Distribution of the Magnetization in a Ferromagnet

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The distribution of the magnetization is calculated for a thick ferromagnetic slab with easy axis transverse to the plane of the slab in a large applied field in the plane of the slab. The calculation predicts a stable nonuniform distribution which has several features suggestive of the domain pattern to be expected in such a system. In particular, the pattern consists of alternating strips having approximately the periodicity expected from conventional domain theory; and incipient flux closure domains appear if the anisotropy field is smaller than the demagnetizing field.

INTRODUCTION

ONE of the unsolved problems of magnetostatics is the prediction of domain structure in a ferromagnetic body. The existence of Bloch walls is predicted by a one-dimensional energy minimization theorem, but it has not been accounted for theoretically in two or three dimensions.

A rigorous technique for calculating nonuniform distributions of magnetization has recently been applied to several simple problems.[1–3] In all these problems, however, it turns out that the nonuniform distributions are unstable and thus cannot be regarded as incipient domain formation.[4]

It is the purpose of this report to show that there is a class of physically interesting problems in which this technique leads to stable nonuniform distributions of the magnetization. The stability is achieved by dealing with specimens nearly saturated in a *hard* direction of magnetization. It is well known that, if such a specimen is held in an applied field nearly large enough to saturate it, then the magnetization will deviate only slightly from alignment with the field. We shall carry out the calculation for a particularly simple arrangement of this type in order to avoid computational difficulties. The resulting distribution of the magnetization has several features that are strongly suggestive of the domain structure to be expected for the configuration. The problem is a nonlinear one and is not carried far enough by the present analytical calculation to show well-developed domain structure. It is apparent from the results, however, that useful progress could be made by further numerical work.

In Sec. II we obtain the initial form of the distribution by the technique used earlier.[3] In Sec. III we show that a stable finite nonuniform distribution results in the configuration we are considering.

II. THE NUCLEATION MODE

In a ferromagnet in equilibrium, the torque acting on the magnetization vanishes everywhere:

$$\mathbf{M}\times\mathbf{H}_{\mathrm{eff}}=0,$$

or, by using $\hat{v}=\mathbf{M}/M=\alpha\hat{x}+\beta\hat{y}+\gamma\hat{z}$, $\alpha^2+\beta^2+\gamma^2=1$,

$$\hat{v}\times[c\nabla^2\hat{v}-(\partial\omega/\partial\hat{v})-M\nabla U+M\mathbf{H}]=0. \tag{1}$$

The terms in the bracket are proportional (in this order) to the exchange field, the anisotropy field, the dipolar field, and the applied field.

In the anisotropy term, $\partial/\partial\hat{v}$ means $(\hat{x}\partial/\partial\alpha+\hat{y}\partial/\partial\beta+\hat{z}\partial/\partial\gamma)$ and the anisotropy energy is assumed written in the form

$$\omega=\omega_0+\omega_1\alpha+\omega_2\beta+\tfrac{1}{2}(\omega_{11}\alpha^2+2\omega_{12}\alpha\beta+\omega_{22}\beta^2)+\cdots.$$

The dipolar energy (magnetostatic potential) obeys the Poisson equation:

$$\nabla^2U=\begin{cases}4\pi M\nabla\cdot\hat{v} & \text{inside}\\ 0 & \text{outside.}\end{cases} \tag{2}$$

The boundary conditions on the surface of the ferromagnet are[4]

$$\hat{v}\times(\partial\hat{v}/\partial n)=0, \tag{3}$$

$$U_{\mathrm{in}}=U_{\mathrm{out}}, \tag{4}$$

$$(\partial U/\partial n)_{\mathrm{out}}=(\partial U/\partial n)_{\mathrm{in}}-4\pi M\hat{v}\cdot\hat{n}, \tag{5}$$

where $\hat{n}$ is the normal to the surface.

In a uniaxial crystal with an easy or hard axis along x, $\omega_{11}=2K$ and all other terms in ω vanish. $\omega=K\alpha^2$; if $K>0$, $\hat{x}$ is a hard axis, if $K<0$, $\hat{x}$ is an easy axis. We will assume $K<0$, and also $\mathbf{H}=H\hat{z}$.

