Production of Pion Pairs—Isospin Analysis

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The reactions $\gamma+N\rightarrow 2\pi+N$ and $\pi+N\rightarrow 2\pi+N$ are analyzed in terms of the relevant isospin amplitudes. An experiment is suggested to measure the phase difference between even and odd $\pi-\pi$ isospin states, with a view toward detecting a resonance in the $\pi-\pi$ system. The experimental determination of the magnitude and phase of the amplitudes for the second reaction is discussed.

INTRODUCTION

In the first two sections of this paper we are concerned with the photoproduction of pion pairs. The interpretation of observed charge state ratios and angular distributions in the reaction

$$\gamma+N\rightarrow 2\pi+N \tag{1}$$

is greatly facilitated by expressing the amplitude in terms of the relevant dynamical variables. The phenomenological analysis of (1) involves both the isospin and angular momentum decomposition of the production amplitude. Here we are concerned with the charge state ratios and so consider only the isospin dependence of (1). The angular momentum analysis of (1) has been given by Peierls[1] and Ciulli and Fischer.[2] Of course the isospin is not conserved in (1); however, photons violate isospin conservation in a well-known way. By this we mean that the interaction inducing the reaction (1) transforms in a known (simple) way under rotations in isospace.

If we let the initial nucleon N in (1) be either a proton or a neutron, then there are six possible reactions. It turns out that the isospin analysis requires six independent matrix elements. Therefore isospin conservation does not play so dramatic a role as in the reaction

$$\pi+N\rightarrow 2\pi+N, \tag{2}$$

in which four isospin amplitudes describe ten possible processes.[3] The isospin analysis, besides being the first step in any "fundamental" dynamical calculation, is also useful for the interpretation of experimental results in terms of simple models. Details of the construction of a dynamical theory for the process (2) may be found in a recent work by the author.[3] In Sec. III we discuss reaction (2).

I. THE PHOTOPRODUCTION AMPLITUDE

We use the following notation: $\mathbf{k}$ is the photon momentum; $\mathbf{p}$, $\mathbf{q}$ are the momenta of the final pions; α, β are the charge indices of the final pions; τ, τ' are the charge indices of the initial and final nucleons; and m, m' are the spin projections of the initial and final nucleons. The nucleon momentum is given by momentum conservation. The index α goes with $\mathbf{p}$, β with $\mathbf{q}$. $\alpha=1, 0, -1$ refers to a positive, neutral, negative pion, respectively. $\tau=\frac{1}{2}$ $(-\frac{1}{2})$ describes a proton (neutron). We use the following coupling scheme: First add the isospin of the two pions to get isospin $\mathbf{t}$; then add this to the nucleon isospin to get total isospin $\mathbf{T}$. The restrictions of Bose symmetry are simplest in this coupling scheme. Clearly, in the final state, for $T=\frac{1}{2}$ only $t=0$, 1 are allowed; for $T=\frac{3}{2}$ only $t=1$, 2.

We are now going to prove that the transition amplitude T for reaction (1) may be written as follows:

$$\langle \mathbf{p}\alpha\mathbf{q}\beta\tau'm'|T|\mathbf{k}\tau m\rangle = \sum_{Ttj} \alpha_{Tt}{}^{j}(\alpha\beta\tau'\tau)\mathfrak{M}_{Tt}{}^{j}(\mathbf{pq},\mathbf{k};m'm). \tag{3}$$

The index j stands for S (scalar) or V (vector), as will become clear presently. The "scalar" amplitude occurs for $T=\frac{1}{2}$ only. The amplitudes $\mathfrak{M}_{Tt}{}^{j}$ may be subjected to an angular momentum and multipole decomposition, but we shall not do that here. The coefficients are just constants, containing all the charge state information.

