

Finite Perturbation Theory in Quantum Electrodynamics*

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It is proved that no infinities appear in the power series expansion of the S matrix in quantum electrodynamics if one uses an improved perturbation procedure which is based on the following property of all renormalizable field theories. The dependence of solutions on the coupling constant has a singular part, nonanalytic at $g=0$. This singular dependence must be treated exactly, whereas the remaining, nonsingular, dependence can be expanded into a power series. This power series coincides with the standard renormalized expansion. All renormalization constants in every order remain finite, provided their singular dependence on the coupling constant is treated exactly. The problem of convergence of the whole series has not been investigated.

I. INTRODUCTION

THE problem of divergencies in quantum field theory is as old as the theory itself. None of the usual methods, which lead to finite renormalized perturbation expansions, remove infinities completely. Merely the relations involving renormalization constants are systematically avoided.

In this paper a new perturbation expansion is introduced. The properties of this modified perturbation expansion may be summarized as follows: (a) The initial Lagrangian remains unchanged, i.e., there is no cutoff. (b) No infinities appear¹ in the perturbation expansion. All renormalization constants are finite in every order. (c) For all finite values of energies, the renormalized results (renormalization is finite) coincide with the old ones.

The modified perturbation expansion is based on the following conclusions which can be drawn from Dyson's classical paper.² Finite values of all renormalized Green's functions indicate that these functions can be expanded into power series in e , which are at least asymptotic. On the other hand, the infinite values of renormalization constants suggest that the dependence of these constants on e is singular at $e=0$. The non-analytic character of quantum electrodynamics at $e=0$ has been conjectured long ago by Dyson.³ Recently this problem was investigated by Redmond⁴ and Frautschi⁵ in connection with the renormalization group and spectral representations.

The purpose of this paper is to show explicitly that all solutions in quantum electrodynamics are singular at $e=0$. It will be proved, moreover, that no infinities appear in every order if one treats the dependence on e of all renormalization constants exactly and uses a perturbation expansion only to evaluate the analytic

part of the functions of e . The renormalization constants, in every order, depend critically on e ; all divergencies, from which standard perturbation theories suffer, are caused by the improper expansion of these constants in terms of e , as will be shown later.

In Sec. II, features of the cutoff renormalization procedure, which are important for the discussion to follow, are briefly reviewed. Section III is devoted to the investigation of a simple mathematical model. This model is treated both by the standard and by the modified perturbation theory. In this elementary case the essential differences of the two methods are clearly displayed. The modified perturbation theory in quantum electrodynamics is introduced in Sec. IV and in Sec. V its physical content is discussed.

II. THE STANDARD PERTURBATION THEORY

The basic property of all renormalizable field theories, which allows of finite results in the standard perturbation theory, consists in that all divergencies appear only through renormalization constants. One may then either introduce a cutoff into the initial Lagrangian,⁶ or treat some intermediate expressions as finite functions, although they are in fact meaningless. There is no basic difference between these two methods, the former being simply a more satisfactory form of the latter. Both will be called the cutoff perturbation expansions.

There are two disadvantages to the cutoff procedure. (a) The theory with a cutoff is different from the original one. (b) The final operation of removing the cutoff, with all infinities compressed into the physical parameters, is unsatisfactory from the mathematical point of view. The recent method⁷ of treating products of distributions as finite but undetermined functions is an alternate description of what is done in the cutoff procedure, rather than a solution of difficulties. Finally, there is the method due to Nishijima⁸ who was able to

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¹ It is assumed that the infrared divergencies are removed by the usual procedure.

² F. J. Dyson, *Phys. Rev.* **75**, 1736 (1949).

³ F. J. Dyson, *Phys. Rev.* **85**, 631 (1952).

⁴ P. J. Redmond, *Nuovo cimento* **14**, 771 (1959).

⁵ S. C. Frautschi, *Progr. Theoret. Phys. Japan* **22**, 882 (1959).

⁶ N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959).

⁷ H. Bremmermann, preprint; J. G. Taylor, *Nuovo cimento* **17**, 695 (1960).

