two respectively. Consequently,

$$\begin{aligned}K^\mu T_{\mu\nu}L^\nu =&4\pi\int dM\int dp_0(p_0{}^2-M^2)^{\frac{1}{2}}\\ &\times\{\mathfrak{N}(p_0,M,x)[K_0L_0p_0{}^2+\tfrac{1}{3}(M^2-p_0{}^2)g_{\alpha\beta}{}^*K^\alpha L^\beta]\\ &+\tfrac{1}{3}\mathfrak{N}^\mu(p_0,M,x)(M^2-p_0{}^2)(K_\mu L_0+L_\mu K_0)\\ &+(2/15)\mathfrak{N}^{\mu\nu}(p_0,M,x)(M^2-p_0{}^2)^2K_\mu L_\nu\}.\end{aligned} \tag{43}$$

The disappearance of the star in quantities transvected on $\mathfrak{N}^\mu$ and $\mathfrak{N}^{\mu\nu}$ is explained by the fact that those tensors are by definition normal to $v^\mu$, and hence need no projection operator to select the components normal to $v^\mu$.

The resolution of $T_{\mu\nu}$ in (43), into parts independently determined by $\mathfrak{N}$, $\mathfrak{N}^\mu$, and $\mathfrak{N}^{\mu\nu}$ is nothing more than the division of $T_{\mu\nu}$ into irreducible parts parallel and normal to $v^\mu$ according to the obvious identity,

$$\begin{aligned}T_{\mu\nu}=&v_\mu v_\nu(v^\alpha v^\beta T_{\alpha\beta})+\tfrac{1}{3}g_{\mu\nu}{}^*(g^{\alpha\beta*}T_{\alpha\beta})\\ &+v_\mu(T_{\alpha\beta}g_\nu{}^{\alpha*}v^\beta)+v_\nu(T_{\alpha\beta}g_\mu{}^{\alpha*}v^\beta)\\ &+T_{\alpha\beta}(g_\mu{}^{\alpha*}g_\nu{}^{\beta*}-\tfrac{1}{3}g^{\alpha\beta*}g_{\mu\nu}{}^*).\end{aligned} \tag{44}$$

Comparison with (30) shows that the density and pressure are given by $v^\alpha v^\beta T_{\alpha\beta}=\rho$ and $\wp=-\frac{1}{3}g^{\alpha\beta*}T_{\alpha\beta}$, while $T_{\alpha\beta}g_\mu{}^{\alpha*}v^\beta$ represents nonconvective energy flow $Q_\mu{}^*$ and $T_{\alpha\beta}(g_\mu{}^{\alpha*}g_\nu{}^{\beta*}-\frac{1}{3}g^{\alpha\beta*}g_{\mu\nu}{}^*)=S_{\mu\nu}{}^*$, the shearing stress. Harmonic analysis of the distribution function correlates these significant quantities with purely har-

monic components as follows:

$$\begin{aligned}
\rho&=4\pi\int dM\int dp_0(p_0^2-M^2)^{\frac{1}{2}}p_0^2\mathfrak{N}(p_0,M,x),\\
\wp&=\frac{4\pi}{3}\int dM\int dp_0(p_0^2-M^2)^{\frac{3}{2}}\mathfrak{N}(p_0,M,x),\\
-Q_\mu^*&=\frac{4\pi}{3}\int dM\int dp_0(p_0^2-M^2)^{\frac{3}{2}}p_0\mathfrak{N}_\mu(p_0,M,x),\\
S_{\mu\nu}^*&=\frac{8\pi}{15}\int dM\int dp_0(p_0^2-M^2)^{\frac{5}{2}}\mathfrak{N}_{\mu\nu}(p_0,M,x).
\end{aligned}\tag{45}$$

Together with

$$T_{\mu\nu}=\rho v_\mu v_\nu-\wp g_{\mu\nu}^*+(Q_\mu^* v_\nu+Q_\nu^* v_\mu)+S_{\mu\nu}^*,$$

(45) gives precise expression to the discussion that began this section. The result is valid for all tensors of matter. Of course, if there are interactions among constituent particles other than gravitation, there occur corresponding contributions to the energy-momentum tensor in addition to the tensor of matter. Those may depend implicitly upon $N$ by the way of the field equations for those interactions and the currents generating those fields. Equation (45) would then not tell the whole story. But it will always represent that part of the energy-momentum tensor that arises from the kinetic energy and mass energy of matter, and from gravitational forces.

Only part of the distribution function determines the tensor of matter, namely certain special moments of the harmonics up to second degree, comprising nine functions of $p_0$, $M$, $x$. For the tensor of matter to take the ideal fluid form, the integrals of $\mathfrak{N}^\mu$ and $\mathfrak{N}^{\mu\nu}$ over $p_0$ and $M$ that enter into (45) must vanish. Unfortunately one cannot argue from the vanishing of $Q_\mu^*$ and $S_{\mu\nu}^*$ to the vanishing of $\mathfrak{N}_\mu$ and $\mathfrak{N}_{\mu\nu}$, since the latter, unlike $\mathfrak{N}$, are not necessarily positive. (The positive character of $\mathfrak{N}$ follows from that of $N$ by averaging the latter over all angles.) It is nevertheless instructive to consider the special case $\mathfrak{N}_\mu=\mathfrak{N}_{\mu\nu}=0$ as representative of general ideal fluid conditions.

Consider now a system of particles interacting only by way of gravitation, the distribution function of which is everywhere dynamically isotropic with respect to $v^\mu(x)$, a time-like vector field. In this case, $N$ reduces to $\mathfrak{N}(v^\mu p_\mu,M,x)$ and the energy-momentum tensor is given entirely by (45), and therefore behaves like that of an ideal fluid. We wish to contrast the claims of Liouville's theorem (25), with their partial expression in (32), (33), the conservation laws of energy and momentum. In the absence of electric forces Eq. (25) reduces to

$$p^\mu(\partial N/\partial x^\mu)_{p,M}+\tfrac{1}{2}g_{\mu\nu,\alpha}p^\mu p^\nu(\partial N/\partial p_\alpha)_{x,M}=0,\tag{46}$$

which further becomes for $N=\mathfrak{N}(p_0,M,x)$,

$$p^\mu p^\nu(g_{\nu\alpha}v^\alpha{}_{,\mu}+\tfrac{1}{2}g_{\mu\nu,\alpha}v^\alpha)+p^\mu(\partial\mathfrak{N}/\partial x^\mu)_{p,M}/(\partial\mathfrak{N}/\partial p_0)_{x,M}=0.$$

With the aid of an elementary identity

$$v_{\mu;\nu}+v_{\nu;\mu}=g_{\nu\alpha}v^\alpha{}_{,\mu}+g_{\mu\alpha}v^\alpha{}_{,\nu}+g_{\mu\nu,\alpha}v^\alpha,$$

we obtain

$$\tfrac{1}{2}p^\mu p^\nu(v_{\mu;\nu}+v_{\nu;\mu})-p^\mu\left(\frac{\partial p_0}{\partial x^\mu}\right)_{N,M}=0.\tag{47}$$

This result suggests the separation of (47) into parts dependent on $p_0$ and $p_\mu^*$ alone. The independently variable functions of $p_\mu^*$ then remaining are clearly the spherical harmonics of degrees zero, one and two. Reference to (42) or (44) shows us how to resolve (47) into its spherical harmonic parts:

$$\tfrac{1}{3}(p_0^2-M^2)v^\alpha{}_{;\alpha}+p_0v^\alpha\left(\frac{\partial p_0}{\partial x^\alpha}\right)_{N,M},\tag{48}$$

$$p_0v^\alpha v_{\mu;\alpha}-g_\mu{}^{\alpha*}\left(\frac{\partial p_0}{\partial x^\alpha}\right)_{N,M}=0,\tag{49}$$

$$\tfrac{1}{2}(v_{\alpha;\beta}+v_{\beta;\alpha})(g_\mu{}^{\alpha*}g_\nu{}^{\beta*}-\tfrac{1}{3}g^{\alpha\beta*}g_{\mu\nu}^*)=0.\tag{50}$$

The intensely restrictive effect of Liouville's theorem is now directly evident in (50), a condition upon the velocity field alone, independent of any assumed properties of the distribution function, and having no analog in the conservation rules or in any equation of state. We conclude that, in a consistent statistical mechanical picture, local dynamical isotropy cannot occur unless the velocity field of streaming satisfies an identical relation similar to the Born-Herglotz[8,9] condition of rigid motion, which in our notation would be the vanishing of the normal components of the rate of strain tensor $\frac{1}{2}(v_{\alpha;\beta}+v_{\beta;\alpha})$, i.e., $\frac{1}{2}(v_{\alpha;\beta}+v_{\beta;\alpha})g_\mu{}^{\alpha*}g_\nu{}^{\beta*}=0$. [Our conclusion (50) asserts the absence of any rate of *shearing* strain. It differs from the Born-Herglotz condition in leaving free, rather than annulling, $v^\alpha{}_{;\alpha}$, the divergence of the stream lines.]

Equations (48) and (49) can be combined into a single equation by adding to the second a multiple of $v_\mu$ times the first. The result is

$$p_0\left(\frac{\partial p_0}{\partial x^\mu}\right)_{N,M}=\frac{1}{2}\left(\frac{\partial p_0^2}{\partial x^\mu}\right)_{N,M}=p_0^2(v^\alpha v_{\mu;\alpha})-\tfrac{1}{3}(p_0^2-M^2)v_\mu v^\alpha{}_{;\alpha}.\tag{51}$$

This is a set of equations fully equivalent to (48) and (49), which in fact can be recovered from it by projection normal and parallel to $v_\mu$. By considering $x^\mu$ as independent variables, $\mathfrak{N}$ and $M$ as parameters, and $p_0^2$ as the dependent variable we have a simple set of

linear equations in one unknown function, of the form

$$\partial p_0^2/\partial x^\mu = p_0^2 a_\mu(x) + b_\mu(x) M^2, \qquad a_\mu = 2v^\alpha v_{\mu;\alpha} - \tfrac{2}{3} v_\mu v^\alpha{}_{;\alpha}, \quad b_\mu = \tfrac{2}{3} v_\mu v^\alpha{}_{;\alpha}. \tag{52}$$

Four equations defining the components of a gradient must have evidently vanishing curls in which the gradients defined by (52) may again be substituted to yield

$$(a_{\mu,\nu} - a_{\nu,\mu})(p_0^2/M^2) + (b_{\mu,\nu} - b_{\nu,\mu} + a_\mu b_\nu - a_\nu b_\mu) = 0.$$

These would make $p_0^2/M^2$ entirely independent of $\mathfrak{M}$, and dependent on $x^\mu$ alone, if we did not separately maintain

$$a_{\mu,\nu} - a_{\nu,\mu} = b_{\mu,\nu} - b_{\nu,\mu} + a_\mu b_\nu - a_\nu b_\mu = 0, \tag{53}$$

which, like (50), are again constraints upon $v_\mu$ alone. The first of these is the vanishing of a curl, which implies the existence of a single function of $x^\mu$ that we prefer to name $\ln(1/q^2)$, of which $a_\mu$ must form the gradient:

$$a_\mu = -(2/q)(\partial q/\partial x^\mu) = 2v^\alpha v_{\mu;\alpha} - \tfrac{2}{3} v_\mu v^\alpha{}_{;\alpha}. \tag{54}$$

This we may project parallel and normal to $v_\mu$, to get

$$v^\alpha{}_{;\alpha} = 3(v^\alpha/q)(\partial q/\partial x^\alpha), \tag{55}$$

$$v^\alpha v_{\mu;\alpha} = -(g_\mu{}^{\alpha*}/q)(\partial q/\partial x^\alpha). \tag{56}$$

The last result we may well compare with (32) and (33) where the same accelerations are determined not by a single function $q$ but rather by $\wp$ and $\rho$, two distinct, and possibly quite independent, functions. In our account, the existence of a functional relation between $\wp$ and $\rho$, or equation of state, follows therefore from the same cause—dynamical isotropy—as the general ideal fluid property.

