Spin-Orbit Interaction in Two-Pion Exchange Nuclear Potential*

SURAJ N. GUPTA
Department of Physics, Wayne State University, Detroit, Michigan
(Received February 15, 1961)

The spin-orbit interaction term in the two-pion exchange nuclear potential is derived. It is found that while the spin-orbit force is quite small in the triplet odd states, it is large and repulsive in the triplet even states. The relationship of the pion theoretical result with the phenomenological spin-orbit interaction is discussed.

1. INTRODUCTION

IN an earlier paper[1] we obtained the two-pion exchange static nuclear potential between two nucleons with nonrelativistic velocities in their center-of-mass system. We have also shown[2] that the neglect of all the velocity-dependent terms in our derivation can be rigorously justified in the nonrelativistic approximation, and therefore the velocity-dependent terms in the two-pion exchange nuclear potential cannot be expected to play a major role in the low-energy nucleon-nucleon interaction. It will, however, be of interest to derive the resulting spin-orbit interaction term, because from an experimental point of view there seems to exist a great deal of uncertainty about the magnitude of spin-orbit interaction in nuclear forces.[4]

2. PION-THEORETICAL SPIN-ORBIT INTERACTION

The contributions of the uncrossed and crossed two-pion exchange diagrams for nucleon-nucleon scattering can be expressed as

$$S_4(a)=(i\pi^2g^4/c^2\hbar^2)\delta(p-p'+q-q')[\bar{\psi}^-(\mathbf{q}')\gamma_\mu\tau_i\tau_j\psi^+(\mathbf{q})][\bar{\psi}^-(\mathbf{p}')\gamma_\nu\tau_i\tau_j\psi^+(\mathbf{p})]$$
$$\times\int_0^1 du\int_0^u dv\int_0^v dw\left[\frac{\delta_{\mu\nu}}{2A}+\frac{v^2(p_\mu-q_\mu)(p_\nu-q_\nu)+w^2(p_\mu+q_\mu)(p_\nu+q_\nu)}{4A^2}\right], \quad (1)$$

$$S_4(b)=(-i\pi^2g^4/c^2\hbar^2)\delta(p-p'+q-q')[\bar{\psi}^-(\mathbf{q}')\gamma_\mu\tau_i\tau_j\psi^+(\mathbf{q})][\bar{\psi}^-(\mathbf{p}')\gamma_\nu\tau_j\tau_i\psi^+(\mathbf{p})]$$
$$\times\int_0^1 du\int_0^u dv\int_0^v dw\left[\frac{\delta_{\mu\nu}}{2B}+\frac{v^2(p_\mu+q_\mu)(p_\nu+q_\nu)+w^2(p_\mu-q_\mu)(p_\nu-q_\nu)}{4B^2}\right], \quad (2)$$

with

$$A=\kappa^2w^2+\lambda^2(1-v)+k^2(u-v)(1-u)-\tfrac{1}{4}(p-q)^2(v^2-w^2), \quad (3)$$

$$B=\kappa^2v^2+\lambda^2(1-v)+k^2(u-v)(1-u)-\tfrac{1}{4}k^2(v^2-w^2)+\tfrac{1}{4}(p-q)^2(v^2-w^2), \quad (4)$$

where the various symbols have the same meaning as in the earlier paper.[3]

In the center-of-mass system of the nucleons the above matrix elements can be simplified by using the relations

$$\mathbf{p}=-\mathbf{q},\quad \mathbf{p}'=-\mathbf{q}',\quad p_0=q_0=p_0'=q_0',$$
$$\mathbf{k}=\mathbf{p}'-\mathbf{p}=-(\mathbf{q}'-\mathbf{q}),\quad k_0=0. \quad (5)$$