⁸ K. Nishijima, *Phys. Rev.* **119**, 485 (1960).

arrive at renormalized Green's functions without even mentioning the renormalization constants in intermediate steps. However, this method lacks a direct connection with the Lagrangian formulation and is rather complicated. All three methods differ only in the derivation of the final results. The renormalized perturbation expansions are in all cases the same and all methods share their main defect: The Lagrangian and the S matrix appear, in a sense, as complementary quantities. If one is made finite, the other becomes infinite. There is no substantial difference between the cutoff procedure and the remaining two methods. In the cutoff perturbation theory all steps are exhibited explicitly, whereas Nishijima and also Bremermann and Taylor ingeniously avoid dangerous places. Therefore, the present method will be compared with the cutoff procedure as the most complete version of the standard perturbation theory. For the purpose of this comparison, the following three operations which characterize the cutoff method are singled out.

(a) The expression for the scattering matrix is changed by introducing a cutoff Λ . When Λ tends to ∞ , the new S matrix approaches its initial form.

(b) S is expanded into a power series in e . All terms are finite, provided the cutoff is kept finite. The dependence on Λ of every term has the following properties. All renormalization constants (and only the renormalization constants) depend critically on Λ , becoming infinite when cutoff is removed. The dependence on Λ of the renormalized Green's functions is nonsingular. They have finite limits when Λ tends to ∞ .

(c) Finally, the experimental data are invoked to replace the singular functions of the cutoff by known finite parameters. The remaining dependence on cutoff is removed by letting Λ tend to ∞ .

In the next section these three operations are performed on a simple model, which then is also treated by the method introduced in this paper.

III. ILLUSTRATIVE MODEL

Let us consider the function $F(\lambda, p)$ defined by the following integral

$$F(\lambda, p) = \int_0^\infty \exp[-\lambda x(1 + \gamma \cos^2 px)] dx, \quad (3.1)$$

where γ , λ , and p are real parameters and $|\gamma| < 1$. Let us assume that the p dependence of the function F for small positive values of λ is to be investigated. In this model, the p dependence represents the physical information and λ is the coupling constant. The improper expansion of (3.1) into the power series of λ produces a series of infinite integrals, just as the usual perturbation procedure does in quantum electrodynamics. The integral (3.1) will now be evaluated using both the cutoff method and the modified perturbation expansion.

The standard method of dealing with divergencies in

quantum electrodynamics can be easily applied to this model by replacing the original function $F(\lambda, p)$ by a cutoff function $F_\Omega(\lambda, p)$,

$$F_\Omega(\lambda, p) = \int_0^\infty \exp(-\Omega x) \times \exp[-\lambda x(1 + \gamma \cos^2 px)] dx, \quad (3.2)$$

$$F(\lambda, p) = F_\Omega(\lambda, p) |_{\Omega=0}. \quad (3.3)$$

The power series expansion in λ of F_Ω is a sum of convergent integrals. When these integrals are evaluated it is found that $F_\Omega(\lambda, p)$ has the form^{8a}

$$F_\Omega(\lambda, p) = \frac{1}{\Omega} F_{-1}(\lambda/\Omega) + \sum_{n=0}^{\infty} \lambda^n f_n(p) + \sum_{m=1}^{\infty} \Omega^m F_m(\lambda, p). \quad (3.4)$$

Suppose the value of the function $F(\lambda, p)$ is known at one point. This assumption corresponds to the introduction of the experimentally observed quantities into the perturbation expansion in quantum electrodynamics. The singular part of (3.4), $\Omega^{-1} F_{-1}(\lambda/\Omega)$, can then be replaced by one finite constant, to be determined from "experiment." The final form of the function F , obtained with the use of cutoff procedure, is

$$F(\lambda, p) = c + \sum_{n=0}^{\infty} \lambda^n f_n(p). \quad (3.5)$$

The three sets of terms in (3.4) correspond, respectively, to the renormalization constants, to the renormalized Green's functions, and to the corrections to these functions for finite values of the cutoff.