Putting together (51), (55), and (56)—all the restrictions on $v^\mu$ obtained up to this point—we may write them as a single equation:

$$0 = q[v_{\mu;\nu} + v_{\nu;\mu}] + [v_\mu(\partial q/\partial x^\nu) + v_\nu(\partial q/\partial x^\mu)] - \tfrac{1}{2} g_{\mu\nu}[v^\alpha(\partial q/\partial x^\alpha) + q v^\alpha{}_{;\alpha}],$$

from which each may be separately recovered by transvection with $v^\mu v^\nu$, and $v^\mu$. We prefer to express this relation in terms of single time-like vector field $q_\mu = q v_\mu$ of unrestricted magnitude $q_\mu q^\mu = q^2$:

$$q_{\mu;\nu} + q_{\nu;\mu} - \tfrac{1}{2} g_{\mu\nu} q^\alpha{}_{;\alpha} = 0. \tag{57}$$

Since the last equation is identically traceless, it is completely equivalent to

$$q_{\mu;\nu} + q_{\nu;\mu} = 2\lambda(x) g_{\mu\nu}, \tag{58}$$

where $\lambda(x)$ is an arbitrary multiplier. [The value $\lambda = \tfrac{1}{4} q^\alpha{}_{;\alpha} = v^\mu(\partial q/\partial x^\mu)$ follows from the trace of (58).] The resemblance of (58) to the equations of Killing,[17] $q_{\mu;\nu} + q_{\nu;\mu} = 0$, well known in differential geometry, suggests the consideration of a one-parameter family of infinitesimal displacements of coordinate numbers $x^\mu$ with generators $q^\mu(x)$, i.e.,

$$\delta x^\mu = q^\mu(x)\delta t. \tag{59}$$

It follows that the distance between points in the neighborhood of $x^\mu$ will be displaced according to $\delta(dx^\mu) = q^\mu{}_{,\nu} dx^\nu \delta t$; and the corresponding interval will alter according to

$$\delta(ds^2) = (g_{\mu\nu,\alpha} q^\alpha + g_{\mu\alpha} q^\alpha{}_{,\nu} + g_{\nu\alpha} q^\alpha{}_{,\mu}) dx^\mu dx^\nu \delta t = (q_{\mu;\nu} + q_{\nu;\mu}) dx^\mu dx^\nu \delta t.$$

Equation (58) now reduces to

$$\delta(ds^2) = \lambda(x) ds^2 \delta t, \tag{60}$$

with $\lambda(x)$ quite arbitrary. We conclude that the displacements defined in (59) form a group that takes every null interval into another null interval. The streaming originally described by a velocity field $v^\mu$ now appears as the continual convection of local intervals so as to magnify or diminish them without shearing or change of their relative magnitudes. The group of rigid motions defined by the Killing equations, or $\delta(ds^2) = 0$, is a subgroup of the *conformal motions* defined by (60). The view of streaming as the unfolding of a continuous transformation illumines the shape of a space-time able to contain a field of such stream lines: successive three dimensional space-like sections normal to $v^\mu$ must differ essentially only by local magnification.

The condition on $v_\mu$ that remains to be explored—the second set of equations in (53)—limits the variation of magnification throughout space-time. By introduction of $q$ from (54) and of $q^\mu = q v^\mu$, and $\lambda = v^\mu(\partial q/\partial x^\mu) = \tfrac{1}{3} q v^\alpha{}_{;\alpha}$, we obtain

$$(\lambda q_\mu)_{,\nu} - (\lambda q_\nu)_{,\mu} = q\lambda(v_{\mu;\nu} - v_{\nu;\mu}) + v_\mu(q\lambda)_{,\nu} - v_\nu(q\lambda)_{,\mu} = 0.$$

Projection of the second form of this equation parallel and normal to $v^\mu$ leads directly to

$$g_\nu{}^{\mu*}\lambda_{,\mu} = 0 \quad \text{and} \quad \lambda\omega_{\mu\nu} = 0, \tag{61}$$

where

$$\omega_{\mu\nu} = \tfrac{1}{2} g_\mu{}^{\alpha*} g_\nu{}^{\beta*}(v_{\alpha;\beta} - v_{\beta;\alpha}) = \tfrac{1}{2} g_\mu{}^{\alpha*} g_\nu{}^{\beta*}(q_{\alpha;\beta} - q_{\beta;\alpha}), \tag{62}$$

thus defined, is the vorticity of the velocity field, or the normal component of circulation of the fluid, or the rate of rotational strain. From the second equation of (62) we see that either the magnification or the vorticity must vanish. In the former case, $\lambda = 0$, the first equation of (61) is automatically satisfied, and the group reduces to the subgroup of rigid motions. In the latter case, $\omega_{\mu\nu} = 0$, and that irrotational condition must be supplemented by the first equation of (61) which ensures that the magnification be constant throughout

[17] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1926), p. 234. Eq. (70.2). This equation was obtained by the authors [see Phys. Rev. **86**, 621(A) (1952)] as the definition of rigid motion exemplified in the streaming permitted in thermal equilibrium. We are grateful to P. G. Bergmann for suggesting that we could simplify our analysis by making use of the group-theoretical arguments classic in this field.

successive sections of space-time normal to $v^\mu$. The resultant subgroup of irrotational homogeneous magnifications is the one generally considered in cosmological models. The two allowed subgroups are not quite disjunct, for they hold the uniform rigid translations in common. Equations (58) and (61) together express all the restrictions on the velocity field arising from the assumption of local dynamical isotropy.

We turn now to the form of $\mathfrak{N}$ itself, to be determined by integration of (52). We observe that the content of (61) may be given fully equivalent expression in the first form of the equation preceding it, which, in turn, asserts the existence of a single function $f$ such that

$$\partial f/\partial x^\mu = 2\lambda q_\mu = 2qv_\mu v^\alpha(\partial q/\partial x^\alpha).$$

From this we derive, with the aid of (52) and (55), the conclusions

$$v^\alpha(\partial/\partial x^\alpha)(f-q^2)=0 \quad \text{and} \quad b_\mu=(1/q^2)(\partial f/\partial x^\mu). \tag{63}$$

Equations (52) and (54) now lead to

$$\left(\frac{\partial}{\partial x^\mu}\right)_{N,M}(p_0{}^2q^2-fM^2)=0,$$

which yields the final form of $N(x,p)$ as an arbitrary (non-negative) function $F$ of only two composite variables, $M^2=g^{\mu\nu}p_\mu p_\nu$ and $p_0{}^2q^2-fM^2=(q^\alpha p_\alpha)^2-fg^{\mu\nu}p_\mu p_\nu$:

$$N(x,p)=\mathfrak{N}(p_0,x,M)=F(p_0{}^2q^2-fM^2,\ M). \tag{64}$$

The splitting of the group of conformal motions into its two permissible subgroups next to be taken into account, causes the distribution function to become further simplified.

In the case of rigid motion, $\partial f/\partial x^\mu=0=b_\mu$; and $f$ is a constant that can be dropped from (64):

$$N=F(qp_0/M,M),\quad \delta q=\delta(ds^2)=0, \tag{65}$$

where we use $F$ as a generic symbol for an arbitrary non-negative function of its explicitly stated arguments. Comparison with (45) shows

$$\begin{aligned}\rho&=\int_1^\infty(\epsilon^2-1)^{\frac{1}{2}}\epsilon^2F(q\epsilon)d\epsilon,\\ \wp&=\frac{1}{3}\int_1^\infty(\epsilon^2-1)^{\frac{3}{2}}F(q\epsilon)d\epsilon.\end{aligned} \tag{66}$$

The equation of state linking $\wp$ and $\rho$ is explicit here in the dependence of those on $x^\mu$ only through $q(x)$. Even though $F(q\epsilon)$ may be chosen arbitrarily non-negative, an elementary integration by parts transforms $\partial\wp/\partial x^\mu$ into $-(1/q)(\wp+\rho)\partial q/\partial x^\mu$, thus showing that the relation

$$qd\wp+(\wp+\rho)dq=0, \tag{67}$$

necessary, indeed, if $\wp$ and $\rho$ are to meet the claims of conservation expressed by the Euler equations (32) and (33), follows from (66). Furthermore, integration by parts, with due regard for the way in which $F(q\epsilon)$ must vanish for large $\epsilon$ if $\wp$ and $\rho$ are to be definable by (66), shows that

$$q(d\rho/dq)+4\rho=-\int_1^\infty\epsilon^2(\epsilon^2-1)^{-\frac{1}{2}}F(q\epsilon)d\epsilon\leqq 0,$$

$$q(d\wp/dq)+4\wp=-\int_1^\infty(\epsilon^2-1)^{\frac{1}{2}}F(q\epsilon)d\epsilon\leqq 0.$$

By combining these results with (67), we obtain

$$\wp/\rho\leqq\tfrac{1}{3}, \tag{68}$$

$$d\wp/d\rho\leqq(\wp+\rho)/4\rho\leqq\tfrac{1}{3}, \tag{69}$$

with all equalities realized together in the limit that constituent particles approach the speed of light. The inequality (68) expresses the limitation of all speeds by the speed of light, and identifies light as the substance of maximum pressure for a given mass density. The inequality (69) states an upper limit for the compressibility, the least compressible substance being light. Both inequalities arise from the non-negative character of the distribution function. Though plausible and not without precedent, (69) must be considered as a new deduction made possible by the explicit form of equation of state, (66), in our consistent statistical theory.

In a later section we will show that in special comoving coordinates ($v^k=0$) a metric that admits a group of rigid motions can be made entirely independent of the time, while $\wp$ and $\rho$ depend on $x^k$ only through $g_{44}=q^2$. Despite the close analogy that then appears between a system in rigid motion and one of complete spherical symmetry like that investigated by Tolman, there is a decisive difference that prevents the gravitational field equations from enforcing spherical symmetry in a self-contained gravitating system in rigid motion, unless that motion reduces indeed to a simple translation. The difference originates in non-vanishing vorticity, $\omega_{jk}\neq 0$, and results in the unavoidable occurrence of $g_{4k}\neq 0$, and in corresponding departures from spatial spherical symmetry. In that case, Coriolis forces pervade all comoving systems of coordinates, and significantly distinguish statically described mass distributions in rigid rotation from systems at rest or in uniform translation.

In the case of irrotational homogeneous magnification, the conditions on $q$ and $f$ in (64) appear simple only in comoving coordinates. There it turns out that one may make $g_{4k}=0$, $g_{44}=q^2\bar{g}_{ij}$, where $\bar{g}_{ij}$ is a function of $q^2=f+\bar{f}$, where $\bar{f}$ is a function of space coordinates alone. The familiar cosmological case corresponds to $\bar{f}=0$. Uniform rigid translation is given by $f=0$.

