electromagnetic interactions only, the theoretical calculation is expected to give an accuracy of about 10%. Within the limits of this theoretical accuracy and the experimental errors already quoted, the experimental scattering distribution is in good agreement with the theoretical predictions.

The experiments on high-energy muon scattering prior to our own work were performed with cosmic-ray muons. These cosmic-ray experiments were cited in our previous paper,[1] and it was pointed out that they are not in agreement with each other regarding the existence of an anomaly in muon scattering. In the latest cosmic-ray experiment by Fukui *et al.*,[6] muons were identified, and their momenta were estimated by requiring the muons to traverse a thick block of iron and stop and decay in a thin layer of carbon. The scattering of these muons was measured by means of a cloud chamber containing lead plates. These authors found no anomaly in the range of momentum transfers up to about 100 Mev/c and cast serious doubt on the anomaly reported by some of the earlier cosmic-ray experiments. Our own work confirms the results of Fukui *et al.*, while avoiding many of the uncertainties connected with cosmic-ray muon experiments.

In summary, the present emulsion experiment supports the conclusion already reached by the counter experiment. The scattering distributions of high-energy muons observed in our experiments are in good agreement with the expected electromagnetic predictions and give no evidence for an anomaly.

[6] S. Fukui, T. Kitamura, and Y. Watase, Phys. Rev. **113**, 315 (1959).

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Cusp Phenomena in the Region of Two Neighboring Thresholds*

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Previous discussions of the cusp phenomena at the threshold for a new reaction are extended to the case of two neighboring thresholds. The S matrix is constructed from an $n\times n$ κ matrix in such a way as to ensure that the physical S matrix is unitary when only r of the n channels are open. As a special application, the amplitude for the reaction $\pi^-+p\rightarrow\Lambda^0+K^0$ is studied in the region of the Σ^- and Σ^0 thresholds.

I. INTRODUCTION

Recently, considerable interest has arisen in the behavior of various cross sections near the threshold for a new reaction. In particular, the possibility of determining the relative parity of the hyperons by study of the production amplitude in a two-body reaction has been pointed out by several authors.[1] Experiments to test this possibility and, hopefully, to measure the (Λ^0,Σ^0) relative parity are in progress.[2]

The previous analyses[3] of the cusp phenomena have been made for a region near a single threshold. However, in the case of actual interest being studied[2] there are really two neighboring thresholds separated in energy by the mass difference $(m_{\Sigma^-}+m_{K^+})-(m_{\Sigma^0}+m_{K^0})$.[4]

It is the purpose of this note to generalize the previous treatments to the case of neighboring thresholds. In so doing, we develop a simple and transparent treatment of the requirements of unitarity on the S matrix in the various energy regions. In Sec. II, the general formalism is set up. For illustration, the well-known results for the case of a single threshold are rapidly rederived in Sec. III. Section IV treats the case when there are two neighboring thresholds, and the results are illustrated and applied to the case of the reaction $\pi^-+p\rightarrow\Lambda^0+K^0$ in the neighborhood of the two ΣK thresholds. Coulomb effects in the (Σ^-,K^+) channel are ignored. A brief summary is given in Sec. V.

II. GENERAL FORMALISM[5]

For simplicity, we consider the case of n coupled two-body channels; each channel consists of one particle of spin zero, and one of spin one-half. We are interested in the submatrix of the S matrix with $J=\frac{1}{2}$ and definite parity.

We adopt the normalization such that the differential cross section for the reaction from channel i to channel j,

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[1] See, for example, R. K. Adair, Phys. Rev. **111**, 632 (1958); A. N. Baz and L. B. Okun, Soviet Phys.-JETP **35**, 526 (1959).

[2] A. M. Schwartz, Bull. Am. Phys. Soc. **5**, 516 (1960); F. S. Crawford, Jr., Bull. Am. Phys. Soc. **5**, 516, (1960).

[3] See, for example, R. G. Newton, Ann. Phys. **4**, 29 (1958), wherein there is a complete list of earlier references; R. G. Newton, Phys. Rev. **114**, 1611 (1959).

[4] W. H. Barkas and A. H. Rosenfeld give 0.6±0.8 Mev. *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), p. 878. F. S. Crawford, Jr., gives 2.2±0.6 Mev, reference 2 above.

