The result is

$$Z_0\{\eta,\bar{\eta},J_\mu\}=N\exp\left[i\int\bar{\eta}(x)\tilde{S}_F(x-y)\eta(y)dxdy+\tfrac{1}{2}i\int J_\mu(x)D_F(x-y)J^\mu(y)dxdy\right], \quad (A7)$$

where

$$N=\int\delta\psi\delta\bar{\psi}\delta A_\mu\exp\left[i\int\tilde{L}_0(x)dx\right].$$

The total Z can be written in the following symbolic form[15]

$$Z\{\eta,\bar{\eta},J_\mu\}=\exp\left[i\int L_I\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)},\frac{1}{i}\frac{\delta}{\delta\eta(x)},\frac{1}{i}\frac{\delta}{\delta J_\mu(x)}\right)dx\right]\times Z_0\{\eta,\bar{\eta},J_\mu\}. \quad (A8)$$

It can be seen that the usual diagram technique can be applied here with all fermion propagators replaced by $\tilde{S}_F$ and the interaction Lagrangian replaced by $\tilde{L}_I$. The specific form of $\tilde{S}_F$ is irrelevant as far as general rules of obtaining the perturbation expansion are concerned.

[15] P. J. Redmond, Phys. Rev. **105**, 1652 (1957).

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Perturbative Treatment of Pairing Forces in Many-Fermion Systems*

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A noncanonical transformation which allows perturbative techniques to be applied to the pairing force problem is introduced. The lowest-order eigenvalue equation gives the standard results for both strong and weak coupling.

The theory of the effects of pairing forces developed by Bardeen, Cooper, and Schrieffer[1] and by Bogolyubov[2] is by now well established. However, despite the elaborate formalism of perturbation theory (in particular, field-theoretic perturbation theory), no perturbation treatment has proved capable of handling pairing forces. This paper introduces a noncanonical transformation which permits the pairing Hamiltonian to be treated by perturbative techniques. Unlike the Bogolyubov canonical transformation,[2] the present transformation maintains particle number conservation. The eigenvalue equation obtained in "lowest order," i.e., by summing the simplest infinite set of graphs, gives the well-known results in both the strong and weak coupling limits simultaneously.

The starting point is the Bardeen-Cooper-Schrieffer[1] pairing Hamiltonian,

$$H=\sum_k(\omega_k/2)(c_{k\uparrow}^\dagger c_{k\uparrow}+c_{-k\downarrow}^\dagger c_{-k\downarrow})-\sum_{k,k'}V_{kk'}b_k^\dagger b_{k'}, \quad (1)$$

where

$$b_k=c_{-k\downarrow}c_{k\uparrow}, \quad (2)$$

and where the $c_{k\sigma}$, $c_{k\sigma}^\dagger$ are Fermion annihilation and creation operators:

$$[c_{k\sigma},c_{k'\sigma'}]_+=0;\quad [c_{k\sigma},c_{k'\sigma'}^\dagger]_+=\delta_{k,k'}\delta_{\sigma,\sigma'}. \quad (3)$$

From (1) it is clear that unpaired particles do not interact. If $k\uparrow$ is occupied and $-k\downarrow$ is not, then the particle in $k\uparrow$ cannot interact with other particles and so has its unperturbed single-particle energy, $\omega_k/2$. Since unpaired particles are unperturbed, we eliminate them and consider only paired particles: That is, we can eliminate from the sums in (1) all values of k which are only singly occupied and write

$$H_{\text{pairs}}=\sum_k{}'(\omega_k/2)(c_{k\uparrow}^\dagger c_{k\uparrow}+c_{-k\downarrow}^\dagger c_{-k\downarrow})=\sum_{k,k'}V_{kk'}b_k^\dagger b_{k'}, \quad (4)$$

where the prime on the sum indicates that pair states $k[=(k\uparrow,-k\downarrow)]$ which are singly occupied are to be omitted. We therefore have for all states in (4) the operator equation,

$$n_{k\uparrow}=c_{k\uparrow}^\dagger c_{k\uparrow}=n_{-k\downarrow}=c_{-k\downarrow}^\dagger c_{-k\downarrow}=n_k=b_k^\dagger b_k, \quad (5)$$

and (2) and (3) give

$$[b_k,b_k]_+=0,\quad [b_k,b_k^\dagger]_+=1, \quad (6)$$

$$[b_k,b_{k'}]_-=0,\quad [b_k,b_{k'}^\dagger]_-=0, \quad (7)$$

the latter two only for $k'\neq k$.

From (6) it follows that $1-2n_k$ anticommutes with b_k and $b_k^\dagger$, and $(1-2n_k)^2=1$. We introduce an order into the set of k's and consider the noncanonical transformation,

$$a_k=[\textstyle\prod_{k'<k}f_{k'}(1-2n_{k'})]b_k, \quad (8)$$

where f_k is $+1$ if the level k is above the unperturbed Fermi sea, -1 if k is below the unperturbed Fermi sea.