A perturbation theory will now be introduced which gives finite terms in all orders, without any modifications in the original function $F(\lambda, p)$. First, let us write our integral in the form

$$F(\lambda, p) = \int_0^\infty \exp(-\lambda x) \exp(-\lambda x \gamma \cos^2 px) dx. \quad (3.6)$$

The dependence on λ in the first exponential is singular whereas in the second it is not. Therefore, this integral may be expanded into a power series of the form

$$F(\lambda, p) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n \gamma^n}{n!} \int_0^\infty \exp(-\lambda x) x^n \cos^{2n} px dx, \quad (3.7)$$

which contains only finite terms. Integration of (3.7) results in

$$F(\lambda, p) = -\frac{1}{\lambda} f_{-1} + \sum_{n=0}^{\infty} \lambda^n f_n(p), \quad (3.8)$$

^{8a} Note added in proof.—The author is indebted to Dr. A. Katz for pointing out an error in the original manuscript.

where the coefficients $f_n(p)$ are identical with those obtained in (3.5). Since f_{-1} is well determined no extraneous constant need be introduced. Comparing (3.8) with (3.4), one can see why the "renormalization constant" $\Omega^{-1}F_{-1}(\lambda/\Omega)$ is a singular function of the cutoff Ω . This is due to the singular dependence of the first term in (3.8) on λ . The principal idea of the modified perturbation expansion consists in the separation of the λ dependence into the singular and nonsingular parts in such a way as to allow an exact treatment of the singular dependence.

IV. THE MODIFIED PERTURBATION THEORY IN QUANTUM ELECTRODYNAMICS

As indicated by the results of the previous section, to get a finite perturbation expansion the singular and the regular dependencies on the coupling constant must be separated out and the singular dependence must be treated exactly. This can be accomplished by separating the total Lagrangian into the free and the interaction parts in a special way. The free Lagrangian will be written as

$$\tilde{L}_0 = \bar{\psi} \left(i\gamma \cdot \frac{\partial}{\partial x} - m \right) \left[1 - \Lambda^{-2} \left(i\gamma \cdot \frac{\partial}{\partial x} - m \right)^2 \right] \psi + \frac{1}{2} A_\mu \square A^\mu. \quad (4.1)$$

The interaction Lagrangian is then equal to

$$\begin{aligned} \tilde{L}_I = & e Z_3 \bar{\psi} \gamma \cdot A \psi + \delta m Z_2 \bar{\psi} \psi + (Z_2 - 1) \bar{\psi} \left(i\gamma \cdot \frac{\partial}{\partial x} - m \right) \psi \\ & + \frac{1}{2} (Z_3 - 1) A_\mu \square A^\mu + \Lambda^{-2} \bar{\psi} \left(i\gamma \cdot \frac{\partial}{\partial x} - m \right)^3 \psi. \end{aligned} \quad (4.2)$$

All fields and charge in (4.1) and (4.2) are renormalized. This separation was found to be the most convenient for the present discussion. However, it is by no means the only one leading to a finite perturbation expansion. The quantity Λ^{-2} will be chosen either as

$$\Lambda^{-2} = m^{-2} e^{N+1}, \quad (4.3)$$

or as

$$\Lambda^{-2} = m^{-2} \exp(1/e^2), \quad (4.4)$$

depending on whether the perturbation expansion up to an order N , or the asymptotic series, is to be evaluated. The S operator has the usual form,⁹

$$S = T \exp \left(i \int \tilde{L}_I(x) dx \right), \quad (4.5)$$

but the interaction Lagrangian in the exponent is built up from $\tilde{\psi}$ fields which satisfy the modified free

equations,

$$\left[1 - \Lambda^{-2} \left(i\gamma \cdot \frac{\partial}{\partial x} - m \right)^2 \right] \left(i\gamma \cdot \frac{\partial}{\partial x} - m \right) \tilde{\psi}(x) = 0. \quad (4.6)$$

The $\tilde{\psi}(x)$ fields depend on e through Λ , and this dependence will be treated exactly. The propagator for the $\tilde{\psi}$ field has the Pauli-Villars structure,

$$\tilde{S}_F(p) = \frac{1}{m - \gamma \cdot p} - \frac{\frac{1}{2}}{m - \Lambda - \gamma \cdot p} - \frac{\frac{1}{2}}{m + \Lambda - \gamma \cdot p}. \quad (4.7)$$

As can be easily verified, the S matrix expanded into the power series in \tilde{L}_I contains only finite terms.