The components of Eq. (1) are

$$c(\beta\nabla^2\gamma-\gamma\nabla^2\beta)-M[\beta(\partial U/\partial z)-\gamma(\partial U/\partial y)-\beta H]=0,$$

$$c(\gamma\nabla^2\alpha-\alpha\nabla^2\gamma)-2K\alpha\gamma-M[\gamma(\partial U/\partial x)-\alpha(\partial U/\partial z)+\alpha H]=0,$$

$$c(\alpha\nabla^2\beta-\beta\nabla^2\alpha)+2K\alpha\beta-M[\alpha(\partial U/\partial y)-\beta(\partial U/\partial x)]=0.$$

We will deal with a slab between $x=\pm l$, nearly saturated in the $+z$ direction, so that α, $\beta\ll1$, $\gamma\cong1$, and we introduce the dimensionless notation

$$\xi=x\pi/l,\quad h=H/2\pi M,\quad l_0=(c/2M^2)^{\frac{1}{2}},$$
$$\eta=y\pi/l,\quad k=K/2\pi M^2,\quad S=l/l_0.$$
$$\zeta=z\pi/l,\quad u=(2/c)^{\frac{1}{2}}U,$$

Then to first order in α and β, the torque equations

[1] W. F. Brown, Jr., Phys. Rev. **105**, 1479 (1957).
[2] E. H. Frei, S. Shtrikman, and D. Treves, Phys. Rev. **106**, 446 (1957).
[3] A. Aharoni and S. Shtrikman, Phys. Rev. **109**, 1522 (1958).
[4] W. F. Brown, Jr., J. Appl. Phys. **30**, 62S (1959).

and boundary conditions are

$$[-\nabla^2+(S^2/\pi)(h+2k)]\alpha+\frac{S}{2\pi}\frac{\partial u}{\partial \xi}=0, \quad (6)$$

$$[-\nabla^2+(S^2/\pi)h]\beta+\frac{S}{2\pi}\frac{\partial u}{\partial \eta}=0, \quad (7)$$

$$\nabla^2 u=\begin{cases}4S[(\partial\alpha/\partial\xi)+(\partial\beta/\partial\eta)] & \text{inside}\\ 0 & \text{outside,}\end{cases} \quad (8)$$

$$\partial\alpha/\partial\xi=\partial\beta/\partial\xi=0, \qquad \xi=\pm\pi, \quad (9)$$

$$u_{\text{in}}=u_{\text{out}}, \qquad \xi=\pm\pi, \quad (10)$$

$$(\partial u/\partial\xi)_{\text{out}}=-4S\alpha+(\partial u/\partial\xi)_{\text{in}}, \quad \xi=\pm\pi. \quad (11)$$

The general solution of the system of Eqs. (6)–(8) is

$$\alpha=A\ \sin(p\xi+\xi_0)\sin(m\eta+\eta_0)\cos(n\zeta+\zeta_0),$$
$$\beta=B\cos(p\xi+\xi_0)\cos(m\eta+\eta_0)\cos(n\zeta+\zeta_0),$$
$$u=U\cos(p\xi+\xi_0)\sin(m\eta+\eta_0)\cos(n\zeta+\zeta_0),$$

where p, m, and n are related by the vanishing of the secular determinant

$$\begin{vmatrix}[p^2+m^2+n^2+(S^2/\pi)(h+2k)] & 0 & -Sp/2\pi\\ 0 & [p^2+m^2+n^2+(S^2/\pi)h] & Sm/2\pi\\ 4Sp & -4Sm & (p^2+m^2+n^2)\end{vmatrix}=0. \quad (12)$$

The boundary value problem of Eqs. (6)–(11) has nontrivial solutions only for certain ranges of the reduced field h. In particular, there is no solution if h is very large. The largest value of h for which a nontrivial solution exists represents the field for which the distribution of the magnetization can first deviate from (uniform) saturation. We will calculate this "nucleation field" h_n and the corresponding eigenfunctions α, β, u for the case of a thick slab ($S\rightarrow\infty$). The restriction to a "thick" slab is not very stringent. In a typical ferromagnet, l_0 is of the order of 10^{-6} cm, so that even the translucent layers that have been used in investigations of domain patterns[5] are thick ($S\sim10^2$) in our sense.