In the "one-photon" approximation the left-hand side of (3) is

$$-\left\langle \Psi^{(-)}(\mathbf{p}\alpha\mathbf{q}\beta,\tau'm')\left|\int \mathbf{j}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})d^3x\right|\Psi(\tau m)\right\rangle. \tag{4}$$

In Eq. (4), $\Psi^{(-)}(\mathbf{p}\alpha\mathbf{q}\beta,\tau'm')$ is a physical eigenstate of the $2\pi-N$ system obeying the incoming-wave boundary condition; $\Psi(\tau m)$ represents a physical nucleon. $\mathbf{j}(\mathbf{x})$ is the charge current density giving rise to the photon field. $\mathbf{A}(\mathbf{x})=\exp(i\mathbf{k}\cdot\mathbf{x})/(2k)^{\frac{1}{2}}$ is the vector potential of the electromagnetic field. [Eq. (4) does not take into account virtual photons or the direct $2\gamma-2\pi$ coupling. Either of these processes can induce isospin changes $\Delta T=2$ or greater. However such processes involve extra factors of e so that the error made is expected to be of the same order as that caused by $\pi^{\pm}-\pi^0$ mass difference, etc., leading to the violations of charge independence in the final state.]

In this approximation one has

$$\mathbf{j}=\mathbf{j}_S+\mathbf{j}_V, \tag{5}$$

where $\mathbf{j}_S$ transforms like a scalar in isospace (i.e. it is

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[1] R. F. Peierls, Phys. Rev. **111**, 1373 (1958).
[2] S. Ciulli and J. Fischer, Nuovo cimento **12**, 264 (1959).
[3] P. Carruthers, Ann. Phys. (to be published).

invariant under rotations and conserves isospin: $\Delta T=0$) and $\mathbf{j}_v$ transforms like the 3-component of a vector in isospace (i.e., transforms like the coordinate z, and gives rise to $\Delta T=0$ or 1). Therefore we write the interaction term as follows:

$$-\int \mathbf{j}\cdot \mathbf{A}d^3x \equiv S+V_3;$$
$$S\equiv -\int \mathbf{j}_S\cdot \mathbf{A}d^3x;\quad V_3\equiv -\int \mathbf{j}_V\cdot \mathbf{A}d^3x. \tag{6}$$

Suppressing the pion momenta and nucleon spin, we have the problem of finding the matrix elements

$$\langle\Psi(\alpha\beta\tau')|S+V_3|\Psi(\tau)\rangle. \tag{7}$$

Our considerations depend only on charge independence and are quite independent of such details of (4) as "physical state vectors," and boundary conditions they might obey, or the precise structure of $\mathbf{j}(\mathbf{x})$. The next step is to express $\Psi(\alpha\beta\tau')$ in terms of eigenstates of $\pi-\pi$ isospin $\mathbf{t}$ and total isospin $\mathbf{T}$. Recalling the previous discussion, we find (χ_τ is the nucleon isospin function, ϕ_α the pion isospin function)

$$\Psi(\alpha\beta\tau')=\phi_\alpha\phi_\beta\chi_{\tau'} \tag{8}$$

$$=\sum_{Tt}\alpha_{Tt}\Phi_{TtT_z}; \tag{9}$$

$$\alpha_{Tt}\equiv C(11t;\alpha\beta)C(t\tfrac{1}{2}T;\alpha+\beta\tau'). \tag{10}$$

Φ_{TtT_z} is the $2\pi-N$ isospin eigenstate of the indicated variables. The notation of Rose[4] is used for the Clebsch-Gordan coefficients. On using (9), Eq. (7) becomes

$$\langle\Psi(\alpha\beta\tau')|S+V_3|\Psi(\tau)\rangle =\sum_{Tt}\alpha_{Tt}\langle\Phi_{TtT_z}|S+V_3|\Psi(T=\tfrac{1}{2},\,T_z=\tau)\rangle. \tag{11}$$

Now, a basic theorem in the theory of angular momentum (Wigner-Eckart theorem[4]) states that the "magnetic" quantum number dependence of matrix elements between eigenstates of J and J' of any quantity that transforms under rotations (in isospace, for our problem) like a "spherical tensor" (i.e. like spherical harmonics) is contained entirely in a Clebsch-Gordan coefficient. Denoting the spherical tensor operator by T_{LM}, the theorem says (j, m are the aforementioned angular momentum quantum numbers)

$$\langle j'm'|T_{LM}|jm\rangle=C(jLj';mM)\langle j'\|T_L\|j\rangle, \tag{12}$$

where the "reduced matrix element" $\langle j'\|T_L\|j\rangle$ is *independent* of m, m' and M. For proton reactions we have $T_z=T_z'=\frac{1}{2}$. For the isotopic scalar operator S [$L=M=0$ in (12)]

$$\mathfrak{M}_{Tt}{}^S\equiv\langle\Phi_{Tt\frac{1}{2}}|S|\Psi(T=T_z=\tfrac{1}{2})\rangle=\delta_{T\frac{1}{2}}\mathfrak{N}_{\frac{1}{2}t}{}^S,$$
$$\mathfrak{N}_{\frac{1}{2}t}{}^S\equiv\langle\Phi_{T=\frac{1}{2},\,t}\|S\|\Psi(T=\tfrac{1}{2})\rangle, \tag{13}$$