[5] The general formalism described herein follows closely the formalism developed by Dalitz and Tuan to treat low-energy K^- p interactions. See R. H. Dalitz and S. F. Tuan, Ann. Phys. **3**, 307 (1960).

in terms of the transition amplitude τ_{ji}, is

$$d\sigma_{ji}/d\Omega_j = k_i^{-2}|\tau_{ji}|^2. \qquad (1)$$

The particles in channel j are assumed to have relative orbital angular momentum zero. Here, k_i denotes the relative wave number of the particles in channel i. The transition amplitudes are related to the S matrix by the matrix equation

$$S = 1 + 2i\tau. \qquad (2)$$

We start by considering the region where all n channels are open. The unitarity of the S matrix is guaranteed if we express τ in terms of a real, symmetric matrix κ by

$$\tau = \kappa(1-i\kappa)^{-1}, \qquad (3)$$

so that

$$S = (1+i\kappa)/(1-i\kappa). \qquad (4)$$

Let M_j be the total rest mass of the particles in channel j. We order the n channels such that $M_1 < M_2 < \cdots < M_n$. Using this ordering, we then wish to distinguish the first r channels from the remaining channels since we shall be interested in the case where only r of the n channels of the S matrix are open.

We write the $n\times n$ κ-matrix in the form

$$\kappa = \begin{pmatrix} L & N \\ \tilde{N} & M \end{pmatrix}, \qquad (5)$$

where L, M, and N are $r\times r$, $(n-r)\times(n-r)$, and $r\times(n-r)$ matrices respectively. $\tilde{N}$ is the transpose of N. We wish to calculate the $r\times r$ submatrix $S^{(r)}$ referring to channels $1\cdots r$ only, in terms of L, M, and N. We use an easily derived formula for inverting a matrix: if A and B are symmetric matrices, then

$$\begin{pmatrix} A & C \\ \tilde{C} & B \end{pmatrix}^{-1} = \begin{pmatrix} X & Z \\ \tilde{Z} & Y \end{pmatrix}, \qquad (6)$$

where

$$\begin{aligned} X &= (A - CB^{-1}\tilde{C})^{-1}, \\ Y &= (B - \tilde{C}A^{-1}C)^{-1}, \\ Z &= -A^{-1}CY. \end{aligned} \qquad (7)$$

Equation (4) then yields

$$S^{(r)} = (1+i\kappa^{(r)})/(1-i\kappa^{(r)}), \qquad (8)$$

where

$$\kappa^{(r)} = L + iN(1-iM)^{-1}\tilde{N}. \qquad (9)$$

We rewrite Eq. (9) as

$$\kappa^{(r)} = L + \Delta\kappa^{(r)}, \qquad (10)$$

where

$$\Delta\kappa^{(r)} = iN(1-iM)^{-1}\tilde{N}. \qquad (11)$$

Let k_i be the relative wave number of the two particles in the ith channel. When all channels are open (all $k_i^2 > 0$), we take[6]

$$\kappa_{ji} = (k_j k_i)^{\frac{1}{2}}\bar{\kappa}_{ji}(E), \qquad (12)$$

where E is the total energy in the center of momentum system; $\bar{\kappa}_{ij}$ is taken to be real and symmetric for all energies $E > M_n$. The proper continuation of k_j through the region where k_j vanishes is well known to be[3]

$$k_j \to +i|k_j|. \qquad (13)$$

If we further assume that $\bar{\kappa}_{ji}(E)$ is real and symmetric for energies $E > M_1$, then in the case when all $(n-r)$ channels are closed, Eq. (12) and (13) imply that M becomes pure imaginary, and that N and $\tilde{N}$ each becomes $\sqrt{i}$ times real matrices. Thus, from Eqs. (10) and (11), $\kappa^{(r)}$ becomes a real, symmetric matrix, and, by Eq. (8), $S^{(r)}$ is manifestly unitary. It should be emphasized that this result is independent of any nonrelativistic approximations and only uses the reality of $\bar{\kappa}_{ji}(E)$ and Eqs. (12) and (13).

Dalitz and Tuan[5] have considered the problem of the analytic continuation of the κ matrix into a region where one or more of the channels is closed. Their results imply that the analyticity requirements on the τ matrix may be incompatible with the assumption that the $n\times n$ κ matrix remains real for all energies $E > M_1$. However, we shall only require that $\bar{\kappa}_{ji}(E)$ be real on the real axis, and analytic in E, in a small region about the thresholds of interest, rather than for all energies $E > M_1$.

