Effect of Interactions on Determination of Fermi Surfaces*

EDWARD A. STERN
Department of Physics, University of Maryland, College Park, Maryland
(Received September 6, 1960)

The effect of both electron-electron and electron-phonon interactions on a degenerate electron gas in a uniform positive background is considered. It is shown that when electron-electron interactions alone are considered the free-electron mass is still measured by cyclotron resonance, the Faraday effect, and optical constants. However, the period of the de Haas-van Alphen oscillations is changed from what one calculates neglecting interactions and is changed in the same way that the specific heat is. When electron-phonon interactions are added everything changes. In particular, it is shown that the cyclotron mass is no longer the free value, and the de Haas-van Alphen period and the specific heat are changed in different ways. Comparison with measurements on aluminum which approximates the model used shows that both electron-phonon and electron-electron effects are important and of the same magnitude.

I. INTRODUCTION

IN recent years various experimental techniques have been used to obtain information on the Fermi surfaces of metals.[1] The results of measurements such as the electronic specific heat, cyclotron resonance, Faraday effect, and optical constants as a function of wavelength can be expressed in terms of effective masses of the electrons. These effective masses can then be related by theory to some properties of the Fermi surfaces. Other techniques such as de Haas-van Alphen effect, anomalous skin effect, magnetoresistance, and magnetoacoustic effect give information directly on the shape of the Fermi surfaces. The interpretation of these experimental measurements in terms of electron band structure has usually neglected the many-body effects such as electron-electron interactions or correlation effects, and electron-phonon interactions. We will consider here first the effects of electron-electron interactions, and for the simplified model of an electron gas imbedded in a uniform smeared-out positively charged background. In this model we will calculate exactly in Sec. II the currents induced by uniform electric and magnetic fields. It is shown that the currents are the same as produced in a noninteracting gas of electrons acted upon by self-consistent fields. Thus the effective masses measured by cyclotron resonance, Faraday effect, and optical constants measure the free electron mass independent of the interactions. By interpreting the currents induced by uniform magnetic fields in terms of quasi-particle states near the Fermi surface, the motion of these quasi-particles in uniform magnetic fields is obtained in Sec. III. From this, by a semiclassical argument, the orbits of the quasi-particles are quantized and the period of the de Haas-van Alphen oscillation is calculated. It is found that in this case electron-electron interactions *do* change the period from what it is in the noninteracting case. Electron-phonon interactions are added in Sec. IV and their effect on the quasi-particle states is determined. From this, expressions are obtained for the specific heat, de Haas-van Alphen period, and the cyclotron frequency. The theory is compared with experiment and it is shown that electron-phonon effects are of the same magnitude as electron-electron effects and both are not negligible. A summary and discussion is included in Sec. V.

II. UNIFORM ELECTRIC AND MAGNETIC FIELDS

The approach employed here uses elementary, non-field-theoretic, ordinary quantum mechanics. The motion of the gas is separated into a motion about the center of mass and the motion of the center of mass. The total current is determined only by the motion of the center of mass. Fortunately, in the electron gas model the motion of the center of mass is unaffected by interactions between electrons and is determined by the external fields alone, a problem that can be solved exactly.

It is important to choose a model for our electron gas with the correct boundary conditions since they are important. We do not choose an infinite gas because it is difficult to account for any effects at infinity. Further on, it will become clear that these boundary effects are not negligible. In fact it will be shown that charge accumulates at the boundary under the action of an external field and modifies the actual electric field seen by the electrons. It is further shown that this effect is correctly taken into account in the usual manner of using self-consistent fields. The model that we will choose is a uniform smeared out positive background of charge density ρ in a spherical volume of radius a. Imbedded in the positive background are N electrons composing the interacting electron gas. The total electronic charge of $-Ne$ is assumed to be much less than the total positive background charge. The electrons will be concentrated in the center of the positive background within a radius small compared to a. In fact the electrons distribute themselves, except for boundary effects which can be neglected if the volume

* This work was performed by the author as a consultant to the RAND Corporation, Santa Monica, California.
[1] An up to date review of the field can be obtained from the *Proceedings of the Conference on Fermi Surfaces in Metals* (John Wiley & Sons, Inc., New York, 1960).

is large enough, so as to produce macroscopic charge neutrality in the region that they occupy. The number of electrons N is taken to be very large. By choosing atomic units in which $\hbar=e=2m=1$ the Hamiltonian for the electrons in uniform electric and magnetic fields is given by

$$H=\sum_i\left(\frac{\nabla_i}{i}+\frac{1}{c}\mathbf{A}(\mathbf{r}_i,t)\right)^2+\tfrac{1}{2}\sum_{ij}\frac{1}{r_{ij}}+4\pi\sum_i n\frac{r_i^2}{6}. \quad (1)$$

The first term on the right is the kinetic energy operator of the electrons in the fields represented by $\mathbf{A}(\mathbf{r},t)$. The position of the ith electron is at $\mathbf{r}_i$. The magnitude of $\mathbf{r}_i$ is denoted by r_i and $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$. The charge of an electron is -1 in atomic units. We will be interested in the case of a uniform electric field oscillating in time as $\sin\omega t$ and a uniform magnetic field. In this case

$$\mathbf{A}(\mathbf{r},t)=\mathbf{E}c\frac{\cos\omega t}{\omega}-\mathbf{j}Hr_1, \quad (2)$$

where $\mathbf{E}$ is the uniform electric field, $\mathbf{j}$ is a unit vector in the y direction, H is the magnitude of the uniform magnetic field in the z direction, and r_1 is the x component of $\mathbf{r}$. The second term on the right of (1) is the Coulomb interaction energy between the electrons themselves, and the last term is the interaction energy between the electrons and the uniform positively charged background. Because of macroscopic charge neutrality $\rho=n$ where n is the number of electrons per unit volume.