We can construct only two independent vectors from the propagation vectors $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{q}$, and $\mathbf{q}'$, and the scattering matrix elements in general can be expressed as a function of these two vectors. One of these vectors must be chosen as $\mathbf{k}$, because the potential is obtained by passing over from the $\mathbf{k}$ space to the $\mathbf{x}$ space. The most natural choice for the other vector would be

$$\mathbf{s}=\tfrac{1}{2}(\mathbf{p}'+\mathbf{p}), \quad (6)$$

because $\mathbf{s}$ is symmetrical with respect to the initial and final states, and it is orthogonal to $\mathbf{k}$. In the nonrelativistic approximation we can regard $\mathbf{k}^2$ and $\mathbf{s}^2=\mathbf{p}^2-\frac{1}{4}\mathbf{k}^2$ as small compared with κ^2, and expand the matrix element in powers of $\mathbf{k}^2/\kappa^2$, $\mathbf{s}^2/\kappa^2$, and $|\mathbf{k}\times\mathbf{s}|/\kappa^2$, wherever such expansions are permissible. Further, as explained in the Appendix, we can transform the Dirac spinors appearing in (1) and (2) into the Pauli spinors, which gives us in the nonrelativistic approximation

* Supported in part by the National Science Foundation.
[1] S. N. Gupta, Phys. Rev. **117**, 1146 (1960).
[2] S. N. Gupta, Nuovo cimento **18**, 823 (1960).
[3] Equations (1) and (2) of this paper are obtained from Eqs. (15) and (16) of reference 1 by replacing one of the variables of integration as $w\to\frac{1}{2}(w+v)$. Also note that the momentum and energy of a nucleon with the propagation four-vector p are $\hbar\mathbf{p}$ and $c\hbar p_0$, respectively.
[4] Some of the important papers on the phenomenological discussion of the spin-orbit interaction are: J. Gammel and R. Thaler, Phys. Rev. **107**, 291, 1337 (1957); P. S. Signell, R. Zinn, and R. E. Marshak, Phys. Rev. Letters **1**, 416 (1958); M. H. Hull, K. D Pyatt, C. R. Fischer, and G. Breit, Phys. Rev. Letters **2**, 264 (1959); R. A. Bryan, Nuovo cimento **16**, 895 (1960).

$$[\psi^{*-}(\mathbf{q}')\psi^{+}(\mathbf{q})][\psi^{*-}(\mathbf{p}')\psi^{+}(\mathbf{p})]=\psi_L{}^{*-}(\mathbf{q}')\psi_L{}^{*-}(\mathbf{p}')\left[1+\frac{i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot(\mathbf{k}\times\mathbf{s})}{4\kappa^2}\right]\psi_L{}^{+}(\mathbf{p})\psi_L{}^{+}(\mathbf{q}), \tag{7}$$

$$[\psi^{*-}(\mathbf{q}')\gamma_4\gamma_i\psi^{+}(\mathbf{q})][\psi^{*-}(\mathbf{p}')\gamma_4\gamma_i\psi^{+}(\mathbf{p})]=\psi_L{}^{*-}(\mathbf{q}')\psi_L{}^{*-}(\mathbf{p}')\left[\frac{i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot(\mathbf{k}\times\mathbf{s})}{2\kappa^2}\right]\psi_L{}^{+}(\mathbf{p})\psi_L{}^{+}(\mathbf{q}), \tag{8}$$

where we have neglected all the lower-order terms except the spin-orbit terms.

Carrying out the nonrelativistic approximations and dropping the non-Hermitian terms in $S_4(a)$, we find that the spin-orbit terms in the scattering matrix elements are given by

$$S_4'(a)=(i/c\hbar)(2\pi)^4\delta(p-p'+q-q')\psi_L{}^{*-}(\mathbf{q}')\psi_L{}^{*-}(\mathbf{p}')\tau_i{}^{(2)}\tau_j{}^{(2)}\tau_i{}^{(1)}\tau_j{}^{(1)}(I_1+I_2)\psi_L{}^{+}(\mathbf{p})\psi_L{}^{+}(\mathbf{q}), \tag{9}$$