For the isovector part V_3 [$L=1$, $M=0$ in (12)]

$$\mathfrak{M}_{Tt}{}^V=C(\tfrac{1}{2}1T;\tfrac{1}{2}0)\mathfrak{N}_{Tt}{}^V;\quad \mathfrak{N}_{Tt}{}^V\equiv\langle\Phi_{Tt}\|V_3\|\Psi(T=\tfrac{1}{2})\rangle;$$
$$\mathfrak{M}_{\frac{1}{2}t}{}^V=(3)^{-\frac{1}{2}}\mathfrak{N}_{\frac{1}{2}t}{}^V,\quad \mathfrak{M}_{\frac{3}{2}t}{}^V=(2/3)^{\frac{1}{2}}\mathfrak{N}_{\frac{3}{2}t}{}^V, \tag{14}$$

Eqs. (11), (13), and (14) thus give

$$\langle\mathbf{p}\alpha\mathbf{q}\beta\tau'm'|T|\mathbf{k}\tau m\rangle=\sum_{t=0,1}\alpha_{\frac{1}{2}t}(\alpha\beta\tau'\tau)[\mathfrak{N}_{\frac{1}{2}t}{}^S+\mathfrak{M}_{\frac{1}{2}t}{}^V] +\sum_{t=1,2}\alpha_{\frac{3}{2}t}(\alpha\beta\tau'\tau)\mathfrak{M}_{\frac{3}{2}t}{}^V, \tag{15}$$

which is of the form (3). If desired, the reduced matrix elements $\mathfrak{N}_{Tt}{}^V$ can be found from (14). Table I gives the α_{Tt} coefficients for $\gamma-p$ reactions.

Now, according to the Bose symmetry of the pions it can make no difference which pion is called number one, e.g., [cf. Eq. (3)],

$$\langle\mathbf{p}\alpha\mathbf{q}\beta\cdots|T|\mathbf{k}\cdots\rangle=\langle\mathbf{q}\beta\mathbf{p}\alpha\cdots|T|\mathbf{k}\cdots\rangle. \tag{16}$$

The $\alpha_{Tt}(\alpha\beta\tau'\tau)$ coefficients have parity $(-1)^t$ under interchange of α and β, as may be seen from (10) since $C(11t;\alpha\beta)=(-1)^tC(11t;\beta\alpha)$. [The 2π isospin wave functions are odd (even) under interchange for $t=1$ ($t=0$, 2).] In order to maintain (16), the amplitudes $\mathfrak{M}_{Tt}{}^j(\mathbf{pq},\mathbf{k})$ must obey [see (3)]

$$\mathfrak{M}_{Tt}{}^j(\mathbf{pq},\mathbf{k})=(-1)^t\mathfrak{M}_{Tt}{}^j(\mathbf{qp},\mathbf{k}). \tag{17}$$

This symmetry is useful in sorting out the contributions of the different isospin channels.

II. CROSS SECTIONS FOR PHOTOPRODUCTION

Using box normalization, setting the volume equal to unity, the differential cross section is

$$d\sigma=\frac{2\pi}{v}|T|^2\delta(\omega_p+\omega_q+E'-k-E)\frac{d^3pd^3q}{(2\pi)^6}, \tag{18}$$

where E (E') is the initial (final) nucleon energy. The "flux" factor v is, for example, in the c.m. system $(1+k/E)$. Owing to the δ function one energy integration can be performed. (Already the nucleon factor d^3p_N has been removed using the total momentum conservation delta function.) Writing $d\sigma=(2\pi/v)|T|^2d\rho$, we have

$$d\rho=\frac{1}{(2\pi)^6}\frac{E'\omega_p\omega_q pq^3d\omega_p d\Omega_p d\Omega_q}{q^2(E'+\omega_q)-\omega_q\mathbf{q}\cdot(\mathbf{q}+\mathbf{p}_{N'})} \tag{19}$$

for the phase space density. It should be remarked that in (18) one generally averages over initial spins and sums over final spins. Also, one frequently needs to average over polarization directions of the photon.

