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Time in the Quantum Theory and the Uncertainty Relation for Time and Energy

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Because time does not appear in Schrödinger's equation as an operator but only as a parameter, the time-energy uncertainty relation must be formulated in a special way. This problem has in fact been studied by many authors and we give a summary of their treatments. We then criticize the main conclusion of these treatments; viz., that in a measurement of energy carried out in a time interval, Δt, there must be a minimum uncertainty in the transfer of energy to the observed system, given by $\Delta(E'-E) \geqslant h/\Delta t$. We show that this conclusion is erroneous in two respects. First, it is not consistent with the general principles of the quantum theory, which require that all uncertainty relations be expressible in terms of the mathematical formalism, i.e., by means of operators, wave functions, etc. Secondly, the examples of measurement processes that were used to derive the above uncertainty relation are not general enough. We then develop a systematic presentation of our own point of view, with regard to the role of time in the quantum theory, and give a concrete example of a measurement process not satisfying the above uncertainty relation.

1. HISTORICAL SUMMARY OF THE STATE OF THE PROBLEM OF TIME MEASUREMENT IN THE QUANTUM THEORY

As is well known, the uncertainty relations in quantum mechanics can be regarded in two closely related ways. First of all, they are a direct mathematical consequence of the replacement of classical numbers by operators, and of adding the basic principle that the statistical distributions of the corresponding observables can be obtained by means of the usual formulas from the wave function and its probability interpretation.[1] Secondly, however, it can be shown by analyses such as that of the Heisenberg microscope experiment that they are also limitations on the possible accuracy of measurements.[2]

These considerations apply to observables such as x, p, and H. With regard to the measurement of time, however, a further problem appears, because time enters into Schrödinger's equation, not as an operator (i.e., and "observable") but rather, as a parameter, which is a "c" number that has a well defined value. Nevertheless, the uncertainty principle, $\Delta E\,\Delta t \geqslant h$, is generally accepted as valid, even though it is not deduced directly from commutation relations in the way described above.

The justification of the time-energy uncertainty relationship has been attempted in several ways. (We shall restrict ourselves here entirely to a discussion of the nonrelativistic case, since the theory of relativity has no essential relationship to the measurement problems that we are going to treat in this paper.)

First, one can begin with the wave function

$$\psi(x,t)=\textstyle\sum_E C_E\psi_E(\mathbf{x})e^{-iEt/\hbar}, \tag{1}$$

where $\psi_E(x)$ is the eigenfunction of the Hamiltonian H of the system belonging to the eigenvalue E, and C_E is an arbitrary coefficient. If we consider a wave packet of width ΔE in energy space (i.e., ΔE is the range in which $|C_E|$ is appreciable), it immediately follows from the properties of Fourier analysis that $(\Delta E)\tau \geqslant h$, where τ is the time during which the wave packet does not change significantly (τ may be regarded as the mean life of the state in question[3]).

The above is a discussion in terms of the Schrödinger representation. Mandelstamm and Tamm[4] have formulated what is, in essence, the same point of view but it is expressed in the Heisenberg representation. They consider a dynamical variable A, which is a function of the time (e.g., the location of the needle on a clock dial or the position of a free particle in motion) and which can therefore be used to indicate time. If ΔA is the uncertainty in A, then the uncertainty in time is

$$\Delta t=\Delta A/|\langle\dot{A}\rangle_{\mathrm{av}}|,$$

provided that $\dot{A}$ does not change significantly during the time, Δt, and that $\Delta\dot{A}/|\langle\dot{A}\rangle_{\mathrm{av}}|$ is negligible. From the relation

$$\Delta A\ \Delta H \geqslant |\langle(A,H)\rangle_{\mathrm{av}}|=\hbar|\langle\dot{A}\rangle_{\mathrm{av}}|,$$

we obtain

$$\Delta A\ \Delta H/\langle\dot{A}\rangle_{\mathrm{av}}=\Delta t\ \Delta H\geqslant h. \tag{2}$$

Since H represents the Hamiltonian of the isolated system, ΔH is also equal to ΔE, the uncertainty in energy of that system.

It should be noted that the method proposed by Mandelstamm and Tamm can actually lead to a determination of time, only when the system is not in a stationary state; i.e., only when the wave function takes the form of a packet, consisting of a linear superposition of stationary states. In other words, the ΔH appearing in equation (2) is determined by the range of energies

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[1] The uncertainty relations are obtained in this way using Schwarz's inequality with the expressions for

$$\langle(A-\bar{A})^2\rangle_{\mathrm{av}}\langle(B-\bar{B})^2\rangle_{\mathrm{av}}=(\Delta A)^2(\Delta B)^2.$$

See, for example, D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1951), pp. 205–206.

[2] W. Heisenberg, *The Physical Principles of the Quantum Theory* (Dover Publications, New York, 1930), Chap. 2.

[3] V. Fock and N. Krylov, J. Phys. (U.S.S.R.) **11**, 112 (1947), present a more detailed account of the lifetime of a state.

[4] L. Mandelstamm and I. Tamm, J. Phys. (U.S.S.R.) **9**, 249 (1945).

in the wave packet. In this way, the relation of the Mandelstamm and Tamm treatment to the Schrödinger representation is made clear.

The above is then a discussion of the relation $\Delta E\,\Delta t \geqslant h$ insofar as this has been obtained from the mathematical formalism of the quantum theory (i.e., the wave function, operators, and probability interpretation). Naturally, as is necessary in the case of observables such as x and p, this uncertainty relation must also be analyzed in terms of the interaction of the measuring apparatus with the observed system. Landau and Peierls,[5,6] for example, do this by considering a special example, in which the momentum of a free particle is measured by means of a collision with a heavy test particle (also free). To simplify the problem they consider a case in which the measuring particle is a perfectly reflecting mirror, and discuss only the movement in one dimension (perpendicular to the mirror). They then apply the laws of conservation of energy and momentum, which are

$$p'+P'-(p+P)=0, \tag{3a}$$

$$E'+\epsilon'-(E+\epsilon)=0, \tag{3b}$$

where lower case letters refer to the observed particle, capitals to the test particle, unprimed quantities to values before collision, and primed quantities to values after collision. Because $E=P^2/2M$, $\epsilon=p^2/2m$, one can solve for the momentum of the observed particle before and after collision, in terms of the corresponding momenta of the test particle.