Let us now consider the first case in which terms up to the N th order are to be evaluated. This whole power series depends on e not only directly, but also through Λ . The dependence on Λ of all renormalized results is analytic at $\Lambda^2 = \infty$. The zeroth-order term in Λ reproduces the usual renormalized functions. All higher-order terms in $1/\Lambda^2$ are negligible in view of the chosen dependence of Λ on e . All renormalization constants and the mass renormalization depend on Λ , and hence on e , logarithmically. To evaluate these constants, the entire series must be summed up because higher terms are of the same order of magnitude as the lower ones. The second choice of Λ , (4.4), allows us to write down the full asymptotic series in e . After renormalization, finite in every order, it coincides with the usual renormalized perturbation expansion. The higher terms in $1/\Lambda^2$ now have vanishing asymptotic expansion. To evaluate the renormalization constants, the summation of all leading diagrams is again necessary.

The separation of the Lagrangian, proposed in this section, is rather artificial. It has no physical meaning and is introduced as a mathematical device only. However, one should expect that this procedure could be improved by using a self-consistent method. For example, one may take the propagator \tilde{S}_F and \tilde{D}_F to include all radiative corrections. This gives finite results, at least in quantum electrodynamics, provided we reject the contribution from complex poles. This problem will be discussed in another publication.

V. DISCUSSION

The failure of all previous attempts to get a finite perturbation expansion was caused essentially by attributing unjustified physical reality to virtual particles. These virtual particles were assumed to have propagators of the free physical particles even for momenta lying far from the mass-shell values. This assumption has no physical basis. One is entitled to require the propagator of the virtual particle to become the propagator of the free particle with a physical mass only near the physical region, $p^2 = m^2$. The notion of the virtual particle, with momentum far from this region, is a purely mathematical concept introduced by

⁹ A consistent way of treating the derivatives in \tilde{L}_I and the higher order field equations is presented in the Appendix.

the perturbation procedure. From the physical point of view, there are no grounds on which to object to indefinite metric for virtual particles; in the present approach they do contain components with negative metric. The final and initial particles, however, show no trace of indefinite metric, in every order of perturbation theory. This is due to the fact that the initial Lagrangian describes a system with positive definite metric.

There still are many questions which remain to be answered. The most interesting of these concerns the values of the renormalization constants. To answer this question, one must sum up the whole series, including at least the leading terms.¹⁰ The problem of values of renormalization constants¹¹ cannot be solved without full examination of the renormalization group and high-energy limits in the modified perturbation expansion. The problem of higher orders is also crucial in applications of the modified perturbation expansion to nonrenormalizable theories.

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APPENDIX

In the functional formulation of quantum field theory,¹² the basic role is played by the vacuum expectation value $Z\{\eta, \bar{\eta}, J_\mu\}$ of the S operator in the presence of external sources.¹³ All Green's functions can be derived from Z by functional differentiation. The concise expression for the Z functional can be written as a functional integral over fields,

$$Z\{\eta, \bar{\eta}, J_\mu\} = \int \delta\psi \delta\bar{\psi} \delta A_\mu \exp \left\{ i \int [L(x) + L_s(x)] dx \right\}, \quad (\text{A1})$$

where $L(x)$ is the usual Lagrangian of quantum electrodynamics and $L_s(x)$ is the Lagrangian of the interaction with sources,

$$L_s(x) = \bar{\psi}(x)\eta(x) + \bar{\eta}(x)\psi(x) + A^\mu(x)J_\mu(x). \quad (\text{A2})$$