In the special case where all the $(n-r)$ channels are near their thresholds, so that the matrix elements of M and N are small compared to those of L, one can treat $\Delta\kappa^{(r)}$ as a small perturbation. Eq. (8) may be rewritten as

$$S^{(r)} = \frac{1+iL}{1-iL} + \Delta S^{(r)}, \qquad (14)$$

where, to lowest order in $\Delta\kappa^{(r)}$,

$$2i\Delta\tau^{(r)} = \Delta S^{(r)} \approx 2i(1-iL)^{-1}\Delta\kappa^{(r)}(1-iL)^{-1}. \qquad (15)$$

We note that the change in cross section corresponding to $\Delta\tau^{(r)}$ is

$$\Delta\sigma_{ji} = k_i^{-2} 2\,\mathrm{Re}\,\tau_{ji}^{*}\Delta\tau_{ji}. \qquad (16)$$

III. SINGLE THRESHOLD

As a simple example, we apply this formalism to the case where only channel n is near its threshold, all other channels being open. Then we take $r = (n-1)$, so that the matrix M reduces to the single element, $k_n\bar{\kappa}_{nn}$; and N becomes a one column matrix with $(n-1)$ elements of the form $(k_n k_i)^{\frac{1}{2}}\bar{\kappa}_{ni}$ $(i=1, \cdots, n-1)$. We wish to display the energy dependence of an element of $S^{(n-1)}$ in the region immediately above and below the threshold for channel n $(E = M_n,\ k_n = 0)$.

[6] The form of Eq. (12) is only appropriate when both channels i and j are in S states. This simplification does not affect the validity of the applications of Sec. III and IV, and there it is only required that the threshold channels be in S states.

Above the threshold, we obtain directly from Eq. (11),

$$\Delta\kappa_{ji}^{(n-1)}=i(N\tilde{N})_{ji}+O(k^2) \\ =ik_n[k_j^{\frac{1}{2}}k_i^{\frac{1}{2}}\bar{\kappa}_{jn}\bar{\kappa}_{ni}]+O(k_n^2). \tag{17+}$$

The quantity in the square bracket is evaluated at the threshold $E=M_n$. Below the threshold, $k_n\to i|k_n|$ and Eq. (17+) becomes

$$\Delta\kappa_{ji}^{(n-1)}=-|k_n|[k_j^{\frac{1}{2}}k_i^{\frac{1}{2}}\bar{\kappa}_{jn}\bar{\kappa}_{ni}]+O(k_n^2). \tag{17−}$$

From Eq. (15) with L evaluated at $E=M_n$, it follows that $\Delta S_{ji}^{(n-1)}$ and hence $S_{ji}^{(n-1)}$, undergoes a 90° right-hand turn when plotted in an Argand diagram, as E passes thru M_n from above. This behavior of S gives rise to the well-known cusp in σ_{ji} at the threshold $E=M_n$.

For the sake of completeness we note[3] that from Eq. (16), using the notation of Eq. (5),

$$\Delta\tau\approx i(1-iL)^{-1}N\tilde{N}(1-iL)^{-1}. \tag{18}$$

The individual elements may be written

$$\Delta\tau_{ji}\approx\tau_{jn}\tau_{ni}, \tag{19}$$

since

$$\tau_{ni}=[(1-iL)^{-1}N]_{ni}+O(k_n^2), \tag{20}$$

where $i=1,\cdots,n-1$. Thus from Eq. (16), we see that the change in the cross section σ_{ji}, due to the existence of the nth channel, is proportional to the product of the amplitudes for the reactions $i\to n$ and $n\to j$ as well as the amplitude for the reaction $i\to j$ itself.

IV. TWO NEIGHBORING THRESHOLDS

In the case of $(n-r)$ channels being near their thresholds, Eq. (18) has the same form, the only change being that N is now an $r\times(n-r)$ matrix, and that, correspondingly, in Eq. (19), the subscript n is summed from $(n-r+1)$ to n.

We illustrate the more general case by application to the reaction $\pi^-+p\to\Lambda^0+K^0$ in the neighborhood of the thresholds, (Σ^0,K^0) and (Σ^-,K^+).