In order to define the time of measurement, Landau and Peierls[5] (and also Landau and Lifshitz)[6] consider the case of a time-dependent interaction between the particle and the mirror, which lasts for some known period of time, Δt. This period Δt, which is the uncertainty in the time of measurement, then implies (e.g., according to perturbation theory) an uncertainty in the energy of the combined system consisting of observed particle and mirror, of magnitude $h/\Delta t$, resulting from the time-dependent interaction. Instead of Eq. (3b) for the exact conservation of energy, we must therefore write

$$|\epsilon'+E'-(\epsilon+E)| \geqslant h/\Delta t. \tag{3c}$$

Evidently the momentum of the test particle before and after collision can be measured with arbitrary accuracy, so that $\Delta P=\Delta P'=0$. As a result, we obtain from Eq. (3a), $\Delta p=\Delta p'$; and from (3c), we have

$$\Delta(\epsilon'-\epsilon) \geqslant h/\Delta t. \tag{4}$$

Since $\epsilon=p^2/2m$, we can also write the above result as

$$(v'-v)\Delta p \geqslant h/\Delta t. \tag{5}$$

(Note that although Δp itself may be very small, there is still a minimum uncertainty in energy transfer, because the change of velocity will then become very large, if Δt is finite.)

Landau and Peierls therefore conclude that there is an uncertainty relation between the energy transferred to the system and the time at which the energy is measured. This means that the energy of the observed system cannot be measured in a short time, without changing it in an unpredictable and uncontrollable way. In other words, *energy measurements carried out in short periods of time are not reproducible.*

Fock and Krylov[3] criticize the derivation of the above results, but come to essentially the same conclusion. In effect, they do not accept the definition of the time of measurement by means of a time-dependent potential of interaction between the two particles. This is because in a real collision, there is no such time-dependent potential. Rather, the time of collision is determined by the movement of the particles themselves, in such a way that one of them serves as a clock. Let us suppose that it is the test particle which fulfills this function. This particle defines the time, t, as that at which it passes a definite point, X, by the equation, $t=X/V$. The time, as defined in this way, has an uncertainty $\Delta t=\Delta X/V$ (provided, as will actually be the case in our example, that $\Delta V/V \ll 1$).

In order to define the time of collision, we must have some information about the initial location of the observed particle, as well as that of the test particle. For simplicity, let us suppose that the initial velocity of the test particle is so much higher than that of the observed particle that the latter can be regarded as essentially at rest until the collision. The mean initial position x of the observed particle will be taken to be at the origin, while the uncertainty in this position is represented by Δx. Evidently we must choose $\Delta x \leqslant \Delta X$, if the location of the test particle is to serve as a definition of the time of collision. Therefore

$$\Delta t=\Delta X/V \geqslant \hbar/(\Delta P)V=\hbar/\Delta E.$$

(This is just the well-known uncertainty relation between the energy of the test particle and the time that is defined by the movement of its coordinate.)

Fock and Krylov then point out that in this case, the laws of conservation of energy and momentum are both satisfied *exactly*, so that Eqs. (3a) and (3b) can be used directly, while the approximate form (3c) for the conservation of energy is not to be applied here (since perturbation theory no longer has any relevance to the problem.) From (3a), we obtain

$$\Delta(p-p')=\Delta(P-P').$$

If V is chosen large enough, we can for a given $\Delta t \geqslant \hbar/V\Delta P$ make ΔP and $\Delta P'$ arbitrarily small, and, as a result, we can likewise make $\Delta(p-p')$ as small as

[5] L. Landau and R. Peierls, Z. Physik **69**, 56 (1931).

[6] See also L. Landau and E. Lifschitz, *Quantum Mechanics* (Pergamon Press, New York, 1958), pp. 150–153.

we please. From Eq. (3b), it then follows that

$$\Delta(\epsilon'-\epsilon)=\frac{(p'-p)}{m}\frac{\Delta(p'+p)}{2}=(v'-v)\Delta p=\Delta(\epsilon'-\epsilon)\geqslant\frac{\hbar}{\Delta t},$$

where we have used the result that $\Delta(p'-p)$ is negligible. The above is exactly the same Eq. (5) as that obtained by Landau and Peierls, but the uncertainty in energy transfer to the observed system is now deduced on the basis of the fact that the (time-dependent) position of one of the particles is used to define the time of collision.

Fock and Krylov then go on to criticize the approach of Mandelstamm and Tamm, suggesting that it is incomplete. They assert that by means of the wave function and the operators of the observed system, one can discuss only the statistical features of any measurement. In order to discuss an *individual* measurement process, they refer to what they call "Bohr's uncertainty relation," $\Delta(\epsilon'-\epsilon)\Delta t\geqslant h$, where ϵ and ϵ' are the actual values of the energy of an individual observed system before and after measurement.

To clarify this distinction between the statistical uncertainty relations discussed by Mandelstamm and Tamm, and the Bohr relation, they point out that, for example, in observation of a state with lifetime τ (as described by its wave function), one can make measurements in times much shorter than τ. Therefore, it is necessary to distinguish between the time intervals defined by the wave function of the observed system, and the time interval representing the actual duration of an individual measurement. The time interval defined by the wave function has in measurements generally only a *statistical* significance.

Even if one treated the measurement process by means of a many-body Schrödinger equation, including the apparatus coordinates, the same distinction would arise. For it would the necessary to observe the combined system by means of additional apparatus; and here too, there will be a "Bohr uncertainty principle" for the individual observation and a statistical uncertainty principle following from the wave function, which applies to an ensemble of cases. To treat the apparatus by quantum theory is, in effect, to push back the well known "cut" between classical and quantum sides another stage. While it is always permissible and sometimes convenient to do this, it cannot change the content of the theory.

Let us now sum up the problem. Mandelstamm and Tamm propose a mathematical operator uncertainty relation between energy and time, as determined by the wave function. Fock and Krylov regard such a treatment as incomplete, because it applies only statistically to a large number of measurements, and because within it one cannot even consider the question of the interval of time needed to carry out an individual measurement. To complete the treatment, they call attention to the "Bohr uncertainty relation" (discussed also by Landau and Peierls,[5] as well as by Landau and Lifshitz[6]) which applies to individual measurements, and which refers to the relation between the error in the measurement of energy and the duration of the measurement process. While criticizing some of the methods of Landau and Peierls, they agree with the essential conclusion that energy cannot be measured in arbitrarily short periods of time, without introducing uncertainties, according to the relation $\Delta(\epsilon'-\epsilon)\geqslant h/\Delta t$.