We now introduce center-of-mass coordinates R and relative coordinates S by the usual relationship,

$$N\mathbf{R}=\textstyle\sum_i\mathbf{r}_i \quad \text{and} \quad \mathbf{S}_i=\mathbf{r}_i-\mathbf{R}. \quad (3)$$

Not all of the $\mathbf{S}_i$ are independent since they satisfy the condition that $\sum_i\mathbf{S}_i=0$. Of the N variables $\mathbf{S}_i$, only $N-1$ of them are independent. In terms of these new coordinates,

$$\nabla_{j\alpha}\equiv\frac{\partial}{\partial r_{j\alpha}}=\frac{1}{N}\frac{\partial}{\partial R_\alpha}+\sum_i\frac{\partial}{\partial S_{i\alpha}}\left(\delta_{ij}-\frac{1}{N}\right). \quad (4)$$

Here α has the values of 1, 2, 3 denoting the three Cartesian components of the vectors and δ_{ij} equals one if $i=j$ and zero otherwise. The partial derivatives $\partial/\partial S_{i\alpha}$ are not exactly components of the momentum operator since the S_i are not all independent variables.

If both sides of (4) are summed with respect to j, we obtain

$$\sum_i\frac{\partial}{\partial r_{j\alpha}}=\frac{\partial}{\partial R_\alpha}, \quad (5)$$

which simply states that the total momentum operator is just the momentum operator of the center of mass.

The vector potential in (2) becomes, in the new coordinates,

$$\mathbf{A}(\mathbf{S}_i,\mathbf{R},t)=\mathbf{E}c\frac{\cos\omega t}{\omega}+\mathbf{j}H(S_{i1}+R_1). \quad (6)$$

The total current operator $\mathbf{J}$ is

$$\mathbf{J}=\frac{-e}{mc}\sum_i\left(\nabla_i/i+\frac{1}{c}\mathbf{A}(\mathbf{r}_i,t)\right)$$
$$=\frac{-e}{cm}\left(\nabla_R/i+\frac{1}{c}\mathbf{A}(\mathbf{R},t)\right), \quad (7)$$

where ∇_R has the components $\partial/\partial R_\alpha$, and $\mathbf{A}(\mathbf{R},t)=\mathbf{E}c(\cos\omega t/\omega)-\mathbf{j}HR_1$. The current depends only on the center-of-mass coordinates so that our goal is to find the equation of motion of the center of mass.

The Hamiltonian of (1) becomes, using the relations in (4) and the subsidiary condition $\sum_i\mathbf{S}_i=0$,

$$H=H_R+H_i,$$

where

$$H_R=\frac{1}{N}\left(\frac{\nabla_R}{i}+\frac{1}{c}N\mathbf{A}(\mathbf{R},t)\right)^2+4\pi\frac{NnR^2}{6},$$

$$H_i=\sum_i\left(\frac{\nabla_{Si}}{i}+\mathbf{j}HS_{i1}\right)^2-\frac{1}{N}\Big(\sum_i\nabla_{Si}\Big)^2$$
$$+\tfrac{1}{2}\sum_{i,j}\frac{1}{S_{ij}}+\sum_i\frac{nS_i^2}{6}, \quad (8)$$

and where ∇_{Si} has the components $\partial/\partial S_{i\alpha}$. The Hamiltonian divides into two parts, one part being a function only of the center of mass coordinates and the other part being a function only of the relative coordinates. It is of interest to note that the electric field has no effect on the relative coordinates. The appearance of the term $(\sum_i\nabla_{Si})^2$ in H_1 is related to the fact that ∇_{Si} are not the true momentum operators for the problem. If N is large enough then one would expect that the subsidiary condition has a negligible effect and the ∇_{Si} are to a good approximation the true momentum operators. In that case the term in question becomes zero since the total momentum about the center of mass is zero.