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[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] N. N. Bogolyubov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 65 (1958) [translation: Soviet Phys.—JETP **34**, 41 (1958)].

Then it is easily seen that

$$b_k=[\prod_{k'<k} f_{k'}(1-2n_{k'})]a_k \tag{9}$$

and

$$a_k^\dagger a_k=b_k^\dagger b_k. \tag{10}$$

Moreover, the operator phase factor in (9) just changes one sign in (7), so that

$$[a_k,a_{k'}]_+=0,\quad [a_k,a_{k'}^\dagger]_+=\delta_{k,k'}, \tag{11}$$

and therefore the a_k, $a_k^\dagger$ are Fermion annihilation and creation operators. Clearly a_k decreases the number of *pairs* by one.

Substitution of (9) into (4) gives

$$\begin{aligned} H_{\text{pairs}}&=\sum_k(\omega_k-V_{kk})a_k^\dagger a_k\\ &\quad-\sum_{k'\neq k} V_{kk'}f_{\min(k,k')}a_k^\dagger \prod_{kk'} a_{k'}, \end{aligned} \tag{12}$$

$$\prod_{kk'}=\prod_{\text{between } k,k'} f_{k''}(1-2n_{k''}).$$

The factor $f_{\min(k,k')}$ in (12) comes from the unsquared $f(1-2n)$ with index equal to the lower of k, k', since $a_k^\dagger n_k=n_k a_k=0$.

Since the a's are Fermi operators, (12) can be treated by standard perturbative techniques. Note that (12) conserves the number of particles.

We note first that

$$f_k(1-2n_k)=1-2f_k:n_k:, \tag{13}$$

where

$$\begin{aligned} :n_k:&=a_k^\dagger a_k, &&k>k_F\\ &=-a_k a_k^\dagger, &&k<k_F \end{aligned} \tag{14}$$

is the normal product of $a_k^\dagger$ and a_k. Thus

$$\begin{aligned} H_{\text{pairs}}&=E_0+\sum(\omega_k-V_{kk}):a_k^\dagger a_k:\\ &\quad-\sum_{k\neq k'} V_{kk'}f_{\min(k,k')}:a_k^\dagger \prod_{kk'} a_{k'}:\\ \prod_{kk'}&=\prod_{\text{between } k,k'}(1-2f_{k''}:n_{k''}:),\\ E_0&=\sum_{k<k_F}(\omega_k-V_{kk}). \end{aligned} \tag{15}$$

The simplest approximation is the replacement of $1-2f_k:n_k:$ by 1, i.e., of the phases by their unperturbed values. This corresponds to the selection of a particular simple set from all the Feynman graphs arising from (15). Then,

$$\begin{aligned} H_{\text{pairs}}^{\text{appr}}&=E_0+\sum(\omega_k-V_{kk}):a_k^\dagger a_k:\\ &\quad-\sum_{k'\neq k} V_{kk'}f_{\min(k,k')}:a_k^\dagger a_{k'}:\\ &=\sum(\omega_k-V_{kk})a_k^\dagger a_k\\ &\quad-\sum_{k'\neq k} V_{kk'}f_{\min(k,k')}a_k^\dagger a_{k'}. \end{aligned} \tag{16}$$

We now solve for the eigenpairs, i.e., find

$$A_l^\dagger=\sum r_{kl}a_k^\dagger, \tag{17}$$

satisfying

$$[H_{\text{pairs}}^{\text{appr}}, A_l^\dagger]_-=\lambda_l A_l^\dagger. \tag{18}$$

Call

$$\begin{aligned} -V_{kk'}f_{\min(k,k')}=W_{kk'}&=-V_{kk'}, \quad k>k_F \text{ and } k'>k_F\\ &=+V_{kk'}, \quad k<k_F \text{ or } k'<k_F \end{aligned} \tag{19}$$

$$\begin{aligned} \mu_k&=\omega_k, &&k>k_F\\ &=\omega_k-2V_{kk}, &&k<k_F. \end{aligned} \tag{20}$$

Then

$$H_{\text{pairs}}^{\text{appr}}=\sum \mu_k a_k^\dagger a_k+\sum W_{kk'}a_k^\dagger a_{k'}, \tag{21}$$

and (18) becomes

$$\lambda_l r_{kl}=\sum_{k'}(\mu_k\delta_{k,k'}+W_{kk'})r_{k'l}. \tag{22}$$

Since the r_{kl} are eigenvectors of a Hermitian matrix, they can be chosen orthonormal and the $A_l^\dagger$ are Fermi creation operators. Moreover, if $V_{kk'}$ is factorable,

$$V_{kk'}=gv_kv_{k'}, \tag{23}$$

then the eigenvalue equation can be found. It is

$$\begin{aligned} 1-\sum_{k<k_F}\frac{gv_k^2}{\lambda-\mu_k}+\sum_{k>k_F}\frac{gv_k^2}{\lambda-\mu_k}&\\ -2\sum_{k<k_F}\frac{gv_k^2}{\lambda-\mu_k}\sum_{k'>k_F}\frac{gv_{k'}^2}{\lambda-\mu_{k'}}&=0, \end{aligned} \tag{24}$$

where the roots, λ, are the eigenvalues of λ_l.