2. CRITICISM OF COMMONLY ACCEPTED INTERPRETATIONS OF THE TIME-ENERGY UNCERTAINTY RELATION

The main conclusion of Landau and Peierls,[5] Landau and Lifschitz,[6] and Fock and Krylov[3] (given in Sec. 1), viz., that energy cannot be measured in a short time without introducing an uncertainty in its value, represents a very widely accepted interpretation of the time-energy uncertainty relation. This conclusion is, as we shall show, erroneous, the error being based in part on an inadequate formulation of Bohr's point of view concerning measurement, and in part on the use of an illustrative example of a measurement process, that was not sufficiently typical of the general case of such a measurement. (In fact, in Sec. 4 we shall give a counterexample, in which the energy of a particle is measured to arbitrary accuracy in as short a time as we please).

With regard to Bohr's point of view concerning measurements, it is important to stress here his continual insistence that the minimum ambiguities in the results of any individual measurement process (in the sense of what Fock and Krylov called the "Bohr uncertainty relations") are always exactly the same as the minimum ambiguities in the possibility of definition in the mathematical theory of the observables that are being measured.[7] (These latter ambiguities are, of course, the "statistical" uncertainty relations referred to by Fock and Krylov.)

The ambiguities in the results of individual measurements are regarded as originating in the indivisible quantum connections of the object under investigation to the apparatus (and indeed to the whole universe), which give rise to a minimum ambiguity in the degree to which well-defined classical properties (e.g., position and momentum) can be assigned to the object as a result of any such measurement.[8] In addition, however, the result of each measurement defines a "quantum state" of the observed system, specified by its wave function. This wave function is, of course, not a representation of an individual system, but it implies, in general, only a set of statistical predictions concerning the results of possible measurements. Nevertheless, if these predictions were such that the minimum ambiguity in the definition of the results of an individual measurement were less than the minimum statistical

[7] N. Bohr, Die Naturwissenschaften, 251 (1928).
[8] W. Heisenberg, *Physics and Philosophy* (Allen and Unwin, London, 1959), Chap. 3.

fluctuation implied by the mathematical theory, then there would be a contradiction. Vice versa, if they were such that the minimum ambiguity in the result of an individual measurement were greater than the minimum statistical fluctuation as described above, then this would lead to an arbitrary restriction, not related in any general way to the mathematical formalism, a restriction that evidently has no place in a coherent over-all framework of theory. Moreover, one could, in general, expect that with sufficient effort, it would be possible to find an example of an individual measurement process with the same minimum ambiguity as that implied by the formalism; and if such a process is found, then the supposedly greater minimum ambiguity in the results of an individual measurement will be contradicted. For these reasons, it is necessary to consider the statistical and individual uncertainty relations as two equally essential sides of what is basically the same limitation on the precise definability and measurability of the state of any system. In other words, as Bohr[9] has stressed, *there can be no limitation on individual measurements that cannot also be obtained from the mathematical formalism and the statistical interpretation.*

There is no question that all the above considerations apply for common examples of the uncertainty principle (e.g., x and p). However, as we pointed out in Sec. 1, time enters into Schrödinger's equation only as a parameter, so that there is no straightforward way to apply these ideas to the time-energy uncertainty relation. Of course, we can, with Mandelstamm and Tamm, obtain an uncertainty relation between the lifetime of a state of the observed system and its energy. But let us recall here that (as pointed out by Fock and Krylov), the operators of the observed system have no connection whatsoever with the duration of the measurement process (which is evidently determined, in general, by the apparatus). Keeping this fact in mind, let us now raise the question of whether there can be a genuine uncertainty relation between the energy transferred to the observed system and the time at which the measurement took place (as has been suggested by Landau and Peierls, Fock and Krylov, and other authors).

In accordance with Bohr's point of view on the subject, as we have described it above, we are led to point out that one cannot safely regard any given uncertainty relations as representing a real limitation on the accuracy of all possible measurements of the quantities under discussion unless the relationship has been shown to follow from the mathematical formalism. On the other hand, all of the authors referred to above seem to be satisfied to establish the time-energy uncertainty relations as applying to individual measurements by what Fock and Krylov called "illustrative examples." Such a point of view would imply, of course, that the uncertainty relations applying to individual measurements could in principle, have a basis that is independent of the statistical relations obtainable from the mathematical formalism. As we have already pointed out, however, such a procedure is arbitrary and therefore subject to the continual danger of being contradicted by the development of new examples of measurement processes, which reduce the ambiguity down to the minimum allowed by the formalism. (For, as is quite evident, there is no way to be sure that conclusions obtained from an illustrative example have universal validity).

[9] N. Bohr (private communication).

It follows from the above discussion that to complete the treatment of the time-energy uncertainty relation, it is necessary to develop a method of showing how the time of measurement and the energy transferred in this measurement are to be expressed in terms of suitable operators. The method that we shall use in this paper starts from our discussion of the example first treated by Landau and Peierls, and then by Fock and Krylov; viz., the one in which the energy of a free particle is measured by collision with another particle. As we saw in Sec. 1, in such an interaction, the time of collision is determined physically by the state of some system which serves as a clock. In the example of Fock and Krylov, it is determined by the position of the test particle (which was taken to be free). Now, for any system, one can define a Hermitian operator representing such a time. In the case of a free particle, this is

$$t_c=\frac{y}{v_y}=\tfrac{1}{2}M\left(y\frac{1}{p_y}+\frac{1}{p_y}y\right), \tag{7}$$

where y and p_y are respectively position and momentum of the particle in question.[10]

The commutation relations between the above operator and the Hamiltonian, H_c, of the "clock" in question are (as can easily be verified for the case of a free particle, for which $H_c=p_y^2/2M$),

$$[H_c,t_c]=i\hbar. \tag{8}$$

This procedure is evidently very similar to that of Mandelstamm and Tamm. However, they discussed only the operators of the observed system, and obtained an uncertainty relation (2) between the energy of this system and the "inner" time as defined by dynamical variables in this system (e.g., the lifetime of a state). On the other hand, we are applying the relations (8) to the energy, E_c, of the "clock" in the apparatus, and to the time, t_c, of measurement as determined by this clock.