The Hamiltonian for the center of mass coordinates H_R describes the motion of a single particle of mass Nm and charge $-Ne$ [in (8) atomic units are used where $2m=e=\hbar=1$] being acted upon by the uniform electric and magnetic fields and the positive background. Because of the large mass of this center-of-mass particle, the motion of the center of mass can accurately be solved for classically. The classical equations of

motion of the center of mass are

$$\begin{aligned} Nm\frac{d^2R_1}{dt^2} &= -\frac{4\pi Ne^2nR_1}{3} - NeE_1\sin\omega t - NeH\frac{dR_2}{dt}, \\ Nm\frac{d^2R_2}{dt^2} &= -\frac{4\pi Ne^2nR_2}{3} - NeE_2\sin\omega t + NeH\frac{dR_1}{dt}, \\ Nm\frac{d^2R_3}{dt^2} &= -\frac{4\pi Ne^2nR_3}{3} - NeE_3\sin\omega t. \end{aligned} \tag{9}$$

It is important to note that the electric field

$$\mathbf{E}=(E_1,E_2,E_3)$$

in (9) is the external applied field and not the actual electric field seen by the electrons. The actual field inside the medium which is in the form of a sphere is given by the well-known result of electrostatics,

$$\mathbf{E}_{\text{in}}=\mathbf{E}[3/(\epsilon+2)], \tag{10}$$

where ϵ is the dielectric constant of the medium. We can solve (9) to obtain $\mathbf{R}$ and thus the polarization of the medium per unit volume given by $\mathbf{P}=-ne\mathbf{R}$. The dielectric constant is defined by

$$\epsilon\mathbf{E}_{\text{in}}=\mathbf{E}_{\text{in}}+4\pi\mathbf{P},$$

or, using (10),

$$\epsilon\left(\mathbf{E}+\frac{4\pi}{3}\mathbf{P}\right)=\mathbf{E}-\frac{8\pi}{3}\mathbf{P}. \tag{11}$$

From symmetry arguments alone it can be shown that the dielectric constant of a uniform gas in a magnetic field must be a tensor of the following form.

$$\epsilon=\begin{bmatrix} \epsilon_{xx} & i\epsilon_{xy} & 0 \\ -i\epsilon_{xy} & \epsilon_{xx} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}. \tag{12}$$

Instead of working with a tensor ϵ it is possible to simplify the problem by using right (r) and left (l) circularly polarized electric fields defined as

$$\begin{aligned} \mathbf{E}_r &= \text{Re}[E(\mathbf{i}+i\mathbf{j})e^{-i\omega t}], \\ \mathbf{E}_l &= \text{Re}[E(\mathbf{i}-i\mathbf{j})e^{-i\omega t}], \end{aligned}$$

where Re means real part of. Defining the displacement vector for right and left circularly polarized cases in an analogous way, the dielectric constants relating $\mathbf{E}_r$ and $\mathbf{D}_r$ and $\mathbf{E}_l$ and $\mathbf{D}_l$ are scalars given by

$$\begin{aligned} &\mathbf{D}_r=\epsilon_r\mathbf{E}_r, \quad \epsilon_r=\epsilon_{xx}+i\epsilon_{xy}, \\ \text{and} \qquad &\mathbf{D}_l=\epsilon_l\mathbf{E}_l, \quad \epsilon_l=\epsilon_{xx}-i\epsilon_{xy}. \end{aligned} \tag{13}$$

It is more convenient to work with ϵ_r and ϵ_l. These scalar dielectric constants can be found from (11) with the understanding that ϵ, $\mathbf{P}$, and $\mathbf{E}$ all have the same subscript of r or l. The circularly polarized polarization vectors $\mathbf{P}_r$ and $\mathbf{P}_l$ are defined analogously to $\mathbf{E}_r$ and $\mathbf{E}_l$.

It is a straightforward but tedious matter to solve (9) and obtain

$$\frac{4\pi P_r}{3E_r}=-\frac{\omega_0^2}{\omega^2\epsilon_0[1+(\omega_c/\omega\epsilon_0)]}, \tag{14}$$

where $\omega_0^2=\omega_p^2/3$, $\omega_p^2=4\pi ne^2/m$, $\epsilon_0=1-\omega_0^2/\omega^2$, and $\omega_c=eH/mc$. The classical plasma frequency of the electron gas is $\omega_p/2\pi$. Finally ϵ_r can be obtained from (11) and (14), giving

$$\epsilon_r=\frac{\epsilon_f+\omega_c/\omega}{1+\omega_c/\omega}, \tag{15}$$

where $\epsilon_f=1-(\omega_p/\omega)^2$.

In the same way one can show that

$$\begin{aligned} \epsilon_l &= \frac{\epsilon_f-\omega_c/\omega}{1-\omega_c/\omega}, \\ \epsilon_{xx} &= \frac{\epsilon_f-(\omega_c/\omega)^2}{1-(\omega_c/\omega)^2}, \\ \epsilon_{xy} &= \frac{\omega_c\omega_p^2}{\omega^3[1-(\omega_c/\omega)^2]}, \end{aligned} \tag{15'}$$

and

$$\epsilon_{zz}=\epsilon_f.$$

Equations (15) and (15′) determine completely and exactly the behavior of the dielectric constant of a electron gas in a uniform magnetic field. The result is correct for all values of magnetic field, large or small; for all densities of the gas, even for the very tenuous region where the electrons form a lattice; and for all temperatures of the gas, from the degenerate region to the classical limit. The cyclotron resonance occurs at the poles of ϵ_l and ϵ_r, giving $\omega_{\text{cyclotron}}=\pm eH/mc$ and showing that the effective mass determined by cyclotron resonance in an electron gas is exactly the free electron value. The long wavelength plasma resonances occur at the zeros of ϵ_l and ϵ_r giving

$$\omega_{\text{plasma}}=\tfrac{1}{2}\omega_c\pm[(\tfrac{1}{2}\omega_c)^2+\omega_p^2]^{\frac{1}{2}}. \tag{16}$$