We define the relative momentum in the (Σ^-,K^+), (Σ^0,K^0) channels as k_- and k_0, respectively, and put

$$\Delta M=(M_{K^+}+M_{\Sigma^-})-(M_{\Sigma^0}+M_{K^0}), \tag{21}$$

which is positive.[4] The three regions of interest are, then,

$$\begin{aligned} &\text{Region I,} && E<M_{\Sigma^0}+M_{K^0};\\ &\text{Region II,} && M_{\Sigma^0}+M_{K^0}<E<M_{\Sigma^-}+M_{K^+};\\ &\text{Region III,} && E>M_{\Sigma^-}+M_{K^+}. \end{aligned} \tag{22}$$

In this special case, the S matrix (S_{ij}) is a 4×4 matrix such that $i=1,2,3,4$ refer to channels π^-p, Λ^0K^0, Σ^0K^0, Σ^-K^+, respectively. Thus, $n=4$, $r=2$, and in the notation of Eq. (5), L, M, and N are all 2×2 matrices.

To the lowest order in k_0 and k_-, we can immediately write the S matrix referring to the π^-p and Λ^0K^0 channels only, in the three energy regions of Eq. (23) as:

$$S\cong\frac{1+iL}{1-iL}-2\frac{1}{1-iL}N\tilde{N}\frac{1}{1-iL}. \tag{23}$$

In analogy to Sec. III, L is held constant at its value for say $E=M_{\Sigma^0}+M_{K^0}$. We break up the 2×2 matrix N as

$$(N)=(N_0N_-), \tag{24}$$

where N_0 and N_- are 2×1 matrices. The two elements of N_0, for example, refer to the processes $\pi^-+p\to\Sigma^0+K^0$ and $\Lambda^0+K^0\to\Sigma^0+K^0$. Referring to Eqs. (12) and (13), each of the elements of N_0, N_- has an explicit factor of $(k_0)^{\frac{1}{2}}$ and $(k_-)^{\frac{1}{2}}$, respectively, in region III. As k_0 and k_- go through their respective thresholds,

$$N_{0,-}\to(\sqrt{i})N_{0,-}', \tag{25}$$

so that the elements of $N_{0,-}'$ are real numbers proportional to $|k_0|^{\frac{1}{2}}$ and $|k_-|^{\frac{1}{2}}$, respectively. We can then write Eq. (23) in the three energy regions as

$$\begin{aligned} S^{\mathrm{III}}&\cong\frac{1+iL}{1-iL}-2\frac{1}{1-iL}(N_0\tilde{N}_0+N_-\tilde{N}_-)\frac{1}{1-iL},\\ S^{\mathrm{II}}&\cong\frac{1+iL}{1-iL}-2\frac{1}{1-iL}(N_0\tilde{N}_0+iN_-'\tilde{N}_-')\frac{1}{1-iL},\\ S^{\mathrm{I}}&\cong\frac{1+iL}{1-iL}-2\frac{1}{1-iL}(iN_0'\tilde{N}_0'+iN_-'\tilde{N}_-')\frac{1}{1-iL}. \end{aligned} \tag{26}$$

Just as in the one threshold case, we see from Eq. (26) that the S-matrix element for the process $\pi^-+p\to\Lambda^0+K^0$, when plotted on an Argand diagram, takes a 90° right-hand turn as the energy E decreases through the point $M_{\Sigma^-}+M_{K^+}$, and again at the point $M_{\Sigma^0}+M_{K^0}$. (We emphasize that these considerations hold only for that part of the $\pi^-+p\to\Lambda^0+K^0$ amplitude whose quantum numbers allow the process to be coupled to the S state (Σ^0,K^0) and (Σ^-,K^+) channels. We are here assuming that the (Σ^0,K^0) and the (Σ^-,K^+) relative parities are equal.)

We wish now to display in more detail the energy dependence in regions I, II, and III of $S_{21}=S$ $(\pi^-+p\to\Lambda^0+K^0)$. Since the (Λ^0K^0) channel has isotopic spin quantum numbers $I=\frac{1}{2}$, $I_z=-\frac{1}{2}$, charge independence implies that we can write

$$\begin{aligned} N_0&=-(1/3)^{\frac{1}{2}}k_0^{\frac{1}{2}}\lambda,\\ N_-&=+(2/3)^{\frac{1}{2}}k_-^{\frac{1}{2}}\lambda, \end{aligned} \tag{27}$$

where λ is a 2×1 matrix describing transitions from the (π^-p) and (Λ^0K^0) channels to the $I=\frac{1}{2}$, $I_z=-\frac{1}{2}$, state of the (ΣK) system. Equation (26) implies that

$$S_{21}\cong\left(\frac{1+iL}{1-iL}\right)_{21}-2\left(\frac{1}{1-iL}\lambda\tilde{\lambda}\frac{1}{1-iL}\right)_{21} \times F(|k_0|,|k_-|), \tag{28}$$