Since the time of measurement can be represented by an operator, t_c, belonging only to the observing apparatus, it follows that this *time must commute with every operator of the observed system and, in particular, with its Hamiltonian.* There is therefore no reason inherent in the principles of the quantum theory why the energy of a system cannot be measured in as short a time as we

[10] There is a singularity for $p_y=0$, but it is easily shown that this will be unimportant if p_y is large enough, as will be the case in our applications.

please. (Recall, however, that in accordance with the treatment of Mandelstamm and Tamm, any such a measurement of the energy of the observed system to an accuracy ΔE must leave the "inner" time undefined to $\Delta t \geqslant \hbar/\Delta E$).

In view of the above discussion, it is evident that the usual treatment of the energy-time uncertainty relation (e.g., as discussed by Fock and Krylov) must be in some way erroneous. Since the particular illustrative example chosen by all the authors cited here (which is, in fact, the one usually given) has in fact been treated correctly, it follows that the mistake must be that this example is not sufficiently typical of the general case. And, indeed, as we shall see in Sec. 4, one can suggest more general methods of measurement of energy, which do not lead to the above limitation. In this way, we confirm our conclusion, based on general considerations regarding the principles of the quantum theory, that it is always possible to obtain true limitations on the measurability of any observable from the mathematical formalism, and that any other limitations that are added to these are arbitrary restrictions, which can eventually be contradicted, if further examples of measurement processes are sought.[11]

Finally, it is instructive to point out that problems similar to those connected with the time-energy uncertainty relations arise in the more familiar example of the position momentum relationship, $\Delta p_x \, \Delta x \geqslant \hbar$. To bring out the analogy, we can ask ourselves whether the momentum of a particle can be measured to arbitrary accuracy by means of an apparatus, which is localized in space. (Here, p_x takes the place of E, while the region of space in which the apparatus is located takes the place of the duration, Δt, of the measurement.) At first sight, it may seem that if the apparatus is localized in a very small region of size ΔX, the momentum of a particle cannot be measured to an accuracy greater than $\Delta p_x = h/\Delta X$. This, however, is not the case, because, what is defined here is the coordinate, X, of the apparatus, and not that of the measured particle, x. Since p_x commutes with X, there is no inherent limitation on how accurately it can be measured, if X is defined.

To illustrate such a possibility, consider the measurement of the momentum of a photon, by measuring its energy and using the relation $p = E/c$. (This is analogous to the measurement of energy of a particle by measuring its momentum and using the relation $E = p^2/2M$.) We can do this by means of an atom which is very highly localized, provided that this atom has a sharp level, excited above the ground state by the amount $E = pc$. If the photon has the appropriate energy, it will be absorbed and eventually reemitted (being delayed and perhaps scattered). It is observable whether this happens or not. If it does happen, then this provides a measurement of the energy, and through this, of the momentum. It is evident that the uncertainty in this momentum has no essential relation to the size of the atom, but only to the lifetime of the excited state. The momentum has therefore been measured by an apparatus, which is as localized in space as we please. (Of course, the position of the photon after the measurement is over is indeterminate, just as happens with "inner" time variables in the analogous case of time measurement.)

[11] This conclusion, the validity of which is fairly evident, will be obtained again in Sec. 3 from a more detailed discussion of the mathematical formalism.

3. TREATMENT OF TIME OF MEASUREMENT IN TERMS OF THE MATHEMATICAL FORMALISM OF THE QUANTUM THEORY

We saw in Sec. 1 that (as pointed out by Fock and Krylov[3]) there is a need to make a careful distinction between the time at which a measurement takes place and the time as defined by the wave function and operators of the observed system (e.g., the lifetime of an excited state). In Sec. 2, we showed how such a distinction can be represented within the mathematical formalism of the quantum theory by considering as operators certain variables that have hitherto usually been associated with the observing apparatus; viz., those variables determining the time at which interaction between the apparatus and the observed system takes place. This implies, of course, that the wave function must now be extended, so as to depend on these latter variables, It is equivalent to placing the "cut" between observing apparatus and observed system at a different point.

It is well known that while there is a certain kind of arbitrariness in the location of this cut, it is not completely arbitrary. For example, in the treatment of the energy levels of a hydrogen atom, one can, in a certain approximation, regard the nucleus as a classical particle in a well-defined position. If, however, the treatment aims at being accurate enough to take the reduced mass into account, both electron and nucleus must be treated quantum-mechanically, and the cut is introduced instead between the atom as a whole and its environment. The place of the cut therefore depends, in general, on how accurate a treatment is required for the problem under discussion. Of course, it follows that once a given place of the cut is justified, then it can always be moved further toward the classical side without changing the results significantly.

If we are interested only in discussing what Fock and Krylov called the "Bohr uncertainty relation" (the one referring to an *individual* measurement of energy and time), then we are justified in placing the physical variables that determine the time of the measurement on the classical side of the cut. For, as is quite evident, in this aspect of the uncertainty relations, these variables are by definition regarded as classical, in the sense that their uncertainty represents only an inherent ambiguity in the possibility of defining the state of an *individual* system. We have seen in Sec. 2, however, that according to the Bohr point of view, every uncertainty

relation that appears in this way must also be able to appear as a statistical fluctuation in a corresponding operator, which must, of course, be calculated from an appropriate wave function. To discuss this side of the uncertainty relations, it is clear that we must change the position of the cut, so that the corresponding variables now fall on the quantum-mechanical side.

In the subsequent discussion of how the uncertainty relations appear in the mathematical formalism, we shall begin with the case in which the time determining variables are placed on the classical side of the cut. In this case, the time variables can be reflected in the Schrödinger equation only in the time parameter t which can, of course, have an arbitrarily well-defined value. This time parameter is related to measurement in several ways.