The results given in (15) and (15′) are exactly the same results one would obtain from (9) if all boundary effects and the background term proportional to R are neglected, and the electric field is treated in a self-consistent way. By a self-consistent treatment of the electric field is meant that the field seen by the electrons is the external field plus the induced electric field assuming no local-field corrections, i.e., the macroscopic induced electric field is the actual induced electric field seen by the electrons. Therefore, if one wants to neglect all boundary effects, as it will be done from now on in this paper, it is possible to do so by replacing the

Hamiltonian H_R in (8) by

$$H_R=\frac{1}{N}\left(\frac{\nabla_R}{i}+\frac{N}{c}\mathbf{A}(\mathbf{R},t)\right)^2, \qquad (8')$$

where now $\mathbf{A}$ is the vector potential of the magnetic field and the self-consistent electric field. All magnetic interactions between electrons have been neglected because in practice these are small. However, if included, they can be taken account of by using a self-consistent magnetic field.

Because the dielectric constant given in (15) and (15′) is the same as if interactions between electrons are neglected except for treating the electric field in a self-consistent way, effective masses measured by the Faraday effect and by the wavelength dependence of the optical dielectric constant are just the free electron value.

In the above discussion an implicit assumption has been made that requires further justification. It has been assumed that the spacial variation of the electric field in the experimental measurements can be neglected. For optical properties such as the Faraday effect and optical constants this is the usual approximation that is used, and we require no further justification. However, in the usual Azbel'-Kaner type of cyclotron resonance experiments in metals the measurements are made in the extreme anomalous skin effect region where it is not possible to relate the currents and electric fields by a local relationship such as a dielectric constant. Nevertheless the dielectric constants in (15) and (15′) do show that in the presence of a uniform magnetic field there are excited states of the system at energies $n\hbar\omega_c$ above the ground state, n an integer, which can be excited by a uniform electric field. The uniform electric field assures that the excited states have the same total momentum as the ground state. A nonuniform electric field, as is present in cyclotron resonance experiments, can transfer momentum to the excited states. However, if the spacial variation of the electric field in the distance of a wavelength corresponding to the Fermi momentum of the electrons can be neglected, as is the case, then the momentum transferred by the electric field can be neglected and an electric field in a cyclotron resonance experiment oscillating near the frequencies $n\omega_c$ should cause transitions to excited states. It follows from the above reasoning that although the calculation at uniform electric and magnetic fields cannot predict the absorption at subharmonics and the shape of the absorption curves in cyclotron resonance experiments in metals, the fact that there is an absorption at ω_c does follow from the uniform-field case.

III. QUASI-PARTICLES AND THE DE HAAS-VAN ALPHEN EFFECT

In this section we use the results of the previous section to derive some of the properties of quasi-particles from which it will turn out to be possible to calculate the period of the de Haas-van Alphen oscillations. A physical picture of a quasi-particle[2] can be obtained by considering the ground state of the electron gas to be analogous to the vacuum state in quantum electrodynamics. Excited states consisting of the excitation of a small fraction of the total number of particles can be described in terms of a "vacuum" state together with a small number of particle-hole pairs. As usual, a hole corresponds to a vacancy in an energy state below the Fermi energy and a particle to a filled energy state above the Fermi energy. Again, in analogy with quantum electrodynamics the electron gas acts like a dielectric medium whose dielectric properties are wave number and frequency dependent. A particle or hole polarizes the electron gas in its vicinity. A quasi-particle consists of an electron and its polarization cloud that it carries along with it. The interaction of an electron with its own polarization cloud produces the difference in energy between a quasi-particle and a bare or free electron. The polarization cloud screens the charge of the electron so that quasi-particles interact with one another via only a weak short-range interaction.

It has been shown[3–5] that it is rigorously correct to talk about quasi-particles only at the Fermi surface. Single-particle or quasi-particle excitations damp out with a lifetime proportional to $(p-p_0)^{-2}$, where p is the momentum of the excited particle and p_0 the Fermi momentum. In this discussion we limit ourselves to quasi-particles very close to the Fermi surface so that the damping is negligible. In fact, it is only in this case that it appears reasonable to talk about quasi-particles at all. In summary, it has been shown elsewhere that in an interacting electron gas in no external fields there exist single-particle states, called quasi-particles, which have a precise momentum and energy for momenta very near the Fermi momentum. The energy of these quasi-particles we will denote by $\epsilon(p)$, where $\mathbf{p}$ is the momentum of the quasi-particle. From the spherical symmetry of the problem ϵ is just a function of the magnitude of $\mathbf{p}$.