$$S_4'(b)=-(i/c\hbar)(2\pi)^4\delta(p-p'+q-q')\psi_L{}^{*-}(\mathbf{q}')\psi_L{}^{*-}(\mathbf{p}')\tau_i{}^{(2)}\tau_j{}^{(2)}\tau_j{}^{(1)}\tau_i{}^{(1)}(J_1+J_2)\psi_L{}^{+}(\mathbf{p})\psi_L{}^{+}(\mathbf{q}), \tag{10}$$

where

$$I_1=(g^4/64\pi^2c\hbar\kappa^2)i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot(\mathbf{k}\times\mathbf{s})\int_0^1 du\int_0^u dv\,v\frac{\frac{1}{2}}{\kappa^2v^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)}, \tag{11}$$

$$I_2=(g^4/64\pi^2c\hbar\kappa^2)i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot(\mathbf{k}\times\mathbf{s})\int_0^1 du\int_0^{2[(1-u)^{\frac{1}{2}}-(1-u)]} dv\int_0^v dw\frac{1}{\kappa^2w^2+\lambda^2(1-v)+\mathbf{k}^2[(u-v)(1-u)-\frac{1}{4}v^2]}, \tag{12}$$

$$J_1=(g^4/64\pi^2c\hbar\kappa^2)i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot(\mathbf{k}\times\mathbf{s})$$
$$\times\int_0^1 du\int_0^u dv\,v\left\{\frac{\frac{1}{2}}{\kappa^2v^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)}-\frac{\kappa^2v^2}{[\kappa^2v^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)]^2}\right\}, \tag{13}$$

$$J_2=(g^4/64\pi^2c\hbar\kappa^2)i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot(\mathbf{k}\times\mathbf{s})\int_0^1 du\int_0^u dv\,v\frac{1}{\kappa^2v^2+\lambda^2(1-v)+\mathbf{k}^2(u-v)(1-u)}. \tag{14}$$

In (9) and (10), I_1 and J_1 arise from the values $\mu=\nu=4$ of the indices μ and ν in (1) and (2), I_2 and J_2 arise from the values $\mu=i$, $\nu=j$, while it is easy to see that the contributions from the values $\mu=i$, $\nu=4$ or $\mu=4$, $\nu=i$ vanish.

The nuclear potential, corresponding to the matrix elements (9) and (10) is

$$V_4'=(2\pi)^{-3}\int d\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{r}}[-\tau_i{}^{(2)}\tau_j{}^{(2)}\tau_i{}^{(1)}\tau_j{}^{(1)}(I_1+I_2)+\tau_i{}^{(2)}\tau_j{}^{(2)}\tau_j{}^{(1)}\tau_i{}^{(1)}(J_1+J_2)]$$

$$=-(2\pi)^{-3}\int d\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{r}}[(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(I_1+I_2)-(3+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(J_1+J_2)]. \tag{15}$$

Carrying out integrations over the $\mathbf{k}$ space, we can express V_4' as a function of $x=\lambda r$ as

$$V_4'=V_{LS}(x)(\mathbf{L}\cdot\mathbf{S}/\hbar^2), \tag{16}$$

where

$$\mathbf{L}\cdot\mathbf{S}/\hbar^2=(\mathbf{r}\times\mathbf{s})\cdot(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})/2=(\mathbf{r}\times\mathbf{p})\cdot(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})/2, \tag{17}$$

$$V_{LS}(x)=(g^2/4\pi c\hbar)^2\lambda c\hbar\{(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})[R_1(x)+R_2(x)]-(3+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})[S_1(x)+S_2(x)]\}, \tag{18}$$

with

$$R_1(x)=\frac{\lambda^2}{16\pi\kappa^2}\int_0^1 du\int_0^u dv\frac{v}{(u-v)(1-u)}\left(f_1+\frac{1}{x}\right)\frac{e^{-f_1x}}{x^2}, \tag{19}$$