Of course, in actual experiments one measures an

[4] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

TABLE I. The coefficients α_{Tt} of the isospin amplitudes $\mathfrak{M}_{Tt}{}^{j}$ for the $\gamma+p\rightarrow 2\pi+N$ reactions.

Reaction	$\alpha_{\frac{1}{2}0}$	$\alpha_{\frac{1}{2}1}$	$\alpha_{\frac{3}{2}1}$	$\alpha_{\frac{3}{2}2}$
$\gamma+p\rightarrow\pi^{+}+\pi^{-}+p$	$(3)^{-\frac{1}{2}}$	$-(6)^{-\frac{1}{2}}$	$(3)^{-\frac{1}{2}}$	$-(15)^{-\frac{1}{2}}$
$\gamma+p\rightarrow\pi^{0}+\pi^{0}+p$	$-(3)^{-\frac{1}{2}}$	0	0	$-2(15)^{-\frac{1}{2}}$
$\gamma+p\rightarrow\pi^{+}+\pi^{0}+n$	0	$(3)^{-\frac{1}{2}}$	$(6)^{-\frac{1}{2}}$	$(3/10)^{\frac{1}{2}}$

average matrix element

$$\Delta\sigma=\frac{2\pi}{v}\,\frac{\int_{\Delta\rho}|T|^{2}d\rho}{\Delta\rho}\Delta\rho\equiv\frac{2\pi}{v}\langle|T|^{2}\rangle_{\Delta\rho}\Delta\rho, \tag{20}$$

where $\Delta\rho=\int d\rho$ is the phase space accepted in a given experiment. An average over beam energy is generally required, as well.

An important consequence of (17) is that even and odd t terms $\mathfrak{M}_{Tt}{}^{j}$ do not interfere in the *total* cross section. This is easily seen from (3) and (18): The even-odd t interference terms in $|T|^{2}$ are odd under interchange of $\mathbf{p}$ and $\mathbf{q}$, according to (17), while the phase space density [cf. (18)] is even under the same operation.

Before proceeding, we remark on the invariance of the transition amplitude. The T is (18) is related to the Lorentz-invariant amplitude T_{inv} by the relation

$$T=\left(\frac{M^{2}}{8\omega_{p}\omega_{q}kEE'}\right)^{\frac{1}{2}}T_{\mathrm{inv}}. \tag{21}$$

Sometimes one writes (18) in the (equivalent) form

$$d\sigma=\frac{1}{(2\pi)^{5}v}\left(\frac{M^{2}}{8kE}\right)|T_{\mathrm{inv}}|^{2}\delta^{(4)}(p+q+p_{N}'-k-p_{N})\times\frac{d^{3}p}{\omega_{p}}\frac{d^{3}q}{\omega_{q}}\frac{d^{3}p_{N}'}{E'}. \tag{22}$$

Now we consider the influence of (17) on the cross section. First of all, according to Eq. (15) and Table I, the total cross section for $\gamma+p\rightarrow 2\pi+N$ is simply

$$\begin{aligned}\sigma_{\mathrm{tot}}(\gamma+p)&=\int(d\sigma_{+-}+d\sigma_{+0}+d\sigma_{00})\\&=(2\pi/v)\int d\rho[|T_{+-}|^{2}+|T_{+0}|^{2}+\tfrac{1}{2}|T_{00}|^{2}]\\&=(\pi/v)\int d\rho\{|\mathfrak{M}_{\frac{1}{2}0}{}^{S}+\mathfrak{M}_{\frac{1}{2}0}{}^{V}|^{2}+|\mathfrak{M}_{\frac{1}{2}1}{}^{S}+\mathfrak{M}_{\frac{1}{2}1}{}^{V}|^{2}\\&\qquad+|\mathfrak{M}_{\frac{3}{2}1}{}^{V}|^{2}+|\mathfrak{M}_{\frac{3}{2}2}{}^{V}|^{2}\}.\end{aligned} \tag{23}$$

A factor $\frac{1}{2}$ has been inserted before $|T_{00}|^{2}$ to prevent counting identical final states twice. Note the lack of interference between the amplitudes for distinct values of Tt.