The advantage of the functional formulation, for our purposes, resides in the fact that once the expression (A1) has been written down—commutation relations,

equations of motion, physical particles, etc., may be disregarded; one then may concentrate on the evaluation of the integral (A1). Besides, this form of the quantum field theory can be treated in complete analogy with the model of Sec. III. Here, as in Sec. III, we have an exponential function under the sign of the integral, whose straightforward expansion into the power series in e produces a series of divergent terms. Looking for a separation of the exponential in (A1) into singular and nonsingular parts, we must bear in mind that the only functional integral amenable to evaluation is the one containing at most a bilinear combination of fields in the exponent. Thus, the Lagrangian \tilde{L}_0 will be chosen in the form

$$\int \tilde{L}_0(x) dx = \int \bar{\psi}(x) \tilde{S}_F^{-1}(x-y) \psi(y) dx dy - \frac{1}{2} \int A_{\mu,\nu}(x) A^{\mu,\nu}(x) dx, \quad (\text{A3})$$

where

$$\tilde{S}_F^{-1}(x-y) = \left(i\gamma \cdot \frac{\partial}{\partial x} - m \right) \delta(x-y) + \sum_{k=0}^m \alpha_k(e) \left(i\gamma \cdot \frac{\partial}{\partial x} \right)^k \delta(x-y), \quad (\text{A4})$$

and the coefficients $\alpha_k(e)$ need not be specified now. It will be assumed only that $m \geq 3$, in order to get finite terms in perturbation expansion. Expanding the integrand into the power series in $\tilde{L}_I = L - \tilde{L}_0$ a series of the following integrals results:

$$Z_n\{\eta, \bar{\eta}, J_\mu\} = \int \delta\psi \delta\bar{\psi} \delta A_\mu \int dx_1 \cdots dx_n \tilde{L}_I(x_1) \cdots \tilde{L}_I(x_n) \times \exp \left[i \int (\tilde{L}_0 + L_s) dx \right]. \quad (\text{A5})$$

All these integrals can be obtained from the simplest of them, i.e., from Z_0 , by functional differentiation with respect to η , $\bar{\eta}$, and J_μ . To evaluate Z_0 one may use the method of Matthews and Salam¹⁴ or simply shift the origin of coordinates in the space of functions ψ , $\bar{\psi}$, and A_μ ,

$$\psi(x) \rightarrow \psi(x) - \int \tilde{S}_F(x-y) dy \eta(y), \quad (\text{A6})$$

$$A_\mu(x) \rightarrow A_\mu(x) - \int D_F(x-y) dy J_\mu(y).$$

¹⁰ For example, one may use the summation procedure, introduced by Landau *et al.* [L. D. Landau, A. A. Abrikosov, and I. M. Halatnikov, Doklady Akad. Nauk S.S.S.R. **95**, 497, 773, 1177 (1954).]

¹¹ It may happen that these constants are after all arbitrary.

¹² K. Symanzik, Z. Naturforsch. **9**, 809 (1954).

¹³ The notation of a previous paper [I. Bialynicki-Birula, Nuovo cimento **17**, 951 (1960)] is used here.

¹⁴ P. T. Matthews and A. Salam, Nuovo cimento **2**, 120 (1955).

The result is

$$Z_0\{\eta, \bar{\eta}, J_\mu\} = N \exp \left[i \int \bar{\eta}(x) \tilde{S}_F(x-y) \eta(y) dx dy + \frac{1}{2} i \int J_\mu(x) D_F(x-y) J^\mu(y) dx dy \right], \quad (\text{A7})$$

where

$$N = \int \delta\psi \delta\bar{\psi} \delta A_\mu \exp \left[i \int \tilde{L}_0(x) dx \right].$$

The total Z can be written in the following symbolic

form¹⁵

$$Z\{\eta, \bar{\eta}, J_\mu\} = \exp \left[i \int L_I \left(\frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)}, \frac{1}{i} \frac{\delta}{\delta \eta(x)}, \frac{1}{i} \frac{\delta}{\delta J_\mu(x)} \right) dx \right] \times Z_0\{\eta, \bar{\eta}, J_\mu\}. \quad (\text{A8})$$

It can be seen that the usual diagram technique can be applied here with all fermion propagators replaced by \tilde{S}_F and the interaction Lagrangian replaced by \tilde{L}_I . The specific form of \tilde{S}_F is irrelevant as far as general rules of obtaining the perturbation expansion are concerned.