We now add electromagnetic fields to the picture of local dynamical isotropy. If we replace $p_\mu$ by $P_\mu=p_\mu-eA_\mu$, then to (46) we must also add the term $-eF_{\mu\nu}P^\mu(\partial N/\partial P_\nu)_{x,M}$, and to (47) we must add $-eF_{\mu\nu}v^\nu P^\mu$. Equation (50) remains unchanged, but

(52), which combines (48) and (49), acquires the additional term $-2eF_{\mu 0}P_0$, on the right. The argument leading to (53) now brings about the conditions:

$$F_{\mu 0,\nu}-F_{\nu 0,\mu}+\tfrac{1}{2}(a_\mu F_{\nu 0}-a_\nu F_{\mu 0})=0,$$

and

$$b_\mu F_{\nu 0}-b_\nu F_{\mu 0}=0.$$

Since Eqs. (54) to (63) follow as before, we may use (54) to derive

$$(qF_{\mu 0})_{,\nu}-(qF_{\nu 0})_{,\mu}=0,$$

or

$$qeF_{\mu 0}=Q_{,\mu}\quad v^\mu Q_{,\mu}=0. \tag{70}$$

We may then use (63) to obtain

$$f_{,\mu}Q_{,\nu}-f_{,\nu}Q_{,\mu}=0. \tag{71}$$

We conclude that $qeF_{\mu\nu}$ is the gradient of a scalar $Q$ that is necessarily functionally dependent on $f$ alone when $f$ is not a constant.

In the important special case of rigid motion, $f$ is, in fact, a constant, and disappears from the problem. The scalar field $Q$ remains to represent the time-independent electrostatic potential energy in the comoving coordinates in which the metric is constant in time. The magnetostatic potential is not limited by these considerations. The analog of (65) in this case is

$$\mathfrak{N}=F[(qP_0+Q)/M,\, M], \tag{72}$$

which reduces exactly to (65) in the special comoving coordinates with $q^4=1$, $q^k=0$. The irrotational solution is only slightly complicated by the presence of electromagnetic fields.

Though we have given special attention to systems with local dynamical isotropy, those are not the only ones of interest. Einstein's model[15] with particles moving in concentric spheres with zero radial velocity and tangential velocities isotropically distributed, and density constant over a sphere but varying arbitrarily among different spheres, is a system with $\mathfrak{N}_\mu=0$ and $\mathfrak{N}_{\mu\nu}$ and $S_{\mu\nu}{}^*\neq 0$. Liouville's theorem applied to this system leads to the determination of tangential momentum and energy as a function of $M$, leaving unspecified the mass in any radial shell. The resultant proportionality of tangential stress $\wp_{\tan}$ and energy density $\rho$ expresses the conservation of energy and momentum in a form very different from (67). The resultant inequality $\wp_{\tan}\leqq\frac{1}{2}\rho$ can be written directly as a relation between $g_{rr}$ and $g_{44}=q^2$, namely

$$q[(\partial g_{rr}/\partial r)+(2g_{rr}/r)]/2g_{rr}(\partial q/\partial r)\geqq 1,$$

which expresses the limitation of all particle speeds to less than the speed of light. In recovering the distinctive conclusions that Einstein drew from a detailed analysis of the trajectories, we confirm the power of Liouville's theorem to disclose the deeper relations among macroscopic quantities that arise from the conformity of their underlying statistical description to relativistic mechanics.

We mention finally another model of special symmetry with $\mathfrak{N}_{\mu\nu}\neq 0$, which has some application to cosmology, namely an isotropic system with vanishing tangential stress. The relation between radial stress $\wp_{\mathrm{rad}}$ and energy density $\rho$ is an equation of state identical with (66) when one replaces $\wp$ by $3\wp_{\mathrm{rad}}$. As a result, the conservation law (67) and the inequalities (68), (69) take very similar form:

$$qd\wp+\tfrac{1}{2}(\wp+3\rho)dq=0,$$

$$\wp\leqq\rho,\quad d\wp/d\rho\leqq\tfrac{3}{4}+\tfrac{1}{4}(\wp/\rho)\leqq 1,$$

the greatest stress and least compressibility again being realized when light is the working substance.

## IV. THERMAL EQUILIBRIUM

In the preceding sections, our statistical method has compounded the precise trajectories of particles moving in the precise combined fields of all other particles. When special symmetry was present, the distribution of trajectories was sufficiently uniform to keep all fields smoothly varying in space and time. In general, however, our description can become complicated far beyond practical needs by the occurrence of fine-grained, but violent, correlated fluctuations of density and field. We turn, therefore, to an approximate statistical theory based on the concept of an average distribution function constrained by the smoothly varying field it generates.

We rarely need to know in detail all the minute fluctuations of density in phase caused by close encounters of particles, nor need we account for the corresponding variations in the field. Instead we imagine the forces divided into a smoothly varying long-range part treated by methods developed up to this point, and a rapidly varying short-range part, the principal effect of which must be to create microscopic disorder, to be treated by the methods of kinetic theory. Among short-range forces we consider not only those of meso-nuclear type, but also that part of gravitation and electromagnetism not already accounted for in the average fields. The separation of short and long-range forces in this sense has never been formulated with absolute sharpness even in the relatively familiar many-body problems—still less in nonlinear gravitational ones. Yet the principle of separation is intuitively persuasive and often correct. We shall therefore adopt one simple form of this hypothesis.

Many proposed equations of state have been the basis for predictions of unusual singular situations, only because they imply or conceal contradictions with the laws of mechanics and probability. Because we are interested in highly concentrated matter, we wish to avoid the *ad hoc* equation of state. Yet we cannot cope with a picture of short-range forces fully realistic in all detail. At this stage, we prefer a model with the power

to represent the emergence of molecular chaos, while conforming to relativistic and statistical mechanics in the manner of Secs. II and III.

We therefore pursue the classic course of Boltzmann, replacing actual short-range forces by discontinuous binary collisions. We renounce, in principle, any prediction of the precise outcome of a collision, confining our account, instead, to a statement of the probability of various final states of mass and motion. We begin with an assumed cross section given as a Lorentz-invariant function of the dynamical variables of colliding particles. Conclusions about thermal equilibrium do not depend on the form of that function but only on its existence. The *ad hoc* character of our assumption will therefore have no effect until we deal explicitly with transport problems in a later paper.

In separating short-range from average long-range effects we tacitly limit the scope of our theory to situations where average effects are never so singular that we cannot find a space-time region near every space-time point, large enough to contain many collisions, yet not so large that the average fields may change appreciably across that domain. We believe that fidelity to relativistic mechanics, the laws of probability, and the nonlinear character of the gravitational field is such a strong deterrent to the formation of singularities, that this will not turn out to be a limitation in practice.

Collisions under the influence of short-range forces can be viewed by a freely falling observer, and can therefore always be made to appear free of effects of the average gravitational field. That is the reason why a Lorentz-invariant formulation of the collision process is sufficient to put the entire theory on a general relativistic footing. We suppose that average electromagnetic effects on collision are included in the assumed cross section as a possible implicit dependence on variables other than the particle momenta. The freely falling observer can see so many collisions in his immediate field-free neighborhood, that he will be able to rediscover the fact that detailed balancing is fully equivalent to thermal equilibrium.

Let the momenta of particles taking part in a standard collision be $p^{\mathrm{I}}\cdots p^{\mathrm{IV}}$, and let there be given a Lorentz-invariant cross section $\sigma(\mathrm{I,II;III,IV})$, a function of these, symmetric in I and II, the initial particles, and in III, IV, the final particles. It is well known that the relative velocity of two particles can be given invariant form:

$$(V_{\mathrm{I,II}})^2=1-(M_{\mathrm{I}}M_{\mathrm{II}}/g^{\mu\nu}p_\mu{}^{\mathrm{I}}p_\nu{}^{\mathrm{II}})^2, \qquad (73)$$

with $(M_{\mathrm{I}})^2=g^{\mu\nu}p_\mu{}^{\mathrm{I}}p_\nu{}^{\mathrm{II}}$, etc., as usual. In the presence of electromagnetic forces $p_\mu{}^{\mathrm{I}}$ must be replaced by $P_\mu{}^{\mathrm{I}}=p_\mu{}^{\mathrm{I}}-e_{\mathrm{I}}A_\mu(x)$. The collision process establishes a direct transfer of representative points between two widely separated locations in the eight-dimensional momentum space for pairs of particles—a sudden disappearange of points at $p^{\mathrm{I}}$, $p^{\mathrm{II}}$ followed by a sudden reappearance at $p^{\mathrm{III}}$, $p^{\mathrm{IV}}$. The process is governed by the four conservation equations

$$p_\mu{}^{\mathrm{I}}+p_\mu{}^{\mathrm{II}}=p_\mu{}^{\mathrm{III}}+p_\mu{}^{\mathrm{IV}}. \qquad (74)$$

Since this transfer is a simplified discontinuous picture of what must be a pair of continuous trajectories viewed in detail, it will obey the laws of mechanics. The Liouville theorem must hold for the process: The invariant volume in phase space associated with the suddenly transferred representative points, must not change.

The collision process can be referred to a twelvefold volume element,

$$dp^{\mathrm{I}}dp^{\mathrm{II}}dp^{\mathrm{III}}dp^{\mathrm{IV}}\delta(p^{\mathrm{I}}+p^{\mathrm{II}}-p^{\mathrm{III}}-p^{\mathrm{IV}})=d^{\mathrm{coll}}p. \qquad (75)$$

In a coordinate system that makes the local metric Minkowskian and sets the particle I at rest, the ordinary phase-space density for particle I is just $N(x,p_{\mathrm{I}})=N_{\mathrm{I}}$, allowing, of course, for the extra mass dimension we have found convenient up to this point. The ordinary phase-space density for particle II is not $N_{\mathrm{II}}$, however, but rather

$$N_{\mathrm{II}}(dx_{\mathrm{II}}{}^4/ds_{\mathrm{II}})=N_{\mathrm{II}}p_{\mathrm{II}}{}^4/M_{\mathrm{II}},$$

to account for the Lorentz contraction of spatial dimensions in the direction of motion of particle I. We recognize this last expression as the invariant $N_{\mathrm{II}}p^{\mathrm{I}}{}_\mu p_{\mathrm{II}}{}^\mu/M_{\mathrm{I}}M_{\mathrm{II}}$. This enables us to write the collision probability for particles I and II in clearly invariant form, namely

$$N_{\mathrm{I}}N_{\mathrm{II}}\sigma(\mathrm{I,II;III,IV})V_{\mathrm{I,II}}p^{\mathrm{I}}{}_\mu p_{\mathrm{II}}{}^\mu/M_{\mathrm{I}}M_{\mathrm{II}} \equiv N_{\mathrm{I}}N_{\mathrm{II}}R(\mathrm{I,II;III,IV}). \qquad (76)$$

$R$ represents the effective volume swept out in space-time by the cross section moving at the relative velocity, and corrected for effects of special relativity on the measurement of density. The differential number of trajectories traversing $dxdp^{\mathrm{I}}$ is continually diminished by collisions with initial momenta close to $p^{\mathrm{I}}$, $p^{\mathrm{II}}$ at the rate

$$-N_{\mathrm{I}}N_{\mathrm{II}}R(\mathrm{I,II;III,IV})d^{\mathrm{coll}}p.$$

It is continually replenished by collisions with initial momenta close to $p^{\mathrm{III}}$, $p^{\mathrm{IV}}$ at the rate

$$+N_{\mathrm{III}}N_{\mathrm{IV}}R(\mathrm{III,IV;I,II})d^{\mathrm{coll}}p.$$

The resultant rate of transfer of representative points into the volume element $dp^{\mathrm{I}}$ by the action of binary collisions is therefore

$$(dN_{\mathrm{I}}/ds_{\mathrm{I}})_{\mathrm{collision}}=\int\int\int dp^{\mathrm{II}}dp^{\mathrm{III}}dp^{\mathrm{IV}}\delta(p^{\mathrm{I}}+p^{\mathrm{II}}-p^{\mathrm{III}}-p^{\mathrm{IV}}) \times[N_{\mathrm{III}}N_{\mathrm{IV}}R(\mathrm{III,IV;I,II})-N_{\mathrm{I}}N_{\mathrm{II}}R(\mathrm{I,II;III,IV})]. \qquad (77)$$

This is essentially the Boltzmann equation.