Now we investigate the possibility of determining the relative phase of the isospin amplitudes. This is of considerable interest in detecting resonances in the final-state interactions of two of the three particles. When an "isobar" is formed in the final state, one may expect the total amplitude to contain as a factor the (resonant) scattering amplitude of the particles forming the isobar. This has been verified for the important case of $\pi+N\rightarrow 2\pi+N$, where the isobar is the $\pi-N$ $3-3$ resonance.[3]

For clarity we consider a specific reaction:

$$\gamma+p\rightarrow\pi^{+}+\pi^{-}+p. \tag{24}$$

We contemplate two experimental situations A and B. In case A the π^{+} has momentum $\mathbf{p}_{1}$, the π^{-} momentum $\mathbf{p}_{2}$. In case B the π^{-} has momentum $\mathbf{p}_{1}$, the π^{+} momentum $\mathbf{p}_{2}$. These situations are described by the amplitudes (e for even, o for odd t)

$$\begin{aligned}T_{+-}(\mathbf{p}_{1},\mathbf{p}_{2})&=T^{e}(\mathbf{p}_{1},\mathbf{p}_{2})+T^{o}(\mathbf{p}_{1},\mathbf{p}_{2}),\quad(A)\\T_{+-}(\mathbf{p}_{2},\mathbf{p}_{1})&=T^{e}(\mathbf{p}_{1},\mathbf{p}_{2})-T^{o}(\mathbf{p}_{1},\mathbf{p}_{2}),\quad(B)\end{aligned} \tag{25}$$

where we have used (17); the meaning of T^{e},T^{o} can be seen from (15).

Now if our counters, subtending solid angles $d\Omega_{1}(d\Omega_{2})$, accepting pions of momentum $\mathbf{p}_{1}(\mathbf{p}_{2})$ cannot distinguish π^{+} from π^{-}, we measure

$$\tfrac{1}{2}(d\sigma_{A}+d\sigma_{B})=(2\pi/v)\{|T^{e}|^{2}+|T^{o}|^{2}\}d\rho, \tag{26}$$

so we cannot say anything about the even-odd t phase difference.

However, if we can distinguish π^{+} from π^{-} (or $\pi^{+}\pi^{0}$ in the reaction $\gamma+p\rightarrow\pi^{+}+\pi^{0}+n$) then we can form the *difference*:

$$\begin{aligned}\tfrac{1}{2}(d\sigma_{A}-d\sigma_{B})&=(2\pi/v)2\,\mathrm{Re}[T^{e*}T^{o}]d\rho\\&=(4\pi d\rho/v)\sum_{TT'jj',t=0,2}\alpha_{Tt}\alpha_{T'1}|\mathfrak{M}_{Tt}{}^{j}|\\&\qquad\times|\mathfrak{M}_{T'1}{}^{j'}|\cos(\delta_{Tt}{}^{j}-\delta_{T'1}{}^{j'}),\end{aligned} \tag{27}$$

where we have written $\mathfrak{M}_{Tt}{}^{j}=|\mathfrak{M}_{Tt}{}^{j}|\exp[i\delta_{Tt}{}^{j}]$ with $j=S$ or V. The meaning of (27) is that, if the kinematical conditions are varied so that, e.g., the odd t amplitude goes through a resonance (as would be the case for a final state $t=1$ $\pi-\pi$ "isobar") and the even t amplitude *does not*, then the cosine will vary rapidly.[5]

III. PION PRODUCTION BY PIONS

Before discussing the content of Eq. (27) we consider the analogous results for reaction (2). Corresponding to Eq. (3), the production amplitude for the process

[5] One must avoid the special arrangement in which $\mathbf{p}_{1}$ and $\mathbf{p}_{2}$ have equal magnitudes, and lie at equal angles (from the beam direction) in a plane parallel to the beam direction. For this case $d\sigma_{A}=d\sigma_{B}$.