First of all such a relation comes about in the preparation of a system in a definite quantum state, and in observations carried out later on that system. Consider, for example, a quantum state prepared at a time determined by means of a shutter (which we are, of course, now regarding as being on the classical side of the cut). There must be some relationship between the time, t_s, at which the shutter functions and the time parameter, t appearing the Schrödinger's equation. Indeed, if the Hamiltonian of the observed system does not depend on time, then it is easily seen that the wave function takes the form

$$\psi=\psi(\mathbf{x},\, t-t_s), \tag{9}$$

where ψ is a solution of Schrödinger's equation for the system in question. The form of ψ is determined by choosing that solution which at $t=t_s$ becomes equal to the function, $\psi_0(\mathbf{x})$, representing the quantum state in which the "preparing" measurement leaves the system. Then, when an observation is made, the time t_m, of the measurement is likewise determined by suitable variables on the classical side of the cut. The probability of any given result is, of course, computed in the well-known way from the wave function, $\psi=\psi(\mathbf{x},\, t_m-t_s)$.

It is clear that as far as this one-body treatment is concerned, there is certainly nothing in the formalism which would prevent the system under discussion from being either prepared or observed in a state of definite energy, when t_m and t_s are as well defined as we please. Thus, if the system is in a state of definite energy, E (so that the uncertainty, ΔE, in its energy is zero, while the lifetime $\tau \gtrsim \hbar/\Delta E$ of the state is infinite), its wave function, $\psi=\psi_E(\mathbf{x})e^{-iEt/\hbar}$ [where $\psi_E(\mathbf{x})$ is the eigenfunction of the Hamiltonian operator belonging to the eigenvalue, E] is evidently able to represent such a state, no matter what value is given to t. A wave function of this kind is therefore evidently compatible with the statement that at some time, $t=t_s$, the system was prepared in the eigenstate of the energy represented by $\psi_E(\mathbf{x})$. Thus, *in the one-body treatment alone*, no reason for an uncertainty relation between the energy of the system and the time of measurement can be found. And this is indeed basically the reason why Fock and Krylov were led to postulate such an uncertainty relation independently, and to try to justify this relationship by means of illustrative examples of measurement processes (see Sec. 1).

Let us now go on to consider the time-energy uncertainty relation from the other aspect, in which the variables determining the time of measurement are placed on the quantum-mechanical side of the cut. In this case, we must introduce these variables into the wave function, so that we are in this way led to a many-body Schrödinger equation. Let us recall, however, that the "cut" has not been abolished, but merely pushed back another stage. Thus, as was pointed out in the discussion of the treatment of Fock and Krylov given in Sec. 1, there is implied an additional observing apparatus on the classical side of the cut, with the aid of which the many-body system under discussion can be observed. The probabilities for the results of such observations are determined by the wave functions, which take the form

$$\Psi=\Psi(\mathbf{x},y_i,t), \tag{10}$$

where y_i, represents the apparatus variables on the quantum-mechanical side of the cut (which include those that describe the time of measurement). The time parameter t here plays a role similar to that which it had in the one-body problem; viz., through it the time frame on the large-scale classical side of the cut is brought into relationship with the quantum-mechanical formalism by means of suitable observations.

We shall now consider as an example of the approach described above, the measurement of energy and time by means of a collision of two particles, as treated in Sec. 1. The initial wave function of the combined system is, for this case, a product of two packet functions, one representing the test particle coming in with a very high velocity, V, and the other representing the observed particle, essentially at rest (with a velocity that is negligible in comparison to V) and with its center at the origin. After the collision, it is well known[12,13] that the wave function becomes a sum of products of such packets, correlated in such a way that an observation of the properties of the test particle can yield information about the particle under discussion.

As far as this particular example is concerned, it will not be relevant here to go into a more detailed discussion of the problem of solving Schrödinger's equation. All that is important here is that as we saw in Sec. 2, the time of collision is given essentially by the operator,

$$t_c=\tfrac{1}{2}M\left(y\frac{1}{p_y}+\frac{1}{p_y}y\right),$$

(where y and p_y refer to the position and momentum of

[12] See reference 1, Chap. 22.

[13] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955), Chap. 6.

the test particle, respectively), so that the operator, t_c, commutes with the Hamiltonian, H_s, of the observed system. As a result, there is, as we have already stated in Sec. 2, no uncertainty relation between the time of measurement and the energy of the observed system. But in the treatment that we are now using, we have obtained this result by allowing the variables determining the time of measurement to fall on the quantum-mechanical side of the cut. (As was to be expected, of course, the actual physical consequences of the theory do not depend on which side of the cut these variables are placed.)

In the example given above, the same apparatus was used to determine both the time of measurement and the momentum of the particle. However, it is possible, and in fact frequently advantageous, to consider a more general situation, in which the time determining variables are separated from those which are used to measure other quantities (such as momentum). This, in fact, would be the correct way of completing the treatment of the example given by Landau and Peierls[5,6] (see Sec. 1), in which the time of measurement was determined by an interaction between the test particle and the observed particle which was assumed to last for some interval Δt.

If there is a time-dependent interaction between apparatus and observed system which lasts for an interval Δt, then the Schrödinger equation will have to have a corresponding potential, which represents this interaction. The form of this potential will depend on where we place the cut.

If the apparatus determining the time of interaction is taken to be on the classical side, then the potential will be a certain well defined function of time, which is nonzero only in the specified interval of length Δt. We may write this potential as

$$V=V(\mathbf{x},y,t), \tag{11}$$

where $\mathbf{x}$ represents the coordinate of the observed particle, and y that of the test particle. This is indeed the usual way by which measurements are represented in the mathematical formalism.[12,13]

In the next section we shall apply this method in order to treat a specific example, in which it will be shown in detail that the energy of a system can be measured in an arbitrarily short time.

If, on the other hand, the variables determining the time of interaction are placed on the quantum mechanical side of the cut, then we cannot regard the potential as a well-defined function of time. Instead, we must write

$$V=V(\mathbf{x},y,z), \tag{12}$$

where z is the variable that determines the time of interaction.