Consider with its help, the change of a four-vector density $K^\mu$ associated with a scalar $K(N,M)$ in the manner of (26):

$$(\partial K^\mu/\partial x^\mu)=\int(\partial K/\partial N)_{\mathrm{I}}(dM_{\mathrm{I}}/ds)_{\mathrm{collision}}dp^{\mathrm{I}}$$

$$=\frac{1}{2}\int\int\int\int d^{\mathrm{coll}}p(\partial K/\partial N)_{\mathrm{I}}[M_{\mathrm{III}}M_{\mathrm{IV}}R(\mathrm{III,IV;I,II})-M_{\mathrm{I}}M_{\mathrm{II}}R(\mathrm{I,II;III,IV})] \tag{78}$$

$$=\frac{1}{4}\int\int\int\int d^{\mathrm{coll}}p[(\partial K/\partial N)_{\mathrm{I}}+(\partial K/\partial N)_{\mathrm{II}}-(\partial K/\partial N)_{\mathrm{III}}-(\partial K/\partial N)_{\mathrm{IV}}]N_{\mathrm{III}}N_{\mathrm{IV}}R(\mathrm{III,IV;I,II}), \tag{79}$$

where we have exploited the symmetry of $d^{\mathrm{coll}}p$ in I, II in III, IV and in the interchange of the pairs I, II and III, IV. By choosing $K=N$, we verify that the total number of representative trajectories cannot change in collision. By explicit introduction of charge conservation in collision according to

$$e_{\mathrm{I}}+e_{\mathrm{II}}=e_{\mathrm{III}}+e_{\mathrm{IV}}, \tag{80}$$

we can obtain the differential law $(\partial/\partial x^\mu)[(-g)^{\frac{1}{2}}j^\mu]=0$. Reference to (28) shows how the formulation of conservation of charge and momentum in a collision according to (80) and (74) applied to the Boltzmann equation (77), leads to the differential conservation laws for momentum density.

Our Lorentz-invariant form of the Boltzmann equation is sufficient to establish the general relativistic picture of thermal equilibrium. A necessary intermediate step is the Lorentz-invariant definition of entropy. That practically requires a sharp distinction between like and unlike particles, which is difficult to achieve when the mass is represented by a continuous variable as heretofore. Merely writing $N=\sum_A\delta(M-m_A)\times N_A(x,p)$ and $R(\mathrm{I,II;III,IV})=\sum_{A,B}\delta(M_{\mathrm{III}}-m_A)\delta\times(M_{\mathrm{IV}}-m_B)R_{AB}(\mathrm{I,II;III,IV})$ effects the necessary adjustment. To avoid complications of notation that are not necessary for our purpose, however, we shall not try to carry through the argument in full generality. We prefer to limit detailed discussion to the simple situation of collisions among particles of the same mass, or between particles of two different masses, assuming that the masses concerned are the same before and after collision. Collisions of more general type, though interesting in other connections, are not essential to thermal equilibrium.

In simple collisions involving at most two distinct masses $m_1$, $m_2$ the volume of phase space concerned is limited by conditions symmetric under interchange of the index pairs I, II and III, IV:

$$M_{\mathrm{I}}+M_{\mathrm{II}}=M_{\mathrm{III}}+M_{\mathrm{IV}}=m_1+m_2,$$
$$M_{\mathrm{I}}M_{\mathrm{II}}=M_{\mathrm{III}}M_{\mathrm{IV}}=m_1m_2,$$

and consequences of these,

$$p^{\mathrm{I}}{}_\mu p_{\mathrm{II}}{}^\mu=p^{\mathrm{III}}{}_\mu p_{\mathrm{IV}}{}^\mu,\quad V_{\mathrm{I,II}}=V_{\mathrm{III,IV}}.$$

In this case, any assumed symmetry of the cross section $\sigma(\mathrm{I,II;III,IV})$ for interchange of index pairs, i.e., microscopic reversibility, will be shared by $R(\mathrm{I,II;III,IV})$. Let us first suppose that

$$N=\mathfrak{N}\delta(M-m)\quad\text{and}\quad R(\mathrm{I,II;III,IV})=\mathfrak{R}(\mathrm{I,II;III,IV})\delta(M_{\mathrm{III}}-m)\delta(M_{\mathrm{IV}}-m).$$

The definition of entropy current density for particles of a single kind obedient to Boltzmann statistics is evidently

$$\mathfrak{S}_\mu=-\int dp(P_\mu/M)\mathfrak{N}\delta(M-m)\ln(\mathfrak{N}h^3). \tag{81}$$

(By taking entropy dimensionless, we identify temperature as energy and obviate the occurrence of Boltzmann's constant.) According to (26) we conclude that

$$(\partial\mathfrak{S}^\mu/\partial x^\mu)_{\mathrm{continuous}}=0 \tag{82}$$

for the continuous streaming in phase space caused by the average fields. For collision processes we may apply (79) with $\partial K/\partial N=-1-\ln(\mathfrak{N}h^3)$, to obtain

$$\partial\mathfrak{S}^\mu/\partial x^\mu=(\partial\mathfrak{S}^\mu/\partial x^\mu)_{\mathrm{collision}}$$
$$=\frac{1}{4}\int\int\int\int d^{\mathrm{coll}}p\ln(\mathfrak{N}_{\mathrm{III}}\mathfrak{N}_{\mathrm{IV}}/\mathfrak{N}_{\mathrm{I}}\mathfrak{N}_{\mathrm{II}})\times N_{\mathrm{III}}N_{\mathrm{IV}}R(\mathrm{III,IV;I,II}). \tag{83}$$

Either by assuming microscopic reversibility, or by following the steps of Boltzmann's demonstration that collisions must form eventually closed cycles through which representative points circulate without accumulation, we reach the conclusion that

$$\partial\mathfrak{S}^\mu/\partial x^\mu\geqq 0. \tag{84}$$

Equality can only hold when there is detailed balancing, i.e.,

$$\mathfrak{N}_{\mathrm{I}}\mathfrak{N}_{\mathrm{II}}=\mathfrak{N}_{\mathrm{III}}\mathfrak{N}_{\mathrm{IV}}. \tag{85}$$

Similar arguments applied to (77) yield for the condition of maximum entropy

$$(dN_{\mathrm{II}}/ds)_{\mathrm{collision}}=0, \tag{86}$$

from which we conclude that maximum entropy means thermal equilibrium.

Detailed balancing, (85), tells us that in thermal equilibrium, $\ln(\mathfrak{N}h^3)$ can only be realized as a linear function of the additive integrals of the collision process with coefficients dependent on $x$ alone. In the absence of electromagnetic fields,

$$\ln(1/\mathfrak{N}h^3)=Q(x)+q^\mu(x)p_\mu. \tag{87}$$

Thus, by steps exactly parallel to those of classical kinetic theory, we are led to the Lorentz invariant generalization of the Maxwell-Boltzmann distribution. The vector field $q^\mu(x)$, represented as $q(x)v^\mu(x)$ with $v^\mu v_\mu=1$, describes massive streaming with velocity field $v_\mu$ and invariant absolute temperature $1/q(x)$, while $Q(x)/q(x)$ is the Gibbs free energy. (If we consider a case with no streaming, $q^k=0$, where particle energies are mostly nonrelativistic, and where gravity is negligible, i.e., $g_{44}\simeq 1$, $p_4\simeq M+\frac{1}{2}M(\dot{x}^4)^2$, then $1/q(x)$ obviously reduces to the usual absolute temperature. In general, $1/q(x)$, is what we would get by collecting a sample of gas in an insulated box moving with the average streaming velocity in its vicinity, removing the sample slowly to a distant field-free laboratory, and then measuring the absolute temperature.)

Conclusions of importance follow from applying the Boltzmann equation (86) to the case of thermal equilibrium. Since

$$(dN/ds)_{\text{collision}}+(dN/ds)_{\text{continuous}}=0$$

is the proper statement of Liouville's theorem in the presence of collisions, and since in thermal equilibrium the collision contribution vanishes, we are left with $(dN/ds)_{\text{continuous}}=0$. This is identical with the earlier form of Liouville's theorem (25) derived there in the absence of any approximately described short-range forces or collisions.

We may therefore apply the methods of Sec. III directly to the case of thermal equilibrium. Up to this point we have been dealing with local phenomena described in Lorentz-invariant fashion by a special, freely falling observer. With the disappearance of any explicit effect of collision processes from the thermal equilibrium form of Liouville's theorem, any reference to special coordinates or special forms of general covariance disappears. The equation $(dN/ds)_{\text{continuous}}=0$ belongs to the domain of general relativity. It describes the curvature of trajectories and of the space-time in which they lie; and its form is generally covariant.

The Maxwell-Boltzmann distribution now displays the property of local dynamical isotropy taken up in Sec. III. Equation (87) is one of the class of distributions described by (72), from which we conclude that the only massive streaming compatible with thermal equilibrium is a rigid motion in the sense of (65). This is the relativistic generalization of the well-known classical theorem that is often stated in the same words.[18] For the sake of clarity, we dissect this property out of the many others developed in III. (The quantities $q$ and $q^\mu=qv^\mu$ have been defined in this section so as to conform to their usage in Sec. III.)

$$q_{\mu;\nu}+q_{\nu;\mu}=0, \tag{88}$$

or, the fully equivalent equations

$$g_\mu{}^{\alpha*}(v_{\alpha;\beta}+v_{\beta;\alpha})g_\nu{}^{\beta*}=0, \tag{89}$$

$$v^\nu v_{\mu;\nu}+(\partial\ln q/\partial x^\mu)=0, \tag{90}$$

together with their necessary consequences

$$v^\mu{}_{;\mu}=v^\mu(\partial q/\partial x^\mu)=0. \tag{91}$$

The temperature field $(1/q)$ is entirely determined according to (90) by the four-vector streaming field, in such a way that the stream lines lie on isothermal surfaces. The entire system in thermal equilibrium is not, however, isothermal. Even in the absence of streaming in a given neighborhood, where $v^k=0$, $v^\mu v_\mu=g_{44}(v^4)^2=1$, (90) requires $g_{44,k}/2g_{44}\simeq\partial\ln q/\partial x^k$, or, (temperature) $=q^{-1}\propto(g_{44})^{-\frac{1}{2}}$, there. The acceleration of streaming along a stream-line is responsive not only to the force of gravitation which enters by way of the covariant derivative in (90), but also to the gradient of the temperature. Unlike the temperature, the quantity $Q$, or Gibbs free energy divided by temperature, is a constant over the whole system, at least in the absence of electromagnetic forces, for which we deduce from (65) or (70) that $\partial Q/\partial x=0$.