(2) has the decomposition[3]

$$\langle \mathbf{p}\alpha\mathbf{q}\beta\tau'm' | T | \mathbf{k}\gamma\tau m\rangle = \sum_{Tt} a_{Tt}(\alpha\beta\gamma,\tau'\tau) T_{Tt}(\mathbf{pq},\mathbf{k}), \tag{28}$$

where $\mathbf{k}$ is the momentum, γ the charge state, of the initial pion. The other symbols have the same meaning as in Eq. (3). The nucleon spin dependence has been suppressed in the four isospin amplitudes $T_{\frac{1}{2}0}$, $T_{\frac{1}{2}1}$, $T_{\frac{3}{2}1}$, $T_{\frac{3}{2}2}$. The constants a_{Tt} are given by[3]

$$a_{Tt}(\alpha\beta\gamma,\tau'\tau) = C(11t;\alpha\beta)C(t\tfrac{1}{2}T;\alpha+\beta\tau') \times C(1\tfrac{1}{2}T;\gamma\tau). \tag{29}$$

Values of a_{Tt} for the $\pi^{\pm}-p$ reactions are given in Table II. Corresponding to (17), we have

$$T_{Tt}(\mathbf{pq},\mathbf{k}) = (-1)^t T_{Tt}(\mathbf{qp},\mathbf{k}). \tag{30}$$

The angular momentum analysis[1–3] of the amplitudes $T_{Tt}(\mathbf{pq},\mathbf{k})$ is similar to the isospin decomposition.

Now define the phases $\delta_{Tt}(\mathbf{pq},\mathbf{k})$ of the production amplitudes by the equation

$$T_{Tt}(\mathbf{pq},\mathbf{k}) \equiv \mathcal{T}_{Tt}(\mathbf{pq},\mathbf{k}) \exp[i\delta_{Tt}(\mathbf{pq},\mathbf{k})], \tag{31}$$

where $\mathcal{T}_{Tt}$ is a positive real number, the amplitude of T_{Tt}.

The differential cross section is of the form (18), so we are led to consider the absolute square of (28) (for a given set of values of $\alpha\beta\gamma$, $\tau'\tau$)

$$|T|^2 = \sum_{TT'tt'} A_{Tt,T't'} \cos[\delta_{Tt}-\delta_{T't'}]; \tag{32}$$

$$A_{Tt,T't'} \equiv a_{Tt}a_{T't'}\mathcal{T}_{Tt}\mathcal{T}_{T't'}. \tag{33}$$

It is of interest to investigate ways of determining the amplitudes $\mathcal{T}_{Tt}$ and phases δ_{Tt}. Only the relative phases occur in (32) so the best we can do is to determine the four amplitudes $\mathcal{T}_{Tt}$ and three relative phases. Making the definitions

$$\begin{aligned} \delta_{\frac{1}{2}0}-\delta_{\frac{1}{2}1} &\equiv \xi, \\ \delta_{\frac{1}{2}1}-\delta_{\frac{3}{2}1} &\equiv \eta, \\ \delta_{\frac{3}{2}1}-\delta_{\frac{3}{2}2} &\equiv \zeta, \end{aligned} \tag{34}$$

the remaining phase differences are given by

$$\begin{aligned} \delta_{\frac{1}{2}0}-\delta_{\frac{3}{2}1} &= \xi+\eta, \\ \delta_{\frac{1}{2}1}-\delta_{\frac{3}{2}2} &= \eta+\zeta, \\ \delta_{\frac{1}{2}0}-\delta_{\frac{3}{2}2} &= \xi+\eta+\zeta. \end{aligned} \tag{35}$$

With the definitions (33)–(35), (32) is given by the sum of all the elements in the symmetrical matrix M:

$$\begin{bmatrix} A_{\frac{1}{2}0\frac{1}{2}0} & A_{\frac{1}{2}0\frac{1}{2}1}\cos\xi & A_{\frac{1}{2}0\frac{3}{2}1}\cos(\xi+\eta) & A_{\frac{1}{2}0\frac{3}{2}2}\cos(\xi+\eta+\zeta) \\ A_{\frac{1}{2}0\frac{1}{2}1}\cos\xi & A_{\frac{1}{2}1\frac{1}{2}1} & A_{\frac{1}{2}1\frac{3}{2}1}\cos\eta & A_{\frac{1}{2}1\frac{3}{2}2}\cos(\eta+\zeta) \\ A_{\frac{1}{2}0\frac{3}{2}1}\cos(\xi+\eta) & A_{\frac{1}{2}1\frac{3}{2}1}\cos\eta & A_{\frac{3}{2}1\frac{3}{2}1} & A_{\frac{3}{2}1\frac{3}{2}2}\cos\zeta \\ A_{\frac{1}{2}0\frac{3}{2}2}\cos(\xi+\eta+\zeta) & A_{\frac{1}{2}1\frac{3}{2}2}\cos(\eta+\zeta) & A_{\frac{3}{2}1\frac{3}{2}2}\cos\zeta & A_{\frac{3}{2}2\frac{3}{2}2} \end{bmatrix}. \tag{36}$$