Consider a system of N electrons in the ground state of zero total momentum. We add one quasi-particle of momentum $\mathbf{p}$ to the system. The momentum of the center of mass of the system is also $\mathbf{p}$. The total current of the system is, by (7),

$$\mathbf{j}=-e\mathbf{p}/mc. \qquad (17)$$

The velocity of a quasi-particle is given by[6]

$$\mathbf{v}=\nabla_p\epsilon(p), \qquad (18)$$

where ∇_p is the gradient operator with respect to the momentum. Since the current $\mathbf{j}$ is all carried by the

[2] J. J. Quinn and R. A. Ferrell (to be published).
[3] J. J. Quinn and R. A. Ferrell, Phys. Rev. **112**, 812 (1958).
[4] N. M. Hugenholtz, Physica **23**, 481 (1957).
[5] D. F. DuBois, Ann. Phys. **7**, 174 (1959); **8**, 24 (1959).
[6] L. D. Landau, Soviet Physics—JETP, **3**, 920 (1957).

quasi-particle, it must have an effective dynamical charge e^* given by

$$e^*=-ep/mv. \tag{19}$$

If we define an effective dynamical mass m^* given by

$$m^*=p/v, \tag{20}$$

then

$$e^*=-em^*/m. \tag{21}$$

Equation (17) tells us that a quasi-particle of momentum $\mathbf{p}$ carries the same current as a free electron of the same momentum. This can be understood physically when one remembers that the momentum is all carried by electrons with the same charge over mass ratio. Any mass flow, no matter how it is distributed among the electrons, must produce a charge flow e/m times the mass flow.

Now consider the same electron gas in its ground state in a magnetic field. At a certain time, say $t=0$, we add to the electron gas a quasi-particle of mechanical momentum $\mathbf{p}$. Transferring to center-of-mass and relative coordinates, at $t=0$ the center of mass also has mechanical momentum $\mathbf{p}$. The center of mass moves like a single particle of mass Nm and charge $-Ne$ as can be seen from (8′). Equation (8′) implies that

$$\frac{d\mathbf{p}}{dt}=\frac{e\mathbf{p}}{mc}\times\mathbf{H}=\frac{e^*}{c}\mathbf{v}\times\mathbf{H}. \tag{22}$$

Since $\mathbf{p}$ is also the momentum of the quasi-particle, the quasi-particle momentum also satisfies (22). This is just the equation of motion we would expect for the quasi-particle of effective dynamical charge e^* and moving with a velocity $\mathbf{v}$.

It is important to emphasize that the effective dynamical charge e^* is not in general the effective charge of the quasi-particle. We have only shown that it is the effective charge in a uniform magnetic field and for calculating the current carried by a quasi-particle. The interaction between two quasi-particles is not like that between two particles of charge e^* but through a screened interaction. In an electric field, the acceleration of the quasi-particle is not that of a charge e^* but of a charge e because not only is the quasi-particle accelerated in an electric field, but so is the rest of the gas. The acceleration of the rest of the gas takes it out of its ground state changing the interaction between the quasi-particle and the rest of the gas and affecting the acceleration of the quasi-particle. This effect does not occur with a uniform magnetic field because stationary ground state solutions for the gas exist in a magnetic field and only the additional quasi-particle state varies.

We quantize the quasi-particle orbits, using the Bohr-Sommerfeld rule in the form

$$\oint \mathbf{\Pi}\cdot d\mathbf{r}=(n+\tfrac{1}{2})h, \tag{23}$$

where $\mathbf{\Pi}=\mathbf{p}+(e^*/2c)\mathbf{H}\times\mathbf{r}$ is the conjugate momentum in a magnetic field. Following Onsager,[7] we integrate (22) and choose an origin so that the constant of integration is zero, obtaining $\mathbf{p}=(e^*/c)\mathbf{r}\times\mathbf{H}$. Substituting into (23) gives the relation

$$A_p c/e^*H=(n+\tfrac{1}{2})h, \tag{24}$$

where $A_p=\pi p_H^2$ and p_H is the component of $\mathbf{p}$ perpendicular to H. The degeneracy of each level in (24) is $H\Omega e^*/hc$, where Ω is the cross section of the electron gas, just as in the noninteracting case.[7] Unfortunately, the concept of quasi-particles is only good near the Fermi surface so that we cannot describe the total state of the gas in terms of the levels of (24), in contrast to the noninteracting gas. However, we can divide the energy of the system into two parts. One part is the quasi-particle states just below the Fermi surface and the other is the rest of the energy. These quasi-particles just below the Fermi surface are located in momentum space on the quantized levels given by (24). It is being assumed that they also act to a good approximation like noninteracting particles, and therefore the usual independent-particle arguments[7] can be used to show that these quasi-particles alone produce an oscillatory part to the susceptibility. The period is determined when, as the magnetic field is varied, states in Eq. (24) differing by one in n pass through the Fermi surface. If we make the further reasonable assumption that the susceptibility produced by the rest of the system varies smoothly with magnetic field, then the period of the de Haas-van Alphen effect is given by the quasi-particles states alone and is

$$\Delta\left(\frac{1}{H}\right)_i=\frac{2\hbar e^*}{p_0^2c}=\frac{2\hbar e}{p_0^2c}\frac{m^*}{m}. \tag{25}$$