Electromagnetic forces affect thermal equilibrium when particles of different charge and mass are present. Consider a simple collision between particles of masses and charges $m_{\text{I}}$, $e_{\text{I}}$ and $m_{\text{II}}$, $e_{\text{II}}$ in which masses and charges remain unchanged. We are led to a result similar to (87) but with an extra term corresponding to the charge which functions, according to (80), as an independent additive integral of the collision process. In order to preserve the explicit gauge invariance of the distribution function, we write the new form of (87) as follows:

$$\ln(1/\mathfrak{N}_{\text{I}}h^3)=Q_0(x)+e_{\text{I}}Q_{\text{I}}(x)+q^\mu(p_\mu-e_{\text{I}}A_\mu). \tag{92}$$

The massive streaming and the invariant temperature, together described by $q^\mu(x)$, are the same for all particles at a given point no matter what their charge or mass, and are governed by Eqs. (88) through (91) just as in the absence of electromagnetic forces. Reference to (70), valid in the case of rigid motion that here obtains, shows that

$$\partial Q_0/\partial x^\mu=0,\quad \partial Q(x)/\partial x^\mu=F_{\mu\nu}q^\nu. \tag{93}$$

We have dropped the index from $Q(x)$ because it is the same field for all species of particle, and is dependent only upon space-time position. Looking ahead to the representation in special comoving coordinates with $q^k=0$, $q^4=1$, $\partial g_{\mu\nu}/\partial x^4=0$ we can see that $\partial Q/\partial x^4=0$, $\partial Q/\partial x^k=\partial A_4/\partial x^k$ so that $Q$ differs by an additive con-

[18] R. H. Fowler, *Statistical Mechanics* (Cambridge University Press, New York, 1929), 1st ed., p. 430, Secs. 17, 33.

stant from $A_4$, the static electric potential. Just as in classical nonrelativistic kinetic theory, the only electromagnetic fields compatible with thermal equilibrium are derivable from time-constant potentials. In these very special coordinates, the terms of (92), $e_I Q_I$ and $-e_I q^\mu A_\mu$, cancel in so far as their space-dependent parts are concerned, leaving

$$\ln(1/\mathfrak{N}_I h^3) = (\text{constant})_I + p_4.$$

Despite its specious independence of temperature $q^{-1}$, and gravitational and electromagnetic forces, this all too simple form of the Maxwell-Boltzmann distribution function actually depends on all of these factors by way of

$$m_I^2 = g^{\mu\nu}(p_\mu - e_I A_\mu)(p_\nu - e_I A_\nu),$$

and

$$q^2 = g_{\mu\nu} q^\mu q^\nu = g_{44}.$$

Because relativistic concentrations of matter are found experimentally together with Fermi degeneracy, we need to formulate the effect of the exclusion principle on our Lorentz invariant model of binary collisions. In nonrelativistic statistical theory the effect is well known: one must count both "holes" and particles, as contributing to the entropy of the system. This means replacing $(\mathfrak{N}h^3)\ln(\mathfrak{N}h^3)$ in the entropy current of (81) by

$$(\mathfrak{N}h^3)\ln(\mathfrak{N}h^3) + (1-\mathfrak{N}h^3)\ln(1-\mathfrak{N}h^3).$$

One must also compute the collision probability of (77) in proportion to the densities of holes in the final state as well as the densities of particles in the initial states. One must therefore replace a term like

$$(\mathfrak{N}_{III}h^3)(\mathfrak{N}_{IV}h^3)R(\text{III,IV};\text{I,II})$$

by

$$(\mathfrak{N}_{III}h^3)(\mathfrak{N}_{IV}h^3)(1-\mathfrak{N}_I h^3)(1-\mathfrak{N}_{II}h^3) \times (p^I{}_\mu p_{II}{}^\mu / M_I M_{II}) R(\text{III,IV};\text{I,II}) \equiv \nu_{III}\nu_{IV}\bar{R}(\text{III,IV};\text{I,II}),$$

with $(1/\nu) \equiv (1/\mathfrak{N}h^3) - 1$. As a result, the statement of (79) must be altered to read:

$$(\partial K^\mu/\partial x^\mu) = \frac{1}{4}\int\int\int\int d^{\text{coll}}p[(\partial K/\partial N)_I + (\partial K/\partial N)_{II} - (\partial K/\partial N)_{III} - (\partial K/\partial N)_{IV}] \times \nu_{III}\nu_{IV}\delta(M_{III}-m)\delta(M_{IV}-m)\bar{R}(\text{III,IV};\text{I,II}). \quad (94)$$

For the entropy, in particular,

$$\partial K/\partial N = \ln\nu,$$

and

$$(\partial \mathfrak{S}^\mu/\partial x^\mu) = \frac{1}{4}\int\int\int\int d^{\text{coll}}p\,\nu_{III}\nu_{IV} \times \ln(\nu_{III}\nu_{IV}/\nu_I\nu_{II})\delta(M_{III}-m)\delta(M_{IV}-m) \times \bar{R}(\text{III,IV};\text{I,II}).$$

Again, the assumption of microscopic reversibility, which renders $\bar{R}$ symmetrical in the index pairs III, IV and I, II, or Boltzmann's collision-cycle argument, permits us to conclude that $\partial\mathfrak{S}^\mu/\partial x^\mu \geqq 0$, with the inequality valid when and only when

$$\nu_I\nu_{II} = \nu_{III}\nu_{IV}, \quad (dN_I/ds)_{\text{collision}} = 0,$$

and

$$-\ln\nu_I = \ln[(1/\mathfrak{N}_I h^3) - 1] = Q_0 + e_I Q + q^\mu(p_\mu - e_I A_\mu). \quad (95)$$

The equilibrium distribution here is related to that derived for Boltzmann statistics in (92) exactly as the Fermi and Maxwell distributions are related in nonrelativistic theory. The reduction of Liouville's theorem to the form $(d\mathfrak{N}_I/ds)_{\text{continuous}} = 0$ again leads to Eqs. (88) through (93) exactly as in the case of classical statistics. In particular, the quantity $Q_0$, which functions as an invariant degeneracy parameter is not affected by the gravitational red-shift factor like the invariant temperature, but instead remains constant throughout the system.

## V. ROTATING FLUIDS

In the last two sections, the streaming of a fluid has appeared as part of the description of internal state inseparably connected with thermodynamic conditions and with the distribution of fields of force. Even in the classical treatment of thermal equilibrium an ideal gas is constrained to move like a rigid body in a conservative field. The identification of inertia and gravitation in general relativity naturally extends these connections, and relates them directly to the metric of space and time. Despite the fact that only first derivatives of the metric tensor appear to enter considerations of this kind, the curvature of space-time is deeply involved. One effect on the curvature results from the self-consistent or codetermined nature of the field equations. But even more directly, the curvature of the stream lines limits the shape of the world in which they must lie embedded.

Consider for example, the case of rigid motion, where space-time necessarily admits a vector field $q_\mu$, such that $q_{\mu;\nu} + q_{\nu;\mu} = 0$. The second covariant derivatives of $q_\mu$ are therefore subject to two identities

$$q_{\mu;\nu;\sigma} + q_{\mu;\sigma;\nu} = 0, \quad (96)$$

$$q_{\mu;\nu;\sigma} - q_{\mu;\sigma;\nu} = R^\alpha{}_{\mu\sigma\nu} q_\alpha. \quad (97)$$

By transposing three times alternately the members of the first pair, and the members of the second pair of indices in $q_{\sigma;\mu;\nu}$ and then applying the well-known cyclic identity for the curvature components $R^\alpha{}_{\mu\sigma\nu}$, one can

obtain

$$q_{\sigma;\mu;\nu}=R^{\alpha}{}_{\nu\sigma\mu}q_{\alpha}, \tag{98}$$

and its contraction,

$$g^{\mu\nu}q_{\sigma;\mu;\nu}=R_{\sigma}{}^{\mu}q_{\mu}. \tag{99}$$

In effect, $q_\mu$ is restricted to multiple periodicity in space-time coordinates, the periods being the corresponding radii of curvature multiplied by $(2\pi)$. Since the conservation laws and the condition of rigid motion determine the variation of $q_\mu$, space-time must have precisely that curvature to make the required variation of $q_\mu$ appear periodic.

The effect of codetermination of the gravitational field can be tested in the case of the rigid motion of an isolated ideal fluid acted upon by its own gravitational force. The law of gravitation there takes the form

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R_{\sigma}{}^{\sigma}=-\kappa T_{\mu\nu}=-\kappa(\wp+\rho)(q_\mu q_\nu/q^2)+\kappa\wp g_{\mu\nu}. \tag{100}$$

Combined with (99), this gives

$$g^{\sigma\mu}q_{\nu;\mu;\sigma}+(\kappa/2)(3\wp+\rho)q_\nu=0, \tag{101}$$

a direct connection between mass density and streaming velocity that emphasizes the isotropic character of stream-line curvature in this case.

As in all codetermined gravitational problems, Newtonian or cosmological, the quantity $c(\kappa\rho)^{\frac{1}{2}}$ is a characteristic reciprocal time that determines the real or imaginary periodicity of stream lines in space-time.

Though all coordinate systems are equally possible, some are more adapted than others for the analysis of streaming. Inertial coordinate systems treat distant matter as unaccelerated from the outset and so facilitate the junction of a solution of the field equations within matter to an outside solution asymptotic to a Minkowskian space-time. By contrast, a noninertial coordinate system comoving with matter at every point has the advantage of presenting a greatly simplified picture of streaming and of identifying the velocity field with a number of gravitational potentials that must in any case appear in the problem. No reference to the state of distant matter need be made until the entire solution interior to streaming matter is constructed. In a general sense, no picture of matter as codetermined with its gravitational field, can be consistently applied to distant regions unless the matter there is supposed accelerated. The cosmologically streaming solutions are always required. The junction of a solution of the interior problem with an exterior solution is thus seen to be an artificial formulation when viewed on a cosmological scale. All problems are interior ones, and all concern streaming matter, near or far. For all regions, the comoving picture is by far the simplest. As for the use of special inertial systems to describe matter that happens to be near, it is as difficult to apply in practice as it is to defend in principle.

Comoving coordinates $x^\mu$ may be defined in terms of some quite arbitrary coordinates $\bar{x}^\mu$ as follows. Let the three functionally independent solutions for $\Phi$ satisfying $\bar{v}^\mu\partial\Phi/\partial\bar{x}^\mu=0$, be called $x^k(\bar{x})$. Then $v^k=\bar{v}^\mu(\partial x^k/\partial\bar{x}^\mu)=0$.