$$R_2(x)=\frac{\lambda^2}{8\pi\kappa^2}\int_0^1 du\int_0^{2[(1-u)^{\frac{1}{2}}-(1-u)]} dv\int_0^v dw\frac{1}{(u-v)(1-u)-\frac{1}{4}v^2}\left(f_2+\frac{1}{x}\right)\frac{e^{-f_2x}}{x^2}, \tag{20}$$

$$S_1(x)=R_1(x)-\frac{1}{16\pi}\int_0^1 du\int_0^u dv\frac{v^3}{(u-v)^2(1-u)^2}\frac{e^{-f_1x}}{x}, \tag{21}$$

$$S_2(x)=2R_1(x), \tag{22}$$

and

$$f_1=\left[\frac{(\kappa^2/\lambda^2)v^2+(1-v)}{(u-v)(1-u)}\right]^{\frac{1}{2}}, \tag{23}$$

$$f_2=\left[\frac{(\kappa^2/\lambda^2)w^2+(1-v)}{(u-v)(1-u)-\frac{1}{4}v^2}\right]^{\frac{1}{2}}. \tag{24}$$

3. DISCUSSION

According to (18), in the triplet odd states

$$^3V_{LS}^-=(g^2/4\pi c\hbar)^2\lambda c\hbar\{[R_1(x)+R_2(x)]-5[S_1(x)+S_2(x)]\}, \tag{25}$$

while in the triplet even states

$$^3V_{LS}^+=(g^2/4\pi c\hbar)^2\lambda c\hbar\{9[R_1(x)+R_2(x)]+3[S_1(x)+S_2(x)]\}. \tag{26}$$

We have evaluated[5] the integrals $R_1(x)$, $R_2(x)$, $S_1(x)$, and $S_2(x)$ numerically for various values of x by taking $\lambda/\kappa=0.15$, and the results obtained are given in Table I. Further, $^3V_{LS}^-$ and $^3V_{LS}^+$ for various values of x are given in Table II, where we have taken $g^2/4\pi c\hbar=14$, and $\lambda c\hbar=140$ Mev.

Table II shows that the pion-theoretical $^3V_{LS}^-$ is small, while $^3V_{LS}^+$ is large and positive. This is in disagreement with the phenomenological result that $^3V_{LS}^-$ is large and negative. It is interesting to note that a reasonable explanation of the phenomenological spin-orbit interaction can be given by assuming that besides the pion-theoretical contribution there exists a large and attractive isotopic-spin-independent spin-orbit contribution from some additional source. The partial cancellation of the two contributions in the triplet even states will also help to overcome the objections raised by Feshbach[6] and by Sessler and Foley.[7] The required isotopic-spin-independent spin-orbit interaction could be due to a neutral heavy scalar meson[8] or a neutral heavy vector meson.[9] It is also possible that such a spin-orbit interaction is due to the pion-pion interaction, although our present knowledge of the pion-pion interaction is not too clear.

TABLE I. Results obtained by numerical integrations of the quantities $R_1(x)$, $R_2(x)$, $S_1(x)$, and $S_2(x)$ for various values of x.

x	$R_1(x)\times10^7$	$R_2(x)\times10^7$	$S_1(x)\times10^7$	$S_2(x)\times10^7$
0.4	12 400	51 300	−5480	24 800
0.5	4015		−2260	8030
0.6	1545	7660	−975	3090
0.7	670		−470	1340
0.8	317	1750	−220	634
0.9	161		−110	322
1.0	86	509	−65	172
1.1	48		−33	96
1.2	27	172	−22	54
1.3	16		−14	32
1.4	10	64	−9	20
1.5	6		−6	12
1.6	4	26	−4	8
1.7	3		−3	6
1.8	2	11	−2	4
1.9	1		−1	2
2.0	1	5	−1	2

TABLE II. V_{LS} in units of Mev in the triplet odd and triplet even states for various values of x.