The period given in (25) is related to the period of the noninteracting gas, $\Delta(1/H)_F=2\hbar e/p_0^2c$, by

$$\Delta\left(\frac{1}{H}\right)_i=\frac{m^*}{m}\Delta\left(\frac{1}{H}\right)_F. \tag{26}$$

The difference in energy between the states in (24) can be easily calculated. For a change in n of one, the change in area ΔA_p is from (24)

$$\Delta A_p=hHe^*/c.$$

But we also have

$$\Delta A_p=\frac{\partial A_p}{\partial p_H}\frac{\partial p_H}{\partial \mathcal{E}}\Delta\mathcal{E}=2\pi m^*\Delta\mathcal{E}.$$

By combining we obtain

$$\Delta\mathcal{E}=\hbar(e^*H/m^*c)=\hbar(eH/mc)=\hbar\omega_c.$$

Cyclotron resonance absorption occurs at a frequency ω_c, in agreement with the results of Sec. II. Quasi-

[7] L. Onsager, Phil. Mag. 43, 1006 (1952).

particles have the same cyclotron frequency as free electrons because $e^*/m^*=e/m$.

The specific heat of an interacting electron gas C_i is given by[6]

$$C_i=(p_0/mv_0)C_f=(m^*/m)C_f, \tag{27}$$

where v_0 is the velocity of the quasi-particles at the Fermi surface, $C_f=(\pi k/p_0)^2mT$ is the specific heat for noninteracting electrons, k is Boltzmann's constant, and T is the absolute temperature. It is seen from a comparison of (26) and (27) that both the period of the de Haas-van Alphen effect and the specific heat are changed in exactly the same way by electron-electron interactions. However, in the next section we will show that this does not remain true when electron-phonon interactions are included. Calculations[5] and experiment[8] indicate that m^*/m can differ significantly from one (for sodium $m^*/m\approx 1.4$), so that the effect of electron-electron interactions on these two quantities is significant.

IV. ELECTRON-PHONON INTERACTIONS

The discussion of this paper has so far been limited to a consideration of the electron-electron interactions alone and electron-phonon interactions have been neglected. However, it is possible to include the effect of electron-phonon interactions on the properties of the quasi-particles, and from this infer the effect on the period of the de Haas-van Alphen effect and on the cyclotron resonance frequency. When both electron-electron and electron-phonon interactions are included, our picture of a quasi-particle is a core electron surrounded by a polarization cloud of electrons and a cloud of phonons. We are again using an idealized model of a metal consisting of a degenerate electron gas together with a uniform background of positive charge which is now capable of propagating sound waves.

As before, quasi-particles at the Fermi surface have momentum p_0 and velocity $v=\nabla_p\epsilon(p)$. We define an effective dynamical mass m' by

$$m'v=p. \tag{28}$$

The current carried by a quasi-particle is

$$j=e'v, \tag{29}$$

where e' is the effective dynamical charge. In this case, however, $e'/m'\neq e/m$ because not all of the momentum is carried by electrons, some of it is carried by the phonons. The phonons to a good approximation do not contribute to the current and thus e', because their charge over mass ratio is much smaller than that for electrons. We can set

$$\frac{e'}{m'}=\frac{e}{m}\frac{1}{(1+\alpha)}, \tag{30}$$

where now α is a measure of the importance of the electron-phonon interaction. When $\alpha=0$ there is no electron-phonon interaction. The fraction of the total dynamical mass contributed by the phonons is given by $\alpha/(1+\alpha)$. Knowing the dynamical charges and masses of the quasi-particle, we can proceed as in Sec. III to quantize the orbits in a magnetic field and obtain

$$\Delta\left(\frac{1}{H}\right)'=\frac{e'}{e}\Delta\left(\frac{1}{H}\right)_F=\frac{m'}{(1+\alpha)m}\Delta\left(\frac{1}{H}\right)_F, \tag{31}$$

$$\omega_c'=\omega_c/(1+\alpha). \tag{32}$$

Here $\Delta(1/H)'$ is the period of the de Haas-van Alphen effect including the electron-phonon interaction, and ω_c' is the cyclotron frequency for the same case. We also have from (27) for C', the specific heat in this case, that

$$C'=(m'/m)C_f. \tag{33}$$

We see that the specific heat and the period of the de Haas-van Alphen effect are affected differently by the electron-phonon interaction.

The importance of α can be estimated by comparing with experiment. Aluminum is a metal which approximates the model used in this paper.[9] As indicated by calculations[10–12] and by comparison with experiment[12] the electronic states for aluminum can be obtained by neglecting the interaction between the electrons and the periodic potential of the lattice except for states in the vicinity of the Brillouin zone boundaries. The effect of the Brillouin zone boundaries is to produce small discontinuities in the energy across the boundaries and preventing an electronic state which is initially in one zone to move out of that zone under the influence of low frequency electric and magnetic fields. This, for instance, complicates the orbits in a constant uniform magnetic field from what they would be if there were no zone boundaries, but this effect can be straightforwardly accounted for by a factor which depends on the geometry of the zone boundaries. In addition, the Fermi surface is changed from a sphere to a multiply connected surface. To summarize, by including geometric factors depending on the geometry of the boundaries, the electrons in aluminum can be considered as approximately the model calculated in this paper.