If the particles determining the time of interaction are heavy enough, then they will move in an essentially classical way, very nearly following a definite orbit, $Z=Z(t)$. To the extent that this happens, we obtain, as a good approximation,

$$V(\mathbf{x},y,z)=V(\mathbf{x},y,z(t)). \tag{13}$$

To treat this problem mathematically, we begin with the Schrödinger equation for the whole system.

$$i\hbar\frac{\partial\Psi}{\partial t}(\mathbf{x},y,z,t)=[H_p+H_y+H_z+V(\mathbf{x},y,z)]\Psi(\mathbf{x},y,z,t), \tag{14}$$

where H_p represents the Hamiltonian of the observed particle, H_y that of the apparatus, H_z that of the time-determining variable, z. We must then show the equivalence of this treatment with that obtained by placing the time-determining variable on the classical side of the cut. To do this, it is sufficient to demonstrate that in a suitable approximation, equation (14) leads to the time-dependent Schrödinger equation for the $\mathbf{x}$ and y variables alone, viz.,

$$i\hbar\frac{\partial\psi}{\partial t}(\mathbf{x},y,t)=[H_p+H_y+V(\mathbf{x},y,z(t))]\psi(\mathbf{x},y,t). \tag{15}$$

We shall simplify the problem[14] by letting the time determining variable be represented by a heavy free particle mass M, for which we have $H_z=p_z^2/2M$. We suppose that the initial state of the time-determining variable can be represented by a wave packet narrow enough in z space, so that $\Delta t=\Delta z/|\dot{z}|$ can be made as small as is necessary. This packet is

$$\Phi_0(z,t)=\sum_{p_z} C_{p_z}\exp\left\{\frac{i}{\hbar}\left[zp_z-\frac{p_z^2}{2M}t\right]\right\}. \tag{16}$$

Because M is very large, the wave packet will spread very slowly, and to a good approximation, we shall have

$$\Phi_0(z,t)=\Phi(z-v_zt)\exp\left\{\frac{i}{\hbar}\left[\bar{p}_zz-\frac{(\bar{p}_z)^2}{2M}t\right]\right\}, \tag{17}$$

where $v_z=\bar{p}_z/M$ is the mean velocity, and $\Phi(z-v_zt)$ is just a form factor for the wave packet which is, in general, a fairly regular function which varies slowly in comparison with the wavelength, $\bar{\lambda}=h/\bar{p}_z$.

If the interaction, $V(\mathbf{x},y,z)$ is neglected, a solution for the whole problem will be

$$\Psi(\mathbf{x},y,z,t)=\Phi_0(z,t)\psi_0(\mathbf{x},y,t), \tag{18}$$

where $\psi_0(\mathbf{x},y,t)$ is a solution of the equation

$$i\hbar\partial\psi_0(\mathbf{x},y,t)/\partial t=(H_p+H_y)\psi_0(\mathbf{x},y,t). \tag{19}$$

When this interaction is taken into account, the solution will, in general, take the form

$$\Psi(\mathbf{x},y,z,t)=\sum_n \Phi_n(z,t)\psi_n(\mathbf{x},y,t)C_n,$$

[14] Our procedure is along lines similar to those developed by H. L. Armstrong, Am. J. Phys. **22**, 195 (1957).

where the sum is taken over the respective eigenfunctions, $\Phi_n(z,t)$ and $\psi_n(\mathbf{x},y,t)$ of H_z and (H_p+H_y). If such a sum is necessary, there will be correlations between the time-determining variable and the other variables, with the result that there will be no valid approximation in which an equation such as (15) involving only $\mathbf{x}$ and y can be separated out. However, if the mass, M, of the time determining particle is great enough, so that the potential $V(\mathbf{x},y,z)$ does not vary significantly in a wavelength, $\lambda=h/p_z$, then, as is well known, the adiabatic approximation will apply. In this case, one can obtain a simple solution, consisting of a single product, even when interaction is taken into account. To show in more detail how this comes about, we first write the solution in the form

$$\Psi=\Phi_0(z,t)\psi(\mathbf{x},y,z,t).$$

When this function is substituted into Schrödinger's equation (15), the result is

$$i\hbar\frac{\partial\psi}{\partial t}=\left[H_p+H_y+V(\mathbf{x},y,z)-\frac{\hbar^2}{M}\frac{\partial\ln\Phi_0}{\partial z}\frac{\partial}{\partial z}-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2}\right]\psi. \quad (20)$$

If M is large, and if the potential does not vary too rapidly as a function of z, the last term on the right-hand side of (20) can be neglected.[15] Moreover,

$$\frac{\partial\ln\Phi_0}{\partial z}=\frac{i}{\hbar}\left[\bar{p}_z+\hbar\frac{\partial\ln\Phi(z-v_zt)}{\partial z}\right].$$

Because $\Phi(z-v_zt)$ does not vary significantly in a wavelength, this term too can be neglected in the above equation, and we obtain

$$i\hbar\frac{\partial\psi(\mathbf{x},y,z,t)}{\partial t}=\left[H_p+H_y+V(\mathbf{x},y,z)-i\hbar v_z\frac{\partial}{\partial z}\right]\psi(\mathbf{x},y,z,t).$$

We then make the substitution, $z-v_zt=u$, and

$$\psi'(\mathbf{x},y,u,t)=\psi(\mathbf{x},y,z,t)=\psi(\mathbf{x},y,u+v_zt,t).$$

With the relation

$$\partial\psi'/\partial t=(\phi\psi/\partial t)+(v_z\partial\psi/\partial z),$$

we have

$$i\hbar\partial\psi'/\partial t=[H_p+H_y+V(\mathbf{x},y,u+v_zt)]\psi'(\mathbf{x},y,u,t). \quad (21)$$

Note that this equation does not contain derivatives of u, so that u can be given a definite value in it.

The complete wave function is, of course, obtained by multiplying $\psi'(\mathbf{x},y,u,t)$ by $\Phi_0(z-v_zt)=\Phi_0(u)$. Now, this was assumed to be a narrow packet centering at $u=0$, such that the spread of u can be neglected. As a result, we can write $u=0$ in the above equation. The result is

$$i\hbar\frac{\partial\psi'(\mathbf{x},y,0,t)}{\partial t}=[H_p+H_y+V(\mathbf{x},y,v_zt)]\psi'(\mathbf{x},y,0,t). \quad (22)$$

In this way, we have obtained the Schrödinger equation for x, y, with the appropriate time-dependent potential $V(\mathbf{x},y,v_zt)$, the relationship between the time parameter t and the time determining variable z being, in this case, $t=z/v_z$. We have therefore completed the demonstration of the equivalence of the two treatments, in which the time-determining variables are placed on different sides of the cut.