The fourth component of $v^\mu$ is defined by $v^\mu v_\mu=1=g_{44}(v^4)^2$, or $v^4=(g_{44})^{-\frac{1}{2}}$; and the covariant comoving components of $v^\mu$ are just

$$v_\mu=g_{\mu\nu}v^\nu=g_{\mu 4}(g_{44})^{-\frac{1}{2}},\quad v^\mu=g_4{}^\mu(g_{44})^{-\frac{1}{2}}. \tag{102}$$

Naturally, the component parallel to $v^\mu$ of any vector $a_\mu$ is

$$a_0=v^\mu a_\mu=a_4(g_{44})^{-\frac{1}{2}},$$

while the component perpendicular to $v^\mu$ is formed by means of the projection operator in (37):

$$\begin{aligned} g_\nu{}^{k*}&=g_\nu{}^k,\quad g_k{}^{4*}=-g_{k4}/g_{44},\quad g_4{}^{4*}=0,\\ g_{ij}{}^*&=g_{ij}-(g_{i4}g_{j4}/g_{44}),\quad g_{k4}{}^*=0,\quad g_{44}{}^*=0, \end{aligned} \tag{103}$$

$$a_k{}^*=g_k{}^{\mu*}a_\mu=a_k-a_4(g_{k4}/g_{44}),\quad a_4{}^*=0. \tag{104}$$

If a comoving coordinate system $x^\mu$ is subjected to a transformation

$$\bar{x}^k=F^{(k)}(x^1,x^2,x^3),\quad \bar{x}^4=F(x^1,\cdots x^4), \tag{105}$$

it remains comoving, as can be seen at once from the fact that

$$\bar{v}^k=v^j[\partial F^{(k)}/\partial x^j]=0.$$

The transformation introduces arbitrary curvilinear space-coordinates and resets clocks at different points—a mere change of measuring conventions. We shall refer to such a transformation as a recalibration.[19] Under it, the comoving space components of a contravariant four-vector form a complete contravariant three-vector, with transformation coefficients depending only on space variables:

$$\bar{a}^k=a^j[\partial F^{(k)}/\partial x^j]. \tag{106}$$

The metric components $g^{kj}$ form a corresponding spatial three-tensor. Since also, according to (103),

$$g^{kj}g_{ij}{}^*=g^{k\mu}g_{i\mu}-g_{i4}(g^{k\mu}g_{\mu 4})=g^k{}_i, \tag{107}$$

so then $g_{ij}{}^*$ is a spatial three-tensor reciprocal to $g^{kj}$, and can be used to make covariant spatial three-vectors out of contravariant ones. The new covariant formed from $a^k$ is just the component of $a^\mu$ normal to $v^\mu$:

$$g_{jk}{}^*a^k=g_{j\mu}{}^*a^\mu=a_j{}^*. \tag{108}$$

In addition to these three-vectors, we also have invariants under recalibration such as $a_4(g_{44})^{-\frac{1}{2}}=a_0$, arising from $a_4=\bar{a}_4(\partial F/\partial x^4)$ and $g_{44}=\bar{g}_{44}(\partial F/\partial x^4)^2$. By elementary transformation of the determinant of $g_{\mu\nu}=g_{\mu\nu}{}^*+v_\mu v_\nu$ expressed in comoving coordinates, we find

$$g=g^*g_{44}, \tag{109}$$

where $g^*$, the three-rowed determinant of the spatial metric $g_{ij}{}^*$, transforms under recalibration like a scalar

[19] See C. Møller, *Theory of Relativity* (Clarendon Press, Oxford, 1952), Secs. 89, 94.

density of three-space. Singling out a time dimension directed along the stream-lines makes especially simple the division of a scalar product into "parallel" and "normal" scalar parts:

$$a^\mu b_\mu = a^k b_k^* + a_0 b_0. \tag{110}$$

In all systems of coordinates comoving with a given velocity field, it is thus possible to analyze phenomena in terms of an effective spatial metric $g_{ik}^*$ reciprocal to $g^{ik}$, and an effectively orthogonal time dimension symbolized by the index 0. One may also construct starred and 0 components beginning with any given $g_{\mu\nu}$ in any given coordinate system: Equations (103) through (110) refer only to the metric and not necessarily to any preassigned velocity field. The latter can, in fact, be made out of the metric by means of Eq. (102); and that construction gives meaning to the quantities that transform in tensor fashion under the group of recalibrations. The importance of that group and its covariants has been recognized by Møller[19] and other as the group of "change of coordinates inside a system of reference"; and many conclusions about these have been stated. Our construction of the velocity field $v_\mu$ associated with a given metric in a given coordinate system enables us to develop simply, and to extend, these conclusions.

Consider for example the vorticity $\omega_{\mu\nu}$ of our velocity field, defined in (62) as the normal projection of the antisymmetric circulation tensor. In all comoving coordinates,

$$\omega_{0i}^* = 0 = \omega_{4i}(g_{44})^{-\frac{1}{2}},$$

and

$$2\omega_{ij}^* = 2\omega_{ij} = (g_{44})^{\frac{1}{2}}(C_{i,j} - C_{j,i} + C_i C_{j,4} - C_j C_{i,4}),$$

where $C_i$, defined by

$$C_i = g_{4i}/g_{44} = v^4 v_i,$$

function as a kind of vector potential for the circulation. The analogy is heightened if one introduces differential operators covariant under recalibration:

$$\partial_i^* = (\partial/\partial x^i) - C_i(\partial/\partial x^4),$$
$$\partial_0 = (g_{44})^{-\frac{1}{2}}(\partial/\partial x^4),$$

with the aid of which

$$2\omega_{ij}^* = (g_{44})^{\frac{1}{2}}[\partial_j^* C_i - \partial_i^* C_j]. \tag{111}$$

Of course, the operators $\partial_i^*, \partial_0$ are not coordinate derivatives as can be seen from their commutation rules:

$$\begin{aligned} \partial_i^*\partial_j^* - \partial_j^*\partial_i^* &= 2\omega_{ij}\partial_0, \\ \partial_k^*\partial_0 - \partial_0\partial_k^* &= v_{k;0}\partial_0. \end{aligned} \tag{112}$$

The second result comes partly from (31) which shows that the acceleration $v_{k;0} = (v_{k;0})^*$, and partly from the direct evaluation,

$$v_{k;0} = C_{k,4} - \partial_k^*(\ln v_4). \tag{113}$$

The Jacobi identities for the commutators in (112) yield interesting relations between vorticity and acceleration in comoving coordinates[20]:

$$\begin{aligned} 2\omega_{ij,0} &= \partial_i^*(v_{j;0}) - \partial_j^*(v_{i;0}), \\ \partial_i^*\omega_{jk} + v_{i;0}\omega_{jk} &+ (\text{cyclic permutation of } i,j,k) = 0. \end{aligned} \tag{114}$$

It is possible to generalize (112) and (114) to fully covariant relations in space-time.

One of the most important considerations concerning vorticity emerges from a study of a freely falling test body with mass too small to disturb the average field in the observer's neighborhood. The forces are obtained by studying the normal and parallel projections of (31), the equation defining $v_{\mu;\nu}$, and the directly verifiable identity for the normal strain in comoving coordinates:

$$\begin{aligned} (v_{\alpha;\beta} + v_{\beta;\alpha}) g_\mu{}^{\alpha *} g_\nu{}^{\beta *} &= (g_{\mu\nu}^*)_{,0}. \\ \Gamma_{00,0} = \partial_0(\ln v_4), \quad \Gamma_{0k,0}^* &= \partial_k^*(\ln v_4), \end{aligned} \tag{115}$$

$$\begin{aligned} \Gamma_{ij,0}^* &= -\tfrac{1}{2}(g_{ij}^*)_{,0} + \tfrac{1}{2} g_i{}^{\mu *} g_j{}^{\nu *}(v_{\mu,\nu} + v_{\nu,\mu}), \\ \Gamma_{00,k}^* &= v_{k;0}, \\ \Gamma_{0j,k}^* &= \tfrac{1}{2}(g_{jk}^*)_{,0} - \omega_{jk}, \\ \Gamma_{ij,k}^* &= \tfrac{1}{2}\{(g_{ik}^*)_{,j}^* + (g_{jk}^*)_{,i}^* - (g_{ij}^*)_{,k}^*\}. \end{aligned} \tag{116}$$

The last quantities mentioned, the "spatial" connection coefficients, when derived from $\Gamma_{\mu\nu,\sigma}$ by externally starring all its indices, turn out to be exactly the same as three-space Γ's derived from the "spatial" metric $g_{ik}^*$, by the spatial operator $\partial_j^*$. The forces $\Gamma_{00,k}^*$ and $\Gamma_{0j,k}^*$ are distinguished by the fact that they transform like tensors under recalibration.

The geodesic equations can be stated as

$$\frac{d}{ds}\left(\frac{dx^k}{ds}\right) = \Gamma_{\mu\nu}{}^k \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$
$$= \Gamma_{ij}{}^{k*}\frac{dx^i}{ds}\frac{dx^j}{ds} + 2\Gamma_{0j}{}^{k*}\frac{dx_0}{ds}\frac{dx^j}{ds} + \Gamma_{00}{}^{k*}\left(\frac{dx_0}{ds}\right)^2.$$

They take a specially suggestive form when the velocities,

$$u^j = dx^j/dx_0 = (dx^j/ds)/(dx_0/ds),$$

are used, since with

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 1 = g_{ij}^*\frac{dx^i}{ds}\frac{dx^j}{ds} + \left(\frac{dx_0}{ds}\right)^2,$$

we have

$$\begin{aligned} dx^i/ds &= u^i/(1+u^m u_m^*)^{\frac{1}{2}}, \\ dx_0/ds &= 1/(1+u^m u_m^*)^{\frac{1}{2}}, \end{aligned} \tag{117}$$

[20] J. Serrin, *Handbuch der Physik* (Springer-Verlag, Berlin, 1959), Vol. 8, Part 1, p. 151ff., see Eqs. (17.1) to (17.4) derived in the special case that an equation of state obtains, and in a nonrelativistic situation.

and

$$g_{kj}^*(1+u^m u_m^*)^{\frac{1}{2}}\left(\frac{d}{dx_0}\right)\left[\frac{u^j}{(1+u^n u_n^*)^{\frac{1}{2}}}\right]=\Gamma_{ij,k}^* u^i u^j+2\Gamma_{0j,k}^* u^j+\Gamma_{00,k}^*. \quad (118)$$

Even in strong fields, the Coriolis force can be separately distinguished[21] as $2\omega_{kj}u^j$, and is thus always determined by the local vorticity in a real or hypothetical reference fluid imagined as streaming in the time direction defined by the given coordinate system. Without appeal to the apparent motions of distant stars, an observer can tell from the fields nearby, e.g., by the precession of a pendulum, to what extent his coordinates are rotating. Of course, under general coordinate transformations, the covariant components of velocity of an imaginary reference fluid will have to be determined afresh from the values of $g_{4\mu}$ according to (102). But if the fluid is real, its velocity is a four-vector and (102) holds only in comoving coordinates. The vorticity of a real fluid is a tensor, and cannot be annulled by any coordinate transformation. For a hypothetical fluid, the vorticity is a tensor only under recalibrations; and therefore they cannot annul $g_{4k}$ unless the vorticity vanishes. Non-null values of $g_{4k}$ are uniquely indicative of local rotation of coordinates wherever the curl of $g_{4k}/g_{44}=C_k$ fails to vanish.

Special interest attaches to a four-vector $\omega_\mu$ constructed from an antisymmetric tensor $\omega_{\mu\nu}$ and the velocity field $v^\mu$ with the aid of the numerical tensor antisymmetric on all index pairs. $\epsilon_{\mu\nu\sigma\tau}=(-g)^{\frac{1}{2}}f(\mu,\nu,\sigma,\tau)$ defined by $f(1,2,3,4)=1$. In comoving coordinates, $\epsilon_{4\mu\nu 0}=0$, $\epsilon_{ijk0}=\epsilon_{ijk}^*$, the corresponding threefold tensor under recalibration. The four-vector

$$\omega_\mu^*=\tfrac{1}{2}\epsilon_{\mu\nu\sigma\tau}\omega^{\sigma\tau}v^\nu, \quad (119)$$

with a structure reminiscent of the ponderomotive force in magnetohydrodynamics, also satisfies

$$\omega_{\mu\nu}=-\epsilon_{\mu\nu\sigma\tau}\omega^{\sigma *}v^\tau, \quad (120)$$

and reduces in comoving coordinates to

$$\omega_k^*=(-g^*)^{\frac{1}{2}}\omega^{ij}, \quad \omega_{ij}=-(-g^*)^{\frac{1}{2}}\omega^k, \quad \omega_0=0, \quad (121)$$

with $i, j, k$ any cyclic permutation of 1, 2, 3. The comoving quantities $\omega_k^*$ and $\omega^k$ transform as space vectors under recalibrations. The direction $\omega^k$ at any point is clearly the axis of local rotation of the reference fluid. If, furthermore, $\omega^k(x)$ is interpreted as the field of tangents to a two-parameter family of curves filling those regions where vorticity exists, the curves represent vortex filaments.