Experimental measurements on aluminum of the electronic specific heat, cyclotron resonance, and de Haas-van Alphen effect are available.[12] Unfortunately the available de Haas-van Alphen data is incomplete and is caused by the electrons on parts of the multiply

[8] E. A. Stern, Phys. Rev. **121**, 397 (1961).

[9] The author is indebted to Professor M. H. Cohen for pointing this fact out to him and the subsequent conclusion that the electron-phonon interaction is important.

[10] V. Heine, Proc. Roy. Soc. (London) **A240**, 340 (1957).

[11] M. H. Cohen and V. Heine, Suppl. Phil. Mag. **7**, 395 (1958).

[12] W. A. Harrison, Phys. Rev. **118**, 1182 (1960).

connected Fermi surface with small cross section. These small cross sections are most sensitive to any deviation of aluminum from our model. For this reason we will not use the de Haas-van Alphen data to determine α, but will use the other experimental data to determine α and then show that the de Haas-van Alphen period calculated from (31) is consistent with experiment. For aluminum $C'/C_f=m'/m=1.6$. Also for the cyclotron resonance of the largest area Fermi surface $\omega_c/\omega_c'=1+\alpha=1.5/0.76=2$. The factor 1.5 is the actually measured cyclotron mass measured in terms of the free electron mass m and is inversely proportional to ω_c'. The factor 0.76 is the cyclotron mass including geometric factors that one calculates for a noninteracting electron gas. One sees immediately that $\alpha=1$ and thus the electron-phonon interaction is as important as the electron-electron interaction and perhaps more so. One can now calculate from (31) that

$$\Delta(1/H)'=0.8\Delta(1/H)_F. \tag{34}$$

This means that the actual area of the Fermi surface is 0.8 of the value that one would calculate using the noninteracting formulas. This factor worsens the original, only semiquantitative agreement with the area calculated from free electron model, but since this result is so sensitive to small deviations from the free electron model it is not significant. Only when the de Haas-van Alphen periods for the large area Fermi surfaces for aluminum, which are relatively insensitive to small deviations from the free electron model, are measured will one be able to determine if (34) is consistent with experiment. The main conclusion remains that electron-phonon interactions are most important and cannot be neglected.[12a]

V. SUMMARY AND DISCUSSION

It has been shown in this paper that electron-electron interactions alone have no effect on the currents induced in an ideal electron gas by uniform electric and magnetic fields. On the basis of this it also followed that the effective masses determined by Faraday effect, optical constants, and cyclotron resonance is just the free electron mass, independent of electron-electron interactions between electrons. The period of the de Haas-van Alphen oscillations is, however, affected by these interactions, the relationship being given by (25) and (26), and is affected in exactly the same way that the specific heat is. When electron-phonon interactions are added to the problem the currents induced by uniform electric and magnetic fields are now affected. The de Haas-van Alphen period and the specific heat are now affected differently and the cyclotron mass is no longer the free value. Comparison with experiments on aluminum shows that the electron-phonon interaction is of the same order as the electron-electron interactions and neither can be neglected. In fact, it is really artificial to separate the electron-electron and electron-phonon interactions. They can be separated only when they are both small or when one is small compared to the other, and it has been shown that this is not the case.

It appears off-hand surprising that in transport type properties such as cyclotron resonance, Faraday effect, and optical constants, interactions between electrons alone appear to have no effect in an electron gas. However, there is a simple explanation for this. Electron-electron interactions do not effect the current. They are momentum conserving interactions and do not change the total momentum of the system. As discussed previously, because all carriers have the same e/m ratio, the current is just e/m times the momentum and thus is not affected by electron-electron interactions. On the other hand properties that depend on the energy of the electron gas such as the de Haas-van Alphen effect and the specific heat, are affected in an important way by electron-electron interactions.

Although the results of this paper apply only to the ideal model of an interacting electron gas, it seems clear that in actual metals the same qualitative conclusions can be drawn. In fact, if the effect of the periodic potential of the lattice can be entirely taken account of by an effective mass, m_l, all of the results of this paper apply if m is replaced by m_l. In the case of actual metals it appears that both electron-phonon and electron-electron interactions should be considered and have important effects on the electronic properties. It is clear from the results of this paper that the size of the Fermi surfaces determined from the de Haas-van Alphen effect using the theory of non-interacting electrons is incorrect. However, more seriously, even the interpretation of the shape using the noninteracting theory may be in error.[13] The interactions may change the relationship between period of the oscillations and area of the Fermi surface in proportionally different ways as the orientation of the crystal is varied. Only actual calculations and further measurements will tell if this possibility is real.

The fact that interactions change the relationship between the area of the Fermi surface and the period of oscillations of the de Haas-van Alphen effect was actually first shown by Dresselhaus,[15] although he did not apparently realize it.[16] Dresselhaus showed that if the first order exchange is included the period is given by $\Delta(\mu/\hbar\omega)=1$, where μ is the Fermi energy. If no interactions at all are included, one also obtains that

[12a] The conclusions of this paragraph are in doubt, particularly Eq. (34), as explained in the *Note added in proof*.