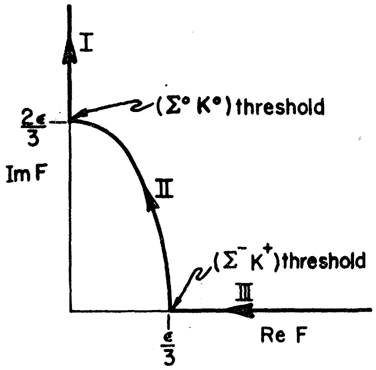

FIG. 1. Argand diagram of the energy-dependent factor $\boldsymbol{F}$ in the S-matrix element for the reaction $\pi^-+p\to\Lambda^0+K^0$ near the (Σ,K) thresholds. [See Eqs. (28) and (29).] The arrows indicate the direction of decreasing total energy E.

where

$$\begin{aligned} F(|k_0|,|k_-|) &= (k_0+2k_-)/3 && \text{in III} \\ &= (k_0+2i|k_-|)/3 && \text{in II} \\ &= (i|k_0|+2i|k_-|)/3 && \text{in I.} \end{aligned} \tag{29}$$

Let $\mu_{\Sigma K}$ equal the average reduced mass of the (Σ^0K^0) and (Σ^-K^+) systems, and $\epsilon^2=2\mu_{\Sigma K}\Delta M$. Then

$$|k_-| = \begin{cases} (k_0^2-\epsilon^2)^{\frac{1}{2}} & \text{in III} \\ (\epsilon^2-k_0^2)^{\frac{1}{2}} & \text{in II} \\ (\epsilon^2+|k_0|^2)^{\frac{1}{2}} & \text{in I.} \end{cases} \tag{30}$$

In Fig. 1, the function $\boldsymbol{F}(|k_-|,|k_0|)$ of Eq. (29) is plotted on an Argand diagram in the energy region near the two (ΣK) thresholds. Note the two successive 90° right-hand turns as E decreases through the two (ΣK) thresholds. In region II, the curve for F is an ellipse, since as follows from Eqs. (29) and (30).

$$(\tfrac{1}{2}\,\mathrm{Im}\,\boldsymbol{F})^2+(\mathrm{Re}\,F)^2=\epsilon^2/9. \tag{31}$$

Note that parts I and III in Fig. 1, when projected, form a right angle.

In a sufficiently small region around the ΣK thresholds the energy variation of S_{21} is dominated by that of F, since all other terms in Eq. (29) may be assumed to be slowly varying with energy. It follows that the Argand diagram for S_{21} will be similar to that of Fig. 1 for F, but displaced and rotated in a way that depends on the dynamical factors L and λ of Eq. (28). Charge independence will still require that the shape of the curve for S_{21} in region II is part of an ellipse of eccentricity $(\sqrt{3}/2)$.

In the limit $\Delta M\to 0$, $\boldsymbol{F}$ and therefore S_{21} exhibits a single 90° right-hand turn corresponding to a single cusp as it must. Note that the over-all turning angle of S_{21} from region III to region I is 90° whether $\Delta M\neq 0$ (two cusps) or $\Delta M=0$ (one cusp).

V. SUMMARY

We have extended the usual treatment of the single cusp phenomena to the case of two or more neighboring thresholds. The process $\pi^-+p\to\Lambda^0+K^0$ near the (Σ^0K^0) and (Σ^-K^+) thresholds has been used to illustrate the formalism. In particular, we have seen that the assumption of charge independence implies that the shape of the curve obtained by plotting the S-matrix element S_{21} on an Argand diagram is elliptical in the region between the 90° right-hand turn taken at each threshold.

It should be emphasized that we have ignored the Coulomb effect in the Σ^-K^+ channel in this analysis. Inclusion of this effect will eliminate the discontinuity in slope of the complex amplitude S_{21} at the Σ^-K^+ threshold.[7] Further work to include the Coulomb effects is in progress. However, for the relatively large mass difference reported by Crawford,[4] we expect that the major features of the energy variation of the amplitude in region II will still be largely as described above.

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We would like to thank M. H. Cha for raising the question as to the effect of two neighboring thresholds on the cusp phenomena.

[7] G. Breit, Phys. Rev. **107**, 1612 (1957); L. Fonda and R. G. Newton, Ann. Phys. **7**, 133 (1959).