With the assumption (which will be shown to be self-consistent later) that in the nucleation mode, (m^2+n^2) is of order S, and $h+2k\equiv\kappa$ is of order S^{-1}, the secular equation becomes approximately

$$p^6+\frac{S^2}{\pi}(2+h)p^4+\frac{2S^4}{\pi^2}hp^2 +\frac{S^2}{\pi}[2m^2+h(m^2+n^2)]\left[\frac{S^2}{\pi}\kappa+m^2+n^2\right]=0, \quad (13)$$

which, to the same approximation, has the roots

$$p_1^2=-(S^2/\pi)h,$$
$$p_2^2=-2S^2/\pi,$$
$$p_3^2=-\frac{2m^2+h(m^2+n^2)}{2h}\left[\frac{(m^2+n^2)\pi}{S^2}+\kappa\right].$$

Thus the eigenfunctions will be of the form

$$\alpha=\sum_{i=1}^{3}A_i\sin(p_i\xi+\xi_{0i})\sin(m\eta+\eta_0)\cos(n\zeta+\zeta_0), \quad (14)$$

with similar forms for β and u. Equations (6)–(8) furnish relations between the A_i, B_i, and U_i:

$$U_i=\frac{2\pi[p_i^2+m^2+n^2+(S^2/\pi)\kappa]}{Sp_i}A_i,$$
$$B_i=-\frac{m}{p_i}\frac{[p_i^2+m^2+n^2+(S^2/\pi)\kappa]}{[p_i^2+m^2+n^2+(S^2/\pi)h]}A_i. \quad (15)$$

(Note that in the vicinity of $h=2$, that is, $k=-1$, B_2 has a singularity and the present approximation breaks down. We shall see below that the form of the eigenfunction changes near this value of the anisotropy field). The reduced potential outside the magnet is of the form

$$u=W\exp[-(m^2+n^2)^{\frac{1}{2}}(\xi-\pi)]\times\sin(m\eta+\eta_0)\cos(n\zeta+\zeta_0),\quad \xi>\pi,$$
$$u=W'\exp[+(m^2+n^2)^{\frac{1}{2}}(\xi+\pi)]\times\sin(m\eta+\eta_0)\cos(n\zeta+\zeta_0),\quad \xi<-\pi, \quad (16)$$

which vanishes properly at infinity. The boundary conditions of Eq. (10) will now be used to determine W and W', and the remaining six boundary conditions of Eqs. (9) and (11) furnish the needed six relations among the A_i and ξ_{0i}. The constants η_0 and ζ_0 only translate the mode in the plane of the slab and thus are not needed.

The tedious work of applying these equations can be simplified a little by a physical argument. From the symmetry of the problem about $\xi=0$, it follows that the ξ_{0i} are either zero or $\frac{1}{2}\pi$ and in fact they must all be zero or all $\frac{1}{2}\pi$. A schematic sketch of the surface pole distributions for these two alternatives is shown in Fig. 1. It is apparent from the figure that $\xi_{0i}=\frac{1}{2}\pi$ is energetically more favorable and thus will lead to the higher value for the nucleation field.

[5] See, for example, C. Kooy and U. Enz, Philips Research Repts. **15**, 7 (1960), which deals with precisely the configuration under investigation here.

Thus the form of the eigenfunctions becomes

$$\alpha=\sin(m\eta+\eta_0)\cos(n\zeta+\zeta_0)\times[A_1\cosh|p_1|\xi+A_2\cosh|p_2|\xi+A_3\cos p_3\xi],$$

$$\beta=-\cos(m\eta+\eta_0)\cos(n\zeta+\zeta_0)\times[iB_1\sinh|p_1|\xi+iB_2\sinh|p_2|\xi+B_3\sin p_3\xi],$$

$$u=-\sin(m\eta+\eta_0)\cos(n\zeta+\zeta_0)\times[iU_1\sinh|p_1|\xi+iU_2\sinh|p_2|\xi+U_3\sin p_3\xi],$$

where the A_i, B_i, and U_i are related according to Eq. (15). Now from Eqs. (9)–(11) and (16)