4. EXAMPLE OF A REPRODUCIBLE MEASUREMENT OF ENERGY IN A WELL-DEFINED TIME

We saw in Secs. 2 and 3 that there is no reason inherent in the principles of the quantum theory why a reproducible and exact measurement of energy cannot be made in an arbitrarily short period of time. Since Landau and Peierls,[5] Fock and Krylov,[6] and many others have considered examples leading to a contrary conclusion, it is necessary to complete the discussion by giving a specific example of a method of measuring energy precisely in as short a time as we please. This we shall do in the present section. Following the development of our example, it will become clear in what way the previous treatments of this problem were inadequate.

As a preliminary step, we discuss the treatment of the measurement of energy by means of the Schrödinger equation for the apparatus and the observed system together. The Hamiltonian of the combined system is

$$H=H_0(p_x,x)+H_0'(p_y,y)+H_I(p_x,x;\,p_y,y,t), \quad (23)$$

where $H_0(p_x,x)$ is the Hamiltonian of the observed system, $H_0'(p_y,y)$, that of the apparatus, and $H_I(p_x,x;\,p_y,y,t)$ is the interaction, which is zero except during a certain interval of time between t_0 and $t_0+\Delta t$. (Here we are adopting the point of view described in Sec. 3, in which we regard the time determining variables as being on the classical side of the cut, so that they do not appear explicitly in Schrödinger's equation.) It will be adequate for our purposes to assume that both the observed system and the apparatus are free particles, with respective Hamiltonians

$$H_0=p_x^2/2M;\quad H_0'=p_y^2/2M. \quad (24)$$

To simplify the problem, we consider the ideal case of a measurement of p_x which does not change p_x. This will happen if H_I is not a function of x. (The satisfaction of this condition will evidently guarantee that reproducible measurements of p_x, and therefore $H_0=p_x^2/2M$ will be possible). The Hamiltonian of the whole system will then be taken as

$$H=(p_x^2/2M)+(p_y^2/2M)+yp_xg(t), \quad (25)$$

[15] If $V(z,y,z)$ varies too rapidly, then $(\hbar^2/2M)\partial^2\psi/\partial z^2$ will not be negligible, even when M is large.

where $g(t)$ is everywhere zero, except between t_0 and $t_0+\Delta t$, where it is constant. (The interaction Hamiltonian is similar to a vector potential in its effects).

With the Hamiltonian (25), p_x is, of course, a constant of the motion. The equations of motion for the remaining variables are then

$$\dot{x}=(p_x/m)+yg(t), \quad \dot{p}=-p_xg(t), \qquad (26)$$
$$\dot{y}=p_y/M.$$

On solving for p_y, we obtain (using p_x=constant)

$$p_y-p_y{}^0=-p_xg(t)\Delta t. \qquad (27)$$

This equation implies a correlation between $p_y-p_y{}^0$ and p_x, such that if $p_y-p_y{}^0$ is observed, we can calculate p_x.

It is important also to consider the behavior of $\dot{x}$. Although p_x is constant, $\dot{x}$ shifts suddenly at $t=t_0$ from p_x/m to $p_x/m+g(t)y$, and remains at this value until $t=t_i$, after which it returns to its initial value. (In a similar way, the velocity and momentum differ in the case of a vector potential.)

The above behavior of the velocity is, as a simple calculation shows, just what is needed to produce the uncertainty in position, which is required by the improved definition of the momentum resulting from the measurement.

It is easily seen that if $g(t)$ is large enough, the measurement described above can be carried out in as short a time as we please. In order that a given accuracy, Δp_x, be possible, the change of deflection of the apparatus, $\Delta(p_y-p_y{}^0)$ due to the shift Δp_x, must be greater than the uncertainty, $\Delta(p_y{}^0)$ in the initial state of the apparatus. This means that we must have

$$\Delta p_x g(t)\Delta t \cong \Delta p_y{}^0,$$

and if $g(t)$ is large enough, both Δt and Δp_x can be made arbitrarily small for a given $\Delta p_y{}^0$.

This hypothetical example confirms our conclusion once again that accurate energy measurements can be reproduced in an arbitrarily short time. We shall now show how to carry out such a measurement by means of a concrete experiment. To guide us in the choice of this experiment, we note that the essential feature of the interaction described in Eq. (25) is that it implies a force that is independent of the x coordinate of the particle, and which alters the velocity suddenly at $t=t_0$ to bring it back to its original value at $t=t_1$.[16] This force is therefore equivalent in its effect to a pair of equal and opposite pulses in a uniform electric field, the first at t_0 and the second at t_1. In order to approximate such pulses, we shall consider two condensers, the fields of which cross the observed particle at the times t_0 and t_1. The condensers are assumed to have a length, l, in the Y direction which is much greater than their thickness, d, in the X direction. Therefore, they will produce a uniform electric field in the X direction, except for edge effects which can be neglected when $l \gg d$. Each condenser will go by the particle at a velocity, V_y in the Y direction, which is assumed to be so great that the electric field acts for a very short time, l/V_y, with the result that the field approximates the one cited in our mathematical example, where the period of action was infinitesimal. If the two condensers follow each other, one at $t=t_0$, the other at $t=t_1$, then we shall approach the case treated in Eq. (25).

As in the case of the collision treated in Secs. 1 and 3, the time of measurement is defined as the time at which the condenser passes the observed particle. (This means that we are now shifting to the point of view in which the time-determining variables are on the quantum-mechanical side of the cut, but as we saw in Sec. 3, both both points of view are equivalent and can be used interchangeably). As in the case of collision, the uncertainty Δt in this time will be given by $\Delta y/V_y$, provided that the observed particle is initially localized in the Y direction, with a velocity v_y much smaller than V_y. This will imply an uncertainty in the energy of the condenser, $\Delta E_c=V_y\Delta P_y \gtrsim \hbar/\Delta t$.