A case of some interest is that where the vortex filaments form a steady pattern in comoving coordinates, so that the direction of $\omega^k$ can be everywhere identified with a single coordinate direction, the comoving $z$ axis, for example. Intuitively, it seems clear that this condition can be met, if and only if $\omega^k{}_{,0}$ is everywhere proportional to $\omega^k$ through a single (scalar) function. Proof goes as follows: if axial comoving coordinates are to be possible there must be some set of comoving coordinates $x^\mu$ in which $\omega^1=\omega^2=0$, and the only non-null component of $\omega^k$ is in the $x^3$ direction. In terms of some arbitrary unlimited coordinates $\bar{x}^\mu$, we must have

$$v^k=(\partial x^k/\partial\bar{x}^\mu)\bar{v}^\mu, \quad \omega^k=(\partial x^k/\partial\bar{x}^\mu)\bar{v}^\mu, \quad (122)$$

and particularly the vanishing of these when $k=1, 2$. There must therefore exist two functionally independent solutions $\Phi=x^1(\bar{x})$ and $\Phi=x^2(\bar{x})$ for the pair of simultaneous equations $\bar{\omega}^\nu\bar{\partial}_\nu\Phi=\bar{v}^\nu\bar{\partial}_\nu\Phi=0$. Since $\omega^\nu v_\nu=\omega_0=0$, these equations must essentially be independent; and, since each has three independent solutions, the pair can have no more than two. Unless the equations for $\Phi$ are compatible as they stand, their commutators can generate new independent restrictions on $\Phi$ and can thus reduce the number of independent solutions to one or zero. In comoving coordinates, the differential operators in the $\Phi$ equations $\omega^k\partial_k^*\Phi=\partial_0\Phi=0$, namely $\omega^k\partial_k^*$ and $\partial_0$, must therefore either commute or give rise to a linear combination of these same operators. Now

$$(\omega^k\partial_k^*)\partial_0-\partial_0(\omega^k\partial_k^*)=(\omega^k v_{k;0})\partial_0-(\omega^k{}_{,0}\partial_k^*),$$

of which the first term meets our requirement, while the second can only do so if $\omega^k{}_{,0}$ is proportional to $\omega^k$. It is easy to verify that in comoving coordinates

$$v^\mu{}_{;\mu}=(-g^*)^{-\frac{1}{2}}\partial_0(-g^*)^{\frac{1}{2}}, \quad (123)$$

so that, by (121),

$$\omega^k{}_{,0}=v^\mu{}_{;\mu}\omega^k-(-g^*)^{-\frac{1}{2}}\omega_{ij,0},$$

with $i, j, k$ a cyclic permutation of 1, 2, 3. Thus, by (114),

$$\omega^k{}_{,0}\partial_k^*=v^\mu{}_{;\mu}\omega^k\partial_k^*-\tfrac{1}{2}(-g^*)^{-\frac{1}{2}}[\partial_i^*v_{j;0}-\partial_j^*v_{i;0}]\partial_k^*,$$

which must be proportional to $-(-g^*)^{-\frac{1}{2}}\omega_{ij}\partial_k^*$. Hence

$$2\omega_{ij,0}=\partial_i^*v_{j;0}-\partial_j^*v_{i;0}=\Psi\omega_{ij}.$$

This may be written as a relation fully covariant in space-time,

$$g_\mu{}^{\sigma *}g_\nu{}^{\tau *}(\partial_\sigma v_{\tau;0}-\partial_\tau v_{\sigma;0})=\Psi\omega_{\mu\nu}, \quad (124)$$

defining the conditions of existence for comoving axial coordinates or vortex filaments stationary for a comoving observer.

In such axial comoving coordinates the metric tensor and associated quantities can be materially simplified. Two of the independent solutions of (122) have already been named $x^1$ and $x^2$. Let us name $x^4$ the third independent solution of $\omega^\nu\partial_\nu\Phi=0$, and let us name $x^3$ the third solution of $v^\nu\partial_\nu\Phi=0$, in keeping with the space-like character of the vorticity field and the time-like character of the velocity field. This characterizes $x^3$ as

[21] A. Einstein, *Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1945), p. 102ff. See also reference 19, Sec. 110. In both these references, the Coriolis force is identified only in the limit of weak fields.

comoving, and $x^4$ so as to make $g_{43}=0$; for $\omega^4=\omega^\nu(\partial x^4/\partial x^\nu)=0$ and therefore

$$\omega_0=\omega^\mu v_\mu=\omega^3 g_{43}/(g_{44})^{\frac{1}{2}}=0,$$

according to (121) and (102). Along with $\omega^1=\omega^2=0$, we conclude from (121) that $\omega_{13}=\omega_{23}=0$ and that therefore in (111), since $C_3=0$ and $\partial_3{}^*=\partial_3$, we must have $C_{1,3}=C_{2,3}=0$. This presents the vortex filaments as orthogonal to the Coriolis potential, and prevents the comoving components of the latter from changing along a filament. From (111) we also conclude that $\omega_{12}$, the sole surviving component of normal circulation—the "strength" of the filament in classical vortex theory—is not quite constant along a filament, but instead is proportional to $(g_{44})^{\frac{1}{2}}$. This is a new general relativistic effect analogous to the red shift and to the variation of temperature throughout a mass in thermal equilibrium. The Coriolis potentials $C_1$ and $C_2$, and the associated strength $\omega_{12}/(g_{44})^{\frac{1}{2}}$, can be assumed to remain constant even where a vortex filament "breaks through" a matter boundary into empty space. This extension of the classical categories of vortex description is without classical analog but expresses instead the characteristic induction effects of rotating matter that resemble magnetic forces originating in a circulation of charge.

The foregoing results apply to fluids with axial comoving coordinates independently of any assumption of the ideal fluid condition and quite apart from the existence of an equation of state. Following our discussion of Sec. III it would seem unlikely that the peculiar and limited dynamical isotropy of the ideal fluid could appear in nature as the result of special accidental circumstance rather than as a result of complete local dynamical isotropy. The latter, as we have seen, goes far beyond the ideal fluid hypothesis however, and also assigns the fluid a particular kind of equation of state, and limits its streaming to one of two quite definite patterns, rigid motion or homogeneous magnification. We therefore incline to the view that the historical division of these combined effects into three distinct steps—the assumption of the ideal fluid (30), the assumption of an equation of state $\wp(\rho)$, and finally the assumption of complete local dynamical isotropy—may be a rather artificial approach to the description of real fluids. Nevertheless, to relate our account to earlier work, we take these three steps in succession.

The introduction of the ideal fluid condition (30) appears particularly simple in components parallel and normal to $v^\mu$,

$$T_{\mu\nu}{}^*=-\wp g_{\mu\nu}{}^*,\quad T_{0\mu}{}^*=0,\quad T_{00}=\rho, \tag{125}$$

and even more simple in comoving coordinates,

$$T_j{}^i=T_j{}^{i*}=-\wp g_j{}^i,\quad T_0{}^k=0,\quad T_{00}=\rho,$$
$$T_k{}^4=(\wp+\rho)C_k=(\wp+\rho)g_{4k}/g_{44}.$$

If $T_k{}^4$ were zero, the tensor would be diagonal and hence would itself discriminate no spatial direction. A codetermined solution of the field equations would therefore have static spherical symmetry, a situation apparently incompatible with the existence of accelerated streaming. The fact that $C_k$ cannot be made to vanish in any comoving coordinate system unless the vorticity vanishes means that it is the presence of vorticity alone that prevents the collapse of a codetermined solution into a spherically symmetric pattern characteristic of a truly static distribution of mass.

The conservation laws for the ideal fluid, (32) and (33), take the form of prescriptions for the time dependence of the two factors making up $\mathfrak{T}_k{}^4$, when expressed in comoving form:

$$\begin{aligned} v^\alpha{}_{;\alpha}&=\partial_0 \ln(-g^*)^{\frac{1}{2}}=-\partial_0\rho/(\wp+\rho),\\ v_{k;0}&=\partial_k{}^*\wp/(\wp+\rho)=C_{k,4}-\tfrac{1}{2}\partial_k{}^*\ln g_{44}; \end{aligned}\tag{126}$$

or alternatively

$$\begin{aligned} \partial_0[(-g)^{\frac{1}{2}}(\wp+\rho)]&=X_0,\\ C_{k,4}&=X_k{}^*, \end{aligned}\tag{127}$$

with

$$X_\mu\equiv\tfrac{1}{2}\partial_\mu \ln g_{44}+\partial_\mu\wp/(\wp+\rho). \tag{128}$$

The importance of the equation of state in providing for the integrability of the above time-differentials is evident in the structure of the quantities $X_\mu$. If and only if there exists a functional relation between $\wp$ and $\rho$, $X_\mu$ may be written as the gradient of a single scalar $X$:

$$X_\mu=\partial X_\mu,\quad X=\tfrac{1}{2}\ln g_{44}+\int d\wp/(\wp+\rho).$$

The quantity $X$ is evidently the hydrodynamic potential and reduces, in weak gravitational fields to the familiar potential function of Bernouilli's theorem. Under recalibration the second term is invariant, while the first can be changed at will, so that $X$ may be transformed into zero, by merely recalibrating the time. That results in the vanishing of $X_0$ and $X_k{}^*$ in (127). In that special coordinate system adapted to the problem of an ideal fluid obedient to an equation of state, the Coriolis potentials $C_k$ as well as the circulating mass density $(-g)^{\frac{1}{2}}(\wp+\rho)$ become quite independent of the time, while $g_{44}$ assumes the standard form

$$g_{44}=\exp\left[-2\int d\wp/(\wp+\rho)\right]. \tag{129}$$

The time independence of $C_k$ makes it possible to recover, in the domain of general relativity, practically every feature of the classic Kelvin-Helmholtz theory of vortices. The first step is to recognize that we are dealing with a stationary pattern of vorticity because $C_k$, and hence $\omega_{ij}/(g_{44})^{\frac{1}{2}}$ generated by the space derivatives of

$C_k$, is independent of the time.[22] We may actually bring our coordinates in which $X=0$ and (129) obtain, into correspondence with the axial comoving coordinates developed earlier for the general fluid with stationary vorticity pattern. Reference to the earlier argument shows that we need to characterize $x^4$ in such a way that $g_{43}=0$; and the development of Bernoulli's theorem for the ideal fluid shows that we must do this while maintaining $g_{44}$ unchanged. Consider a simple recalibration of the time by addition to $x^4$ of a function $\tau(x^1,x^2,x^3)$:

$$x^{4'}=x^4+\tau(x^1,x^2,x^3), \quad x^{k'}=x^k.$$

From

$$g_{\mu\nu}=g_{\sigma\tau}'(\partial x^{\sigma'}/\partial x^\mu)(\partial x^{\tau'}/\partial x^\nu),$$

we derive

$$g_{44}=g_{44}', \quad g_{4k}=g_{44}'(\partial\tau/\partial x^k)+g_{4k}',$$

or

$$C_k'=C_k-\partial\tau/\partial x^k.$$

Since by (129), $C_3$ is, like $\tau$, a function only of space variables $x^k$, $C_3'$ can be made to vanish by suitable choice of $\tau$ without altering the Bernoullian value of $g_{44}$. The remaining properties of comoving axial coordinates can now be demonstrated exactly as in the general case, from which we conclude that we may establish a comoving coordinate system with the striking special properties:

$$\omega^1=\omega^2=C_3=C_{1,3}=C_{1,4}=C_{2,3}=C_{2,4} = (\omega_{12}/(g_{44})^{\frac{1}{2}})_{,3}=(\omega_{12}/(g_{44})^{\frac{1}{2}})_{,4}=0, \quad (130)$$

in addition to (128) and $[(\wp+\rho)(-g)^{\frac{1}{2}}]_{,4}=0$.