$$|p_1|A_1\sinh|p_1|\pi+|p_2|A_2\sinh|p_2|\pi-p_3A_3\sin p_3\pi=0, \quad (17)$$

$$i|p_1|B_1\cosh|p_1|\pi+i|p_2|B_2\cosh|p_2|\pi+p_3B_3\cos p_3\pi=0, \quad (18)$$

$$(m^2+n^2)^{\frac{1}{2}}[iU_1\sinh|p_1|\pi+iU_2\sinh|p_2|\pi+U_3\sin p_3\pi]=[i|p_1|U_1\cosh|p_1|\pi+i|p_2|U_2\cosh|p_2|\pi+p_3U_3\cos p_3\pi]+4S[A_1\cosh|p_1|\pi+A_2\cosh|p_2|\pi+A_3\cos p_3\pi]. \quad (19)$$

Equations (17)–(19), after elimination of B_i and U_i by means of Eq. (15), form a linear homogeneous set which has a solution A_1, A_2, A_3 only if the determinant of the coefficients vanishes.

It can be shown that, to the approximation in which we are working, the determinant will vanish if

$$-(m^2+n^2)^{\frac{1}{2}}\left(\frac{8h}{2m^2+h(m^2+n^2)}\right)^{\frac{1}{2}}\left(-\frac{(m^2+n^2)\pi}{S^2}\right)^{\frac{1}{2}}-\kappa\sin p_3\pi+4\cos p_3\pi=0. \quad (20)$$

Since the coefficient of $\sin p_3\pi$ is of order $S^{-\frac{1}{2}}$, this implies $p_3\cong\frac{1}{2}$ and hence

$$\kappa=-\{h/2[(2+h)m^2+hn^2]\}-[(m^2+n^2)\pi/S^2],$$

which is to be maximized by adjusting m and n. To maximize, we take

$$d\kappa=dh=(\partial\kappa/\partial h)dh+(\partial\kappa/\partial m)dm+(\partial\kappa/\partial n)dn=0,$$

or

$$\frac{\partial\kappa}{\partial m}\Big/\left(1-\frac{\partial\kappa}{\partial h}\right)=0,\quad \frac{\partial\kappa}{\partial n}\Big/\left(1-\frac{\partial\kappa}{\partial h}\right)=0.$$

Since $\partial\kappa/\partial h$ has no poles, κ has stationary points at

$m=0$, $n=0$ (minimum);

$m=0$, $n=(S^2/2\pi)^{\frac{1}{4}}$,

$$\kappa=-(2\pi/S^2)^{\frac{1}{2}} \quad \text{(local maximum)}; \quad (21a)$$

$$m=\left(\frac{S^2}{2\pi}\frac{h}{2+h}\right)^{\frac{1}{4}},\quad n=0,$$

$$\kappa=-\left(\frac{h}{h+2}\frac{2\pi}{S^2}\right)^{\frac{1}{2}} \quad \text{(local maximum)}. \quad (21b)$$

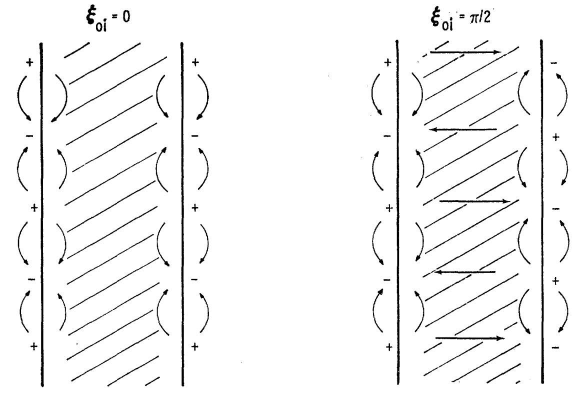

FIG. 1. Distribution of surface poles and dipolar fields for two alternative types of mode.

There is no stationary point $m\neq0$, $n\neq0$, and Eq. (21b) gives the absolute maximum. It is the mode belonging to this eigenvalue that may be expected to nucleate. We note here, incidentally, that we have shown the self-consistency of the assumption made earlier that, for the nucleation mode, (m^2+n^2) is of order S, and κ is of order S^{-1}.