x	$^3V_{LS}^-$	$^3V_{LS}^+$
0.4	−90.28	1728.72
0.6	−3.76	244.27
0.8	−0.01	54.61
1.0	0.16	15.57
1.2	0.11	5.17
1.4	0.05	1.92
1.6	0.03	0.80
1.8	0.01	0.34
2.0	0.00	0.16

APPENDIX

Any scattering matrix element involving the Dirac spinors in the interaction representation can be reduced to an element involving the Pauli spinors without any approximation, for we can express the Dirac matrices γ_μ and the Dirac spinor $\psi^+(\mathbf{p})$ in terms of the Pauli matrices σ_i and the Pauli spinors $\psi_L^+(\mathbf{p})$ and $\psi_S^+(\mathbf{p})$ as[10]

$$\gamma_i=\begin{pmatrix}0 & -i\sigma_i\\ i\sigma_i & 0\end{pmatrix},\quad \gamma_4=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{A.1}$$

and

$$\psi^+(\mathbf{p})=C\begin{pmatrix}\psi_L^+(\mathbf{p})\\ \psi_S^+(\mathbf{p})\end{pmatrix}, \tag{A.2}$$

where C is a constant. Substituting (A.1) and (A.2) in

$$(ip\gamma+\kappa)\psi^+(\mathbf{p})=0, \tag{A.3}$$

we obtain

$$(\mathbf{p}\cdot\boldsymbol{\sigma})\psi_S^+(\mathbf{p})+(\kappa-p_0)\psi_L^+(\mathbf{p})=0,$$
$$-(\mathbf{p}\cdot\boldsymbol{\sigma})\psi_L^+(\mathbf{p})+(\kappa+p_0)\psi_S^+(\mathbf{p})=0,$$

which gives us

$$\psi_S^+(\mathbf{p})=\frac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{\kappa+p_0}\psi_L^+(\mathbf{p}), \tag{A.4}$$

[5] The numerical integration was carried out by the Computing Center of the Wayne State University on an IBM 650 digital computer.

[6] H. Feshbach, Phys. Rev. **107**, 1626 (1957).

[7] A. M. Sessler and H. M. Foley, Phys. Rev. **110**, 995 (1958).

[8] S. N. Gupta, Phys. Rev. Letters **2**, 124 (1959).

[9] G. Breit, Proc. Natl. Acad. Sci. **46**, 746 (1960); Phys. Rev. **120**, 287 (1960).

[10] The superscripts + and − refer to the positive- and negative-frequency parts. Thus, $\psi^+(\mathbf{p})$ and its Hermitian conjugate $\psi^{*-}(\mathbf{p})$ refer to the particle, while $\psi^-(\mathbf{p})$ and $\psi^{*+}(\mathbf{p})$ refer to the anti-particle.

so that $\psi_L^+(\mathbf{p})$ and $\psi_S^+(\mathbf{p})$ can be interpreted as the large and small components of $\psi^+(\mathbf{p})$.

We can now write $\psi^+(\mathbf{p})$ in terms of $\psi_L^+(\mathbf{p})$ as

$$\psi^+(\mathbf{p})=C\begin{bmatrix}\psi_L^+(\mathbf{p})\\ \dfrac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{\kappa+p_0}\psi_L^+(\mathbf{p})\end{bmatrix}. \tag{A.5}$$

By choosing C in such a way that $\psi^+(\mathbf{p})$ and $\psi_L^+(\mathbf{p})$ satisfy the normalization condition

$$\psi^{*-}(\mathbf{p})\psi^+(\mathbf{p})=\psi_L{}^{*-}(\mathbf{p})\psi_L^+(\mathbf{p}), \tag{A.6}$$

we find

$$C=\left(\frac{\kappa+p_0}{2p_0}\right)^{\frac{1}{2}}, \tag{A.7}$$

so that (A.5) becomes

$$\psi^+(\mathbf{p})=\left(\frac{\kappa+p_0}{2p_0}\right)^{\frac{1}{2}}\begin{bmatrix}\psi_L^+(\mathbf{p})\\ \dfrac{(\boldsymbol{\sigma}\cdot\mathbf{p})}{\kappa+p_0}\psi_L^+(\mathbf{p})\end{bmatrix}. \tag{A.8}$$