For g small, (24) can be written

$$\begin{aligned} \sum_{k<k_F}\frac{v_k^2}{\lambda-\mu_k}-\sum_{k>k_F}\frac{v_k^2}{\lambda-\mu_k}&\\ =\frac{1}{g}-2g\sum_{k<k_F}\frac{v_k^2}{\lambda-\mu_k}\sum_{k'>k_F}\frac{v_{k'}^2}{\lambda-\mu_{k'}}&. \end{aligned} \tag{25}$$

A graphical solution is shown in Fig. 1. When the two terms on the right-hand side are equal, the two central

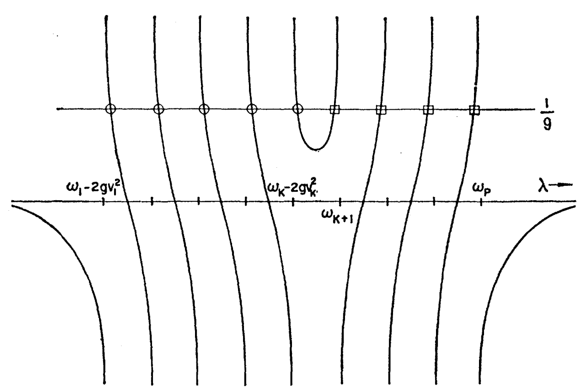

FIG. 1. Graphical solution of Eq. (25) with the second term on the right-hand side neglected. ω_1 to ω_k are the unperturbed energies below the Fermi sea, ω_{k+1} to ω_p above the Fermi sea. The circles (boxes) are energy eigenvalues of eigenpairs which are occupied (unoccupied) in the ground state.

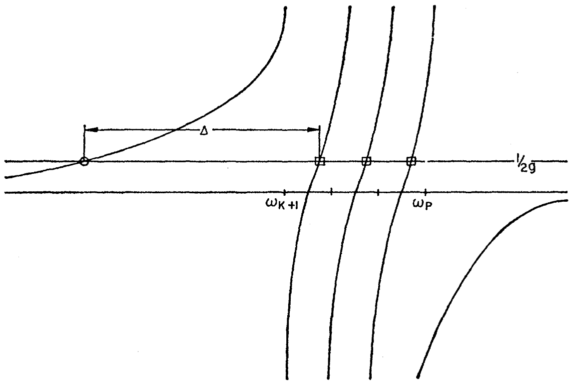

FIG. 2. Graphical solution for the roots of the second factor in Eq. (26). ω_i and the circles and boxes have the same significance as in Fig. 1.

roots are stationary with respect to g, and for larger g they move apart again. For large g, (24) can be written

$$\left(\sum_{k<k_F}\frac{v_k^2}{\lambda-\mu_k}-\frac{1}{2g}\right)\left(\sum_{k>k_F}\frac{v_k^2}{\lambda-\mu_k}+\frac{1}{2g}\right)=\frac{1}{4g^2}, \quad (26)$$

and the solutions are very nearly the values of λ which make one of the factors of the left-hand side vanish. These are shown for the second factor in Fig. 2. The amount by which the collective root is split off is just the pair binding energy Δ.

The perturbed eigenstates are now easily constructed. They are just

$$A_i^\dagger A_j^\dagger\cdots|0\rangle \quad (27)$$

with the number of $A_i^\dagger$'s equal to one-half the number of paired particles. In the strong-coupling case the ground state has an energy lower than the normal state,

$$a_1^\dagger a_2^\dagger\cdots a_{N/2}^\dagger|0\rangle, \quad (28)$$

by approximately 2Δ. However, the excited states are obtained by promoting eigenpairs, and there is no gap in the spectrum of energy eigenvalues of the system. In addition to the states (27), there are eigenstates in which there are unpaired particles occupying various single-particle states.

It is clear that other approximations can be contemplated, corresponding to the selection of a larger set of Feynman graphs. The simplest self-consistent approximation would involve replacing the $:n_k:$'s in $\prod_{kk'}$, Eq. (12), by the expectation values calculated in the perturbed ground state. The character of the spectrum is not expected to change with this procedure, since it depends basically on the distribution of signs in $W_{kk'}$, Eq. (19), and this latter is not changed by the self-consistent technique.

These techniques are currently being used in a study of pairing forces and particle-hole forces in nuclei (the same noncanonical transformation applies to both types of force).