We see from Eq. (21b) that the nucleation mode has the form of incipient strip domains parallel to the z axis. The nucleation field $h_n=-2k+\kappa$ is, of course, very close to the anisotropy field, and the reduced nucleation wavelength $2\pi/m$ agrees reasonably well with measured domain widths.[5] (Comparison with experimental values is difficult because the exchange constants for most materials are not accurately known.) If $h\gg2$ ($k\ll-1$), as is true of several uniaxial ferrites, Eqs. (21a) and (21b) yield nearly the same nucleation fields, and the appearance of meandering strip domains would be consistent with the present calculation.

In order to study the nucleation mode in more detail, we use Eqs. (17) and (18) to obtain the constants A_1 and A_2. Using some further simple approximations, we find

$$A_2\cong\frac{\exp[-|p_2|\pi]}{2|p_2|}A_3=\left(\frac{\pi}{8}\right)^{\frac{1}{2}}\frac{\exp[-|p_2|\pi]}{S}A_3, \quad (22)$$

$$A_1=-\frac{\pi}{2(h-2)[h(h+2)]^{\frac{1}{2}}}\exp[-|p_1|\pi]A_3,\quad k\neq-1.$$

If $k<-1$ ($h>2$), $|p_2|<|p_1|$, and $A_1\ll A_2$; thus A_1 and all the B_i are negligibly small. Also we know from Eq. (20) that p_3 is not exactly $\frac{1}{2}$, but rather

$$p_3=\tfrac{1}{2}-[h/2(h+2)]^{\frac{3}{4}}(\pi S)^{-\frac{1}{2}}.$$

With this information we can get a picture of the mode; a schematic plot of α and β vs ξ appears in Fig. 2.

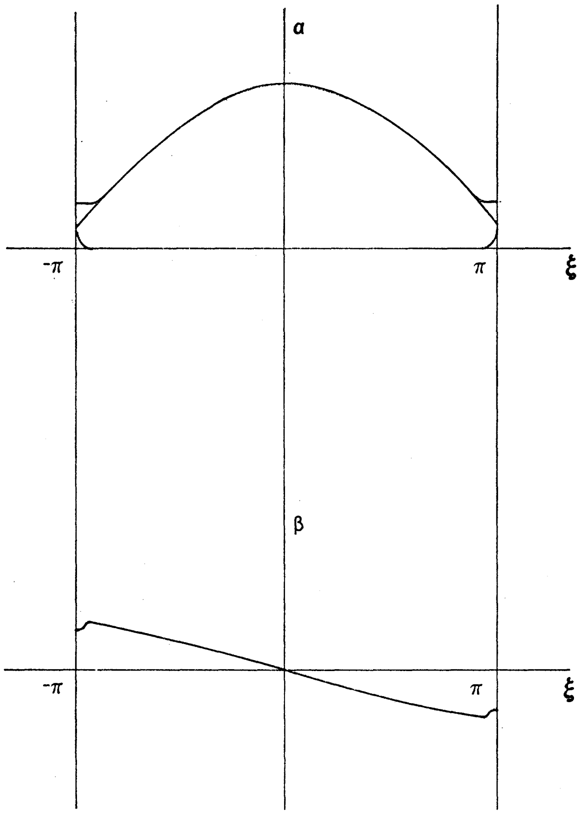

FIG. 2. x and y components of the magnetization in the nucleation mode; $k<-1$ (variation near walls exaggerated).

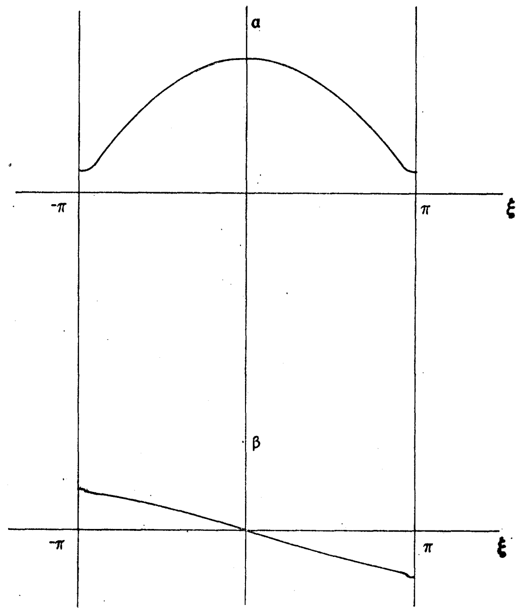

FIG. 3. x and y components of the magnetization in the nucleation mode; $k>-1$ (variation near walls exaggerated).