The shape of a vortex filament appears steady to the observer comoving with matter, and its strength, $\omega_{12}/(g_{44})^{\frac{1}{2}}$, is constant not only along the length of the filament, but also in time. From the point of view of general relativity, the Coriolis potentials $g_{4k}/g_{44}=C_k$ have been reduced to a very simple form constant along a filament and in time, while the pseudo-Newtonian gravitational potential $\frac{1}{2}\ln g_{44}$, which determines the red shift, is exactly equal to the hydrodynamic potential. These profound simplifications permit us to take a further step beyond the classical theory of vortices, made possible by our general coordinates. A recalibration will enable us to make the strength of all vortex filaments the same, as though the motion were rigid with constant angular velocity $\varpi$.

Of course we do not imply that the motion is actually rigid; we are merely using coordinates that make it appear quasi-rigid. (A true rigid motion, as we shall shortly prove, leads to time constancy of *all* metric components. In quasi-rigid motion, $g_{ij}^*$ retain a time dependence in general, and moreover display an intricate pattern of "centrifugal" force that describes the departure of the motion from actual rigidity.) The analogy between the quasi-rigid picture and the familiar notion of nonrelativistic rigid motion is nevertheless so close that we shall be able to define "radial" coordinates $x^1$ and $x^2$ in which distance is essentially determined by the magnitude of the components of Coriolis potential.

We construct the quasi-rigid representation by a recalibration to a system of coordinates $\bar{x}^\mu$, that leaves intact all the special simplifications of (130), achieved up to this point.

$$\bar{x}^1=\xi(x^1,x^2); \quad \bar{x}^2=\eta(x^1,x^2); \quad \bar{x}^3=\zeta(x^1,x^2,x^3); \quad \bar{x}^4=x^4+\tau(x^1,x^2).$$

It is easy to verify the maintenance of (130), (128), and (127) with $X_0=X_k^*=0$ in the new system, and also that

$$C_1=(\partial\tau/\partial x^1)+\bar{C}_1(\partial\xi/\partial x^1)+\bar{C}_2(\partial\eta/\partial x^1),$$

$$C_2=(\partial\tau/\partial x^2)+\bar{C}_1(\partial\xi/\partial x^2)+\bar{C}_2(\partial\eta/\partial x^2),$$

$$\bar{\omega}^3=(\partial\zeta/\partial x^3)\omega^3,$$

$$(-g^*)^{\frac{1}{2}}=(-\bar{g}^*)^{\frac{1}{2}}(\partial\zeta/\partial x^3)\begin{vmatrix}(\partial\xi/\partial x^1) & (\partial\xi/\partial x^2)\\ (\partial\eta/\partial x^1) & (\partial\eta/\partial x^2)\end{vmatrix},$$

and therefore that

$$\bar{\omega}_{12}=\omega_{12}\Big/\left(\begin{vmatrix}(\partial\xi/\partial x^1) & (\partial\xi/\partial x^2)\\ (\partial\eta/\partial x^1) & (\partial\eta/\partial x^2)\end{vmatrix}\right).$$

By choosing $\eta=x^2$, $\xi=\varpi^{-1}\int(C_{2,1}-C_{1,2})dx^1$, and $\tau=\int dx^1[C_1+\frac{1}{2}x^2(C_{2,1}-C_{1,2})]$, it is possible to satisfy all these relations and obtain

$$\bar{C}_1=-\tfrac{1}{2}\varpi\bar{x}^2, \quad \bar{C}_2=+\tfrac{1}{2}\varpi\bar{x}^1, \quad \bar{\omega}_{12}/(g_{44})^{\frac{1}{2}}=\varpi, \quad (131)$$

with $\varpi$ merely a constant preserving the dimensionless definition of Coriolis potential. The achievement of this simple picture does not prejudice $\zeta(x^1,x^2,x^3)$, which still remains arbitrary. The conservation of mass may be used to determine $\zeta$ as follows: from (127) with $X=0$, $(\wp+\rho)(-g)^{\frac{1}{2}}$ is a function of $x^1$, $x^2$, $x^3$ that is equal to $(\bar{\wp}+\bar{\rho})(-\bar{g})^{\frac{1}{2}}(\partial\zeta/\partial x^3)(C_{2,1}-C_{1,2})/\varpi$. It is therefore possible to reduce $(\bar{\wp}+\bar{\rho})(-\bar{g})^{\frac{1}{2}}$ to a constant $\epsilon$ by suitable choice of $\zeta$, namely

$$(\bar{\wp}+\bar{\rho})(-\bar{g})^{\frac{1}{2}}=\epsilon,$$

$$\zeta=(\varpi/\epsilon)\int dx^3[(\wp+\rho)(-g)^{\frac{1}{2}}(C_{2,1}-C_{1,2})].$$

In sum, our standard comoving picture of a rotating ideal fluid satisfying an equation of state throws all the effects of streaming upon the metric, but nevertheless pictures $\frac{1}{2}\ln g_{44}$ in accordance with (129), presents the circulatory mass density $(\wp+\rho)(-g)^{\frac{1}{2}}$ as a constant in space and time, and depicts the vortex filaments as stationary and parallel with a uniform strength that is constant over all space and time occupied by matter and gives the $g_{4k}$ components of metric the definite form

$$g_{43}=0, \quad g_{41}=-\tfrac{1}{2}\varpi x^2 g_{44}, \quad g_{42}=+\tfrac{1}{2}\varpi x^1 g_{44}.$$

[22] See references 11 and 12 which are concerned with the "holonomic" or "barotropic" case with an equation of state. Our results in this case derived after this point have not been anticipated in the cited references.

Of the fifteen unknowns $g_{\mu\nu}$, $\wp$, and $\rho$ with which we began, only seven, $g_{ij}^*$ and $\wp$, remain. These must be codetermined from the field equations. It is interesting that the transformation to constant circulating mass density makes the actual boundary where $(\wp+\rho)$ falls to zero, a singular region in which $(-g^*)^{\frac{1}{2}}$ may become infinite. The interior solutions may possess, at least in some directions, a natural space-time boundary.

We now proceed to the last step towards complete dynamical isotropy, the assumption of one of the patterns of motion deduced in Sec. III. Since the irrotational cases are likely to have their main application in cosmology, we consider only the case of rigid motion, characterized in (57) by the existence of a scalar function $q$ such that

$$q_\mu = qv_\mu, \quad q_{\mu;\nu}+q_{\nu;\mu}=0.$$

We can make $q^k$ vanish by choosing any comoving coordinates $\bar{x}^\mu$; and then we can bring $\bar{q}^4$ to a constant value $(1/\tau)$ by a transformation to another comoving coordinate system $x^\mu$ such that $\bar{x}^k=x^k$ and $x^4=\int d\bar{x}^4/\bar{q}^4\tau$. For then $q^4=(\partial x^4/\partial\bar{x}^4)\bar{q}^4=(1/\tau)$. In this case we can set all derivatives $g_{\mu\sigma}q^\sigma{}_{,\nu}+g_{\nu\sigma}q^\sigma{}_{,\mu}$ equal to zero in the equation $g_{\mu\sigma}q^\sigma{}_{;\nu}+g_{\nu\sigma}q^\sigma{}_{;\mu}=0$, from which we conclude that

$$0=(\Gamma_{\mu\alpha,\nu}+\Gamma_{\nu\alpha,\mu})q^\alpha=g_{\mu\nu,\alpha}q^\alpha=g_{\mu\nu,4}/\tau. \qquad (132)$$

This means that in rigid motion there is a class of comoving coordinates in which all metric components are time constant, and this conclusion is quite independent of the additional conclusion that we derived in Sec. III from local dynamical isotropy, namely the existence of an equation of state of the form implied by (66). In fact, the static quality of the metric, coupled with the ideal-fluid hypothesis, leads immediately to the conservation equations in the form (127) with $X_\mu$ necessarily vanishing. Since, with $X_\mu=0$, derivatives of $\wp$ become proportional, by the same factor $(\wp+\rho)$, to the corresponding derivatives of $\frac{1}{2}\ln g_{44}$, the Jacobians of $\wp$ and $g_{44}$ on all variable pairs must vanish; and $\wp$ and $g_{44}$ must be functionally dependent. Therefore $\wp$ and $(\wp+\rho)$ must be functionally dependent; or rigid motion, in itself, requires the existence of an equation of state. Since $g_{44}$, $\wp$, and $\rho$ are now all independent of time, a simple time recalibration $x^k\rightarrow x^k$, $x^4\rightarrow x^4+f(x^1,x^2,x^3)$ can reduce $X$ to zero without changing the constant value of $q^4-1/\tau$. In this way we connect the invariant $q$ with Bernoulli's theorem

$$q=(g_{44}q^4q^4)^{\frac{1}{2}}=\tau^{-1}\exp\left[-\int d\wp/(\wp+\rho)\right]; \qquad (133)$$

and this agrees exactly with the differential identity (67). The functional dependence of $\wp$ and $\rho$ on $g_{44}$ now means that $\wp$ and $\rho$ are also entirely dependent on $q$. In general coordinates, $\wp$ and $\rho$ can depend on $x^\mu$ only through the invariant $q(x)$, just as we concluded earlier in (66), on other grounds. There is little difference, indeed, between an ideal fluid in rigid motion and a system subject to local dynamical isotropy. It may be well to recall that in thermal equilibrium, streaming is rigid motion, and

$$\tau=(g_{44})^{-\frac{1}{2}}/q=T(g_{44})^{-\frac{1}{2}},$$

which thus appears as the temperature reduced by exactly that red-shift factor required to bring it to a constant value throughout the mass.

Naturally, the time independence of $g_{4k}$ and $g_{44}$ and thence of $C_k$, leads, as before, to the possibility of constructing axial comoving coordinates corresponding to uniform and constant angular velocity $\varpi=\omega_{12}(g_{44})^{-\frac{1}{2}}$, and uniform and constant mass density $\epsilon=(\wp+\rho)\times(-g)^{\frac{1}{2}}$. The added feature here is the thorough disappearance of all time-dependence from the functions $g_{ij}^*$ and $\wp$ that remain to be codetermined by the field equations. It is worth emphasizing that, despite the completely static character of the problem, codetermined solutions of spherical symmetry are still impossible as long as $\varpi\neq 0$, simply because the energy tensor is not diagonal:

$$\mathfrak{T}_1{}^4=-\tfrac{1}{2}\epsilon\varpi x^2, \quad \mathfrak{T}_2{}^4=+\tfrac{1}{2}\epsilon\varpi x^1, \quad \mathfrak{T}_3{}^4=0. \qquad (135)$$

Under these circumstances, the field equations, especially the one involving $R_{00}$, clearly reveal the presence of centrifugal energy terms, and show that even in weak gravitational fields such terms are still present and come from the quadratic dependence of curvature components on the forces.

We have here set down the simple standard forms that solutions of the field equations must take in most cases of interest. The development of explicit solutions will be reported in a forthcoming paper by the authors.


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