In the interaction between particle and condenser, the transfer of X component of the momentum (neglecting edge effects) is

$$\Delta p_x=p_x{}^0=F\tau=e\mathcal{E}\tau=e\mathcal{E}l/V_y, \qquad (28)$$

where τ is the time taken by the condenser to pass the particle. (Note that τ and Δt are different quantities). This transfer is independent of initial conditions, and is calculable to arbitrary accuracy. (Since V_y is as large as you please, the Y component of the velocity of the particle can be neglected in the above calculation.) The above transfer of momentum implies a transfer of energy to the particle,

$$\epsilon-\epsilon_0=\frac{(p_x{}^0+\Delta p_x)^2}{2m}-\frac{p_x{}^2}{2m}=\frac{p_x{}^0}{m}\Delta p_x+\frac{(\Delta p_x)^2}{2m}. \qquad (29)$$

By conservation, this transfer must be equal to the energy loss E_0-E_1 of the condenser. Since the initial momentum P_x of the condenser in the X direction is zero, and since the mass M of the condenser is large, the term $(\Delta p_x)^2/2M$ which represents the energy of the condenser due to momentum transfer in the X direction, will be negligible. The energy change must therefore be the result of alteration in the Y component of the condenser momentum, so that it will be equal to $E_0-E_1 \cong V_y\Delta P_y$.

Equation (29) can now be used to permit $p_x{}^0$ to be measured if $E_0-E=\epsilon-\epsilon_0$ is known. For since $\epsilon-\epsilon_0$ depends on $p_x{}^0$ and since Δp_x can be obtained from Eq. (28), $p_x{}^0$ can be calculated in terms of E_0-E_1.

There is, however, a limitation on how accurately E_0-E_1 can be measured because we require that the

[16] In the hypothetical example of Eq. (25), this force resulted from the time-dependent interaction, which was equivalent to a corresponding vector potential, which would produce a field, $\mathcal{E}=-(1/c)\partial\mathbf{A}/\partial t$, that is nonzero only when $\mathbf{A}(t)$ changes; i.e., at the beginning and the end of the interval.

Y coordinate of the condenser shall serve as a clock to an accuracy, δt, with the result that the uncertainty relation, $\delta E \geqslant \hbar/\delta t$ will hold for the energy of the condenser. By conservation, the same uncertainty relation must hold for the energy $\epsilon - \epsilon_0$ transferred to the particle.

By evaluating $\delta(\epsilon - \epsilon_0)$ from Eq. (29), we obtain

$$\delta(\epsilon_0 - \epsilon)\delta t = \delta p_x{}^0 \frac{\Delta p_x}{m} \delta t = \delta p_x{}^0 \Delta v_x \delta t = \delta p_x'(v_x' - v_x)\delta t \geqslant h.$$

This is exactly the same relation as was obtained in the collision example given in Sec. 1. In other words, the measurement on the first condenser alone, must satisfy the condition that if it is carried out in a time defined as δt, there will be an uncertain energy transfer, $\delta E \geqslant h/\delta t$. It is at this point, however, that the second condenser plays an essential role. For immediately after the interaction with the first condenser is over, it will bring about a transfer of X component of the momentum, which is equal and opposite to that transferred to the first condenser. As a result, the velocity of the observed particle will return to its initial value, just as happened in the mathematical example discussed in the beginning of this section. Thus, the momentum and the energy have been measured without their being changed. There is, therefore, no limitation on the accuracy with which the energy of the particle can be measured, regardless of the value of δt, which can be made as small as we please by making V_y very large.

A similar two-stage interaction can be carried out in the collision example described in Sec. 1. To do this, we recall that the uncertainty in energy transfer, $\delta(\epsilon' - \epsilon) = |v - v'|\delta p$ is large because $|v - v'|$ is large. Nevertheless $v - v'$ can be determined with arbitrary accuracy from the results of the measurement. After this is done, one can then send in a second test particle, with initial momentum calculated to be such as to change v' back to v. After the two collisions, there will be, as in the case of the condensers, no uncertainty in energy of the observed particle. In the collision experiment, the change of velocity depends on the value of the momentum of the observed particle, so that the initial conditions in the second collision must be arranged, in accordance with this value, which is learned from the first collision. On the other hand, in the condenser experiment, $v_x' - v_x$ is independent of initial conditions so that the second condenser can be prearranged to cancel out this shift of velocity without any information from the results of the first interaction.

At first sight, one might raise the question as to whether our conclusions could be invalidated by effects of radiation, or by currents which might be induced in the condenser. Since we are discussing only the problem of nonrelativistic quantum mechanics, we can assume that the velocity of light is infinite. In this case, radiation and relativistic effects, in general, can be made negligible, no matter how sudden the shift of potential is. As for currents induced in the condenser, these can be avoided by charging an insulator instead of a metal plate. The field will still be uniform, but the charges will not be mobile, so that no currents will be induced in the condenser.

The error in the treatments of Landau and Peierls, Fock, and Krylov, and others, as discussed in Sec. 1, is now evident. For in all of these treatments, the example used was that of a single collision of a pair of particles. For this case, our own treatment also gave the result that energy transfer in a short time must be uncertain. But as shown in our general canonical treatment of the problem [see Eqs. (25)–(28)], it is clear that this is not the correct way to measure momentum and energy without changing them. To accomplish this purpose we need an interaction of the kind described in the above equations, which changes the velocity only while interaction is taking place, but which brings it back to the initial value after interaction is over. And, as we have seen, it is possible to realize such a measurement in a concrete example.

5. SUMMARY AND CONCLUSIONS

There has been an erroneous interpretation of uncertainty relations of energy and time. It is commonly realized, of course, that the "inner" times of the observed system (defined as, for example, by Mandelstamm and Tamm[4]) do obey an uncertainty relation $\Delta E\,\Delta t \geqslant h$, where ΔE is the uncertainty of the energy of the system, and Δt is, in effect, a lifetime of states in that system. It goes without saying that whenever the energy of any system is measured, these "inner" times must become uncertain in accordance with the above relation, and that this uncertainty will follow in any treatment of the measurement process. In addition, however, there has been a widespread impression that there is a further uncertainty relation between the *duration* of measurement and the *energy transfer* to the observed system. Since this cannot be deduced directly from the operators of the observed system and their statistical fluctuation, it was regarded as an additional principle that had to be postulated independently and justified by suitable illustrative examples. As was shown by us, however, this procedure is not consistent with the general principles of the quantum theory, and its justification was based on examples that are not general enough.

Our conclusion is then that there are no limitations on measurability which are not obtainable from the mathematical formalism by considering the appropriate operators and their statistical fluctuation; and as a special case we see that energy can be measured reproducibly in an arbitrarily short time.

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