We next consider the condition $k>-1$ $(h<2)$ so that $|p_2|>|p_1|$. We now find that $A_1 \gg A_2$, so that the term in A_2 is negligible. The form of α is not greatly affected, apart from being smaller near the walls.

Thus a plot of α and β vs ξ will appear as shown in Fig. 3. The change in the form of β shown in the figure is highly suggestive of incipient flux closure domains. Such domains would indeed be expected to form if $H_{\text{anisotropy}}<2\pi M$.

III. FINITE DEVIATIONS FROM SATURATION

The linearized problem we have solved in the preceding section admits superposition and hence cannot provide a description of stable finite deviations from saturation. On the other hand, the nonlinear boundary-value problem of Eqs. (1)–(5) presents considerable mathematical difficulties even to large scale numerical computation. It is the purpose of this section to show that in the present problem some finite amplitude results can be obtained with modest mathematical means. These results may serve as a guide for more ambitious numerical work.

We will now take advantage of the fact (mentioned in Sec. I, above) that, if a ferromagnet is saturated in the hard direction, and the applied field is then reduced slightly below the value required to maintain saturation (the nucleation field h_n), it will remain nearly saturated. This is equivalent to saying that the hysteresis loop in the hard direction is not rectangular; a rectangular loop implies that as soon as the applied field is lowered below the nucleation value, the magnetization abruptly changes to a radically different distribution (complete reversal). Thus in describing magnetization processes in the hard direction one might hope to use some kind of perturbation technique which would not be applicable in a situation leading to a rectangular hysteresis loop.

In the notation which we have been using, if $h=h_n-\delta$ $(\delta \ll h_n)$, then

$$\alpha, \beta \ll 1; \quad \gamma = 1-\tfrac{1}{2}\alpha^2-\tfrac{1}{2}\beta^2. \tag{23}$$

This approximation allows us to write Eq. (1) in a form containing no radicals and no power higher than the third of the dependent variables.

To develop a perturbation procedure, assume (as is possible in principle) that u and β have been eliminated between the torque and field equations. Then we must solve

$$T\alpha = 0,$$

where T is an operator containing linear and nonlinear terms. Assume that the solution is of the form

$$\alpha = \sum_{ij} A_{ij}\alpha_{ij}, \quad A_{ij} \ll 1,$$

where the α_{ij} are normalized functions. We use the double subscript notation because we will want to write

$$\alpha_{ij} = f_i(\xi)g_j(\eta).$$

Now let

$$T = T_0 + \delta + T_1,$$

where T_1 contains all the nonlinear terms, T_0 is linear, and δ is the deviation of the applied field from the nucleation value. In this notation, the linear problem of Sec. II is $T_0\alpha_{00}=0$.

We choose that α_{ij} to form a complete orthogonal set over a suitable rectangle. This can be done easily in one direction: Let $g_j(\eta)=\cos(2j+1)m\eta$. The appropriate $f_i(\xi)$ are complicated, but they can be approximated by trigonometric functions. Then

$$T_0\alpha=T_0\sum A_{ij}\alpha_{ij}=\sum A_{ij}a_{ij}\alpha_{ij}.$$

The a_{ij} are of order unity or higher except $a_{00}=0$; and

$$T\alpha=\sum a_{ij}A_{ij}\alpha_{ij}+\delta\sum A_{ij}\alpha_{ij}+T_1(\sum A_{ij}\alpha_{ij})=0, \quad (24)$$

but in the approximation of Eq. (23) all the terms in $T_1\alpha$ are of the third degree. Thus using the completeness of the α_{ij} we can write schematically

$$T_1(\sum A_{ij}\alpha_{ij})=\sum [A_{ij}^3]\alpha_{ij}, \quad (25)$$

where the brackets mean [of the order of magnitude of]. In this way we can classify the A_{ij} as to order of magnitude in powers of δ, using the orthogonality of the α_{ij}

$$a_{ij}A_{ij}+\delta A_{ij}+[A_{ij}^3]=0. \quad (26)$$

Since $a_{00}=0$, we have from Eq. (25)

$$[A_{00}]=\delta^{\frac{1}{2}}, \quad [A_{ij}]\leq\delta^{\frac{3}{2}}, \quad i+j\neq 0, \text{ etc.}$$