[13] Experimental results of reference 14 show a disagreement between the period measured by the de Haas-van Alphen effect in copper and the predicted period from Pippard's model for copper using the noninteracting theory. If we interpret this disagreement in terms of (31) and assume that Pippard's model is correct, we obtain $e'/e=0.96$ in [111] and $e'/e=0.92$ in [100]. This indicates that e'/e is a function of orientation of the crystal.

[14] D. Shoenberg, Phil. Mag. **5**, 105 (1960).

[15] G. Dresselhaus, Phys. Rev. **114**, 736 (1959).

[16] The author is indebted to Dr. Donald DuBois of The RAND Corporation for pointing this fact out to him.

$\Delta(\mu/\hbar\omega)=1$, and from this Dresselhaus concluded that the de Haas-van Alphen effect is unchanged by the exchange term except for a possible shift in the phase of the oscillations. With no interactions, $\mu=3.69/r_S^2$ rydbergs[17] and is proportional to the maximum area of the Fermi sphere. The periodicity condition that $\Delta(\mu/\hbar\omega)=1$ gives the correct relationship between period and area for no interactions. However, when first order exchange is included, $\mu=(3.69/r_S^2-1.22/r_S)$ rydbergs and is not any longer simply related to the maximum area of the Fermi surface. Then $\Delta(\mu/\hbar\omega)=1$ when exchange is included is a changed relationship, and the period in the de Haas-van Alphen effect *is changed* by first order exchange because μ is changed.

It has been previously shown that the dielectric constant of an interacting electron gas in the long-wavelength limit reduces to the nonintersecting case.[5,18–21] However, the previous result was proved by using calculations that depended on a perturbation approach and were only correct at high densities. The results of this paper are correct for all densities and are thus more general.

ACKNOWLEDGMENTS

The author is glad to acknowledge many stimulating discussions with Dr. Donald DuBois and Mr. Sidney Coleman, both of The RAND Corporation.

Note added in proof. Since the writing of this paper, before the meeting of the *Conference on Fermi Surfaces in Metals* held in Cooperstown, New York, on August 22–24, 1960, additional related papers have been published. Falicov at the Conference[22] and Falicov and Heine[23] afterwards have also independently introduced the concept of an effective dynamical charge for quasi-particles when electron-electron interactions are considered. Their conclusions about the consequences of an effective dynamical charge in magnetic fields are in essential agreement with that presented in this paper. Luttinger has, at the Conference,[24] and subsequently,[25] put forth a proof that the concept of e^* and its consequences for the period of the de Haas-van Alphen effect are incorrect. He maintains that from arguments of gauge invariance the quasi-particle interacts with the magnetic field via the electronic charge e, not e^*, and thus the de Haas-van Alphen period is not changed. Luttinger's arguments imply that the energy difference between the quantized levels of the quasi-particles is not $\hbar\omega_c$ but is instead $\hbar(m^*/m)\omega_c$. Since $m^*\neq m$ it is hard to reconcile Luttinger's result with the result that the cyclotron frequency is $\hbar\omega_c$. Also, it can be shown on the basis of Landau's theory of quasi-particles,[26] though it will not be done here for reasons of economy of space, that the core particle part of the quasi-particle *does* interact with both uniform electric and magnetic fields via the electron charge e. However, because the core particles does *not* carry all of the momentum of the quasi-particle it turns out from Landau's theory that the motion of the core particle gives *exactly identical results* with saying that the whole quasi-particle has an effective dynamical charge e^* as was done in this paper. Interestingly, if it is *incorrectly* assumed that the core particle *does* carry all of the quasi-particle momentum then one obtains Luttinger's results. The possibility then presents itself that by some subtle means Luttinger *may* have identified the core particle with the whole quasi-particle.

Recently another calculation of the effect of first order exchange on the de Haas-van Alphen effect has been published[25] and results on the period agree with those of Dresselhaus[15] which then give added weight to his calculation that the period is not simply related to the area of the Fermi surface.

Finally, the correctness of the measured cyclotron mass in aluminum of 1.5 times the free electron value has been questioned at the Conference by Fawcett.[27] The conclusions reached in the last paragraph of Sec. IV are to be considered only tentative pending an experimental verification of this mass.

[17] r_s is the radius measured in units of the Bohr radius of a sphere that has a volume equal to the average volume per electron.
[18] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953).
[19] R. Brout, Phys. Rev. **108**, 515 (1957).
[20] K. Sawada, K. A. Brueckner, N. Fukuda, and R. Brout, Phys. Rev., **108**, 507 (1957).
[21] D. Pines, in *Solid-State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, New York, 1955), Vol. 1, p. 367.
[22] L. M. Falicov, reference 1, pp. 39–49.
[23] L. M. Falicov and V. Heine, Suppl. Phil. Mag. **10**, 57 (1961).
[24] J. M. Luttinger, reference 1, pp. 2–8, 67–68.
[25] J. M. Luttinger, Phys. Rev. **121**, 1251 (1961).
[26] H. Ichimura and S. Tanaka, Progr. Theoret. Phys. **25**, 315 (1961).
[27] E. Fawcett, reference 1, p. 166.