¹⁵ P. J. Redmond, Phys. Rev. **105**, 1652 (1957).

Perturbative Treatment of Pairing Forces in Many-Fermion Systems*

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A noncanonical transformation which allows perturbative techniques to be applied to the pairing force problem is introduced. The lowest-order eigenvalue equation gives the standard results for both strong and weak coupling.

THE theory of the effects of pairing forces developed by Bardeen, Cooper, and Schrieffer¹ and by Bogolyubov² is by now well established. However, despite the elaborate formalism of perturbation theory (in particular, field-theoretic perturbation theory), no perturbation treatment has proved capable of handling pairing forces. This paper introduces a noncanonical transformation which permits the pairing Hamiltonian to be treated by perturbative techniques. Unlike the Bogolyubov canonical transformation,² the present transformation maintains particle number conservation. The eigenvalue equation obtained in "lowest order," i.e., by summing the simplest infinite set of graphs, gives the well-known results in both the strong and weak coupling limits simultaneously.

The starting point is the Bardeen-Cooper-Schrieffer¹ pairing Hamiltonian,

$$H = \sum_k (\omega_k/2) (c_{k\uparrow}^\dagger c_{k\uparrow} + c_{-k\downarrow}^\dagger c_{-k\downarrow}) - \sum_{k,k'} V_{kk'} b_k^\dagger b_{k'}, \quad (1)$$

where

$$b_k = c_{-k\downarrow} c_{k\uparrow}, \quad (2)$$

and where the $c_{k\sigma}$, $c_{k\sigma}^\dagger$ are Fermion annihilation and creation operators:

$$[c_{k\sigma}, c_{k'\sigma'}]_+ = 0; \quad [c_{k\sigma}, c_{k'\sigma'}^\dagger]_+ = \delta_{k,k'} \delta_{\sigma,\sigma'}. \quad (3)$$

From (1) it is clear that unpaired particles do not

interact. If $k\uparrow$ is occupied and $-k\downarrow$ is not, then the particle in $k\uparrow$ cannot interact with other particles and so has its unperturbed single-particle energy, $\omega_k/2$. Since unpaired particles are unperturbed, we eliminate them and consider only paired particles: That is, we can eliminate from the sums in (1) all values of k which are only singly occupied and write

$$H_{\text{pairs}} = \sum_{k'} (\omega_{k'}/2) (c_{k'\uparrow}^\dagger c_{k'\uparrow} + c_{-k'\downarrow}^\dagger c_{-k'\downarrow}) = \sum_{k,k'} V_{kk'} b_k^\dagger b_{k'}, \quad (4)$$

where the prime on the sum indicates that pair states $k = (k\uparrow, -k\downarrow)$ which are singly occupied are to be omitted. We therefore have for all states in (4) the operator equation,

$$n_{k\uparrow} = c_{k\uparrow}^\dagger c_{k\uparrow} = n_{-k\downarrow} = c_{-k\downarrow}^\dagger c_{-k\downarrow} = n_k = b_k^\dagger b_k, \quad (5)$$

and (2) and (3) give

$$[b_k, b_k]_+ = 0, \quad [b_k, b_k^\dagger]_+ = 1, \quad (6)$$

$$[b_k, b_{k'}]_- = 0, \quad [b_k, b_{k'}^\dagger]_- = 0, \quad (7)$$

the latter two only for $k' \neq k$.

From (6) it follows that $1 - 2n_k$ anticommutes with b_k and b_k^\dagger , and $(1 - 2n_k)^2 = 1$. We introduce an order into the set of k 's and consider the noncanonical transformation,

$$a_k = [\prod_{k' < k} f_{k'} (1 - 2n_{k'})] b_k, \quad (8)$$

where f_k is $+1$ if the level k is above the unperturbed Fermi sea, -1 if k is below the unperturbed Fermi sea.

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¹ J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

² N. N. Bogolyubov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 65 (1958) [translation: Soviet Phys.—JETP **34**, 41 (1958)].