In principle, this procedure will yield all the A_{ij} from the coefficients of successive powers of δ. In practice, the procedure would be exceedingly tedious and not very useful, since we already know that the higher-order terms are exceedingly small. Hence we will only work out A_{00}, A_{01}, A_{10}, and A_{11} in order to show that the distribution of α does tend in the direction to be expected if domains are going to be formed. The equations are

$$\begin{gathered}\left[-\nabla^2+\frac{S^2}{\pi}(h+2k)\right]\alpha-\frac{S^2}{\pi}\delta\alpha+\frac{S}{2\pi}\frac{\partial u}{\partial\xi}\\ -\tfrac{1}{2}\alpha\nabla^2\alpha-\alpha(\nabla\alpha)^2-\frac{S^2}{\pi}k\alpha^3-\frac{S}{4\pi}\alpha^2\frac{\partial u}{\partial\xi}=0,\\ \left[-\nabla^2+\frac{S^2}{\pi}h\right]\beta+\frac{S}{2\pi}\frac{\partial u}{\partial\eta}=0,\\ \nabla^2u=4S\left(\frac{\partial\alpha}{\partial\xi}+\frac{\partial\beta}{\partial\eta}\right).\end{gathered} \quad (27)$$

We cannot neglect $\partial\beta/\partial\eta$ in the field equation and hence we need a linear approximation for the B_{ij}. The trial function is

$$\begin{aligned}\alpha=A_{00}\cos p\xi\cos m\eta+A_{10}\cos 3p\xi\cos m\eta\\ +A_{01}\cos p\xi\cos 3m\eta+A_{11}\cos 3p\xi\cos 3m\eta,\end{aligned} \quad (28)$$

with analogous forms for β and u; $p=\frac{1}{2}$ and m has the value of Eq. (21b). The omission of the term in $\cosh|p_2|\xi$ does not affect the accuracy of the trial function except near the surfaces $\xi=\pm\pi$, where it is not a good approximation.

Substitution of the trial function into Eq. (27) finally yields the values of the A_{ij} as follows:

$$\begin{aligned}A_{00}&=\tfrac{4}{3}[\delta/(-k)]^{\frac{1}{2}};\\ A_{10}&=-[S/9(2\pi)^{\frac{1}{2}}][(1-k)/k^2]^{\frac{1}{2}}\delta^{\frac{3}{2}};\\ A_{01}&=-[3S/8(2\pi)^{\frac{1}{2}}][(1-k)/k^2]^{\frac{1}{2}}\delta^{\frac{3}{2}};\\ A_{11}&=-[S/3(2\pi)^{\frac{1}{2}}][(1-k)/k^2]^{\frac{1}{2}}\delta^{\frac{3}{2}}.\end{aligned} \quad (29)$$

These results are valid only if $\delta\ll S^{-1}$, that is to say, in a very narrow range near saturation. However, they do show the behavior one would expect of incipient domain formation. The sign of the "third harmonic" terms is opposite to that of the fundamental, which indicates a "squaring off" of the distribution. Also, the initial decrease of M_z is linear in the applied field, as is to be expected.

IV. CONCLUSION

It has been the primary purpose of Sec. III to demonstrate the existence of a finite solution for the distribution of magnetization and to outline a method of attack on one aspect of the problem of domain formation. These results are evidently applicable to a fairly wide range of problems in which the hysteresis loop is not rectangular.

The detailed calculation of the nonlinear solution in the specific problem we have treated here is not completely rigorous. For example, it is possible that the fundamental period of the distribution, as well as its shape, will change as the applied field is reduced. It would not be especially difficult to modify the mathematical procedure for this and similar contingencies. However, in view of the limited range of validity of the approximations which have been made in order to permit an analytical treatment, it does not appear very useful to add such refinements. Further progress can probably be made most readily by means of numerical work. Also, other geometrical and magnetic configurations of greater practical interest should be amenable to a similar treatment.

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