By using (A.1), (A.8), and the commutation relation

$$(\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}+i\boldsymbol{\sigma}\cdot(\mathbf{A}\times\mathbf{B}), \tag{A.9}$$

we can always obtain a result of the form

$$\psi^{*-}(\mathbf{p}')O\psi^+(\mathbf{p})=\psi_L{}^{*-}(\mathbf{p}')[a+b_i\sigma_i]\psi_L^+(\mathbf{p}), \tag{A.10}$$

where O is a product of the Dirac matrices, and a and b_i are numbers.

PHYSICAL REVIEW VOLUME 122, NUMBER 6 JUNE 15, 1961

Fermi-Thomas Type Approximation for Nuclei

KAILASH KUMAR*
Institute of Advanced Studies, Australian National University, Canberra, Australia

AND

RAJAT K. BHADURI
Tata Institute of Fundamental Research, Bombay, India

(Received January 16, 1961; revised manuscript received March 10, 1961)

The Hartree-Fock to Fermi-Thomas reduction is carried through for finite nuclei, starting with the K-matrix formulation. The resulting expression represents the nuclear energy to good accuracy in terms of the density and its first derivatives only; it differs in detail from the semiempirical expressions previously proposed for this purpose. This improved expression shows the inadequacy of the "semi-infinite" approximation used often in earlier studies.

1. INTRODUCTION

IT is well known that a Hartree-Fock like energy expression for a many-fermion system can always be converted to a form depending only on the density and its derivatives. On minimizing such an expression with respect to variations, one obtains a Fermi-Thomas type equation for the density. For problems involving many electrons, this situation has been well explored. In general, however, it is not immediately clear just how rapidly the procedure converges, or whether the resulting expression will be simple enough to be handled by a reasonable amount of labor.

In this work, we report upon a reduction of a Hartree-Fock to Fermi-Thomas type equation for finite nuclei and some of its applications. The Hartree-Fock type basis in this case is provided by the K-matrix theory of Brueckner and collaborators,[1,2] and in the resulting expression, terms up to $(\nabla\rho)^2$ provide a good approximation for the energy density. In obtaining this result (Sec. 2), we have used an approximate form of the K matrix, which is substantially nonlocal, in conjunction with a square-well potential. Very probably, this overestimates the neglected higher-derivative terms, which are a measure of nonlocality.

Having thus established the unimportance of higher-derivative terms, we go on to calculate a form of energy density, up to terms in $(\nabla\rho)^2$, which is independent of these approximations. In fact, this form of energy-density is as good as may be profitably extracted from the best currently available K matrices.

It is important for the present applications that the density dependence of the K matrix be accurately known. In the published literature,[1,2] the K matrices are given numerically as functions of r and r' for a fixed density. Their density dependence is given in an approximate formula[1] expected to be valid roughly for $\rho=3\rho_0$ to $\rho_0/3$, where ρ_0 is the equilibrium density of nuclear matter. In view of this and the doubts recently raised[3] concerning the Gammel-Thaler potentials, which form the basis of BGW K matrices, we do not consider

* On leave from Tata Institute of Fundamental Research, Bombay, India.

[1] K. A. Brueckner, J. L. Gammel, and H. Weitzner, Phys. Rev. **110**, 431 (1958), referred to as BGW in the text. K. A. Brueckner, A. M. Lockett, and M. Rotenberg, Phys. Rev. **121**, 255 (1961).

[2] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

[3] G. Derrick, D. Mustard, and J. M. Blatt, Phys. Rev. Letters, **6**, 69 (1961).