If desired, the five proton-pion reactions may be identified by the charge of the final pions. Then the total cross section for π^+-p $\{\pi^--p\}$ single pion production is proportional to

$$\mathrm{Tr}[M_{+0}+\tfrac{1}{2}M_{++}]\{\mathrm{Tr}[M_{+-}+M_{-0}+\tfrac{1}{2}M_{00}]\},$$

where Tr denotes the trace.

The simplest reaction is of course $\pi^++p\to\pi^++\pi^++n$. Only $A_{\frac{3}{2}2\frac{3}{2}2}=\frac{4}{5}\mathcal{T}_{\frac{3}{2}2}^2$ is non zero in this case. For a specific $\mathbf{pqk}$, $\mathcal{T}_{\frac{3}{2}2}$ is found from

$$d\sigma_{++} = (2\pi/v)(\tfrac{2}{5}\mathcal{T}_{\frac{3}{2}2}^2)d\rho. \tag{37}$$

From the reaction $\pi^++p\to\pi^++\pi^0+p$, one may find $\mathcal{T}_{\frac{3}{2}1}$ and $\cos(\delta_{\frac{3}{2}1}-\delta_{\frac{3}{2}2})=\cos\zeta$. Recalling Eqs. (25)–(27) we have

$$\tfrac{1}{2}(d\sigma_{+0}{}^A+d\sigma_{+0}{}^B) = (2\pi/v)[\tfrac{1}{2}\mathcal{T}_{\frac{3}{2}1}^2+\tfrac{1}{10}\mathcal{T}_{\frac{3}{2}2}^2]d\rho. \tag{38}$$

$\mathcal{T}_{\frac{3}{2}1}$ follows from (38) once $\mathcal{T}_{\frac{3}{2}2}$ has been found from (37). The magnitude of the phase difference ζ is found from

$$\tfrac{1}{2}(d\sigma_{+0}{}^A-d\sigma_{+0}{}^B) = -(2\pi/v)\mathcal{T}_{\frac{3}{2}1}\mathcal{T}_{\frac{3}{2}2}\cos\zeta\, d\rho/(5)^{\frac{1}{2}}. \tag{39}$$

Eq. (39) gives two possible solutions $\zeta=\pm|\zeta|$. Then from the relations

$$\frac{\frac{1}{2}(d\sigma_{-0}{}^A+d\sigma_{-0}{}^B)}{d\rho} = \frac{2}{9}\mathcal{T}_{\frac{1}{2}1}^2+\frac{1}{18}\mathcal{T}_{\frac{3}{2}1}^2 +\frac{2}{9}\mathcal{T}_{\frac{1}{2}1}\mathcal{T}_{\frac{3}{2}1}\cos\eta+\frac{1}{10}\mathcal{T}_{\frac{3}{2}2}^2, \tag{40}$$

$$\frac{\frac{1}{2}(d\sigma_{-0}{}^A-d\sigma_{-0}{}^B)}{d\rho} = [2\mathcal{T}_{\frac{1}{2}1}\mathcal{T}_{\frac{3}{2}2}\cos(\eta+\zeta) +\mathcal{T}_{\frac{3}{2}1}\mathcal{T}_{\frac{3}{2}2}\cos\zeta]/[3(5)^{\frac{1}{2}}], \tag{41}$$

one can find solutions for $\mathcal{T}_{\frac{1}{2}1}$ and η, for each of the two signs of ζ. Note that the sign of η is coupled to the choice of ζ: if $\zeta\to-\zeta$ then $\eta\to-\eta$ gives a solution. Rearranging (40) and (41), dividing (41) by (40) gives $\tan\eta=f(\mathcal{T}_{\frac{1}{2}1}^2)$, where f is of the form

$$a-b(c-\mathcal{T}_{\frac{1}{2}1}^2)^{-1}.$$

Substitution into (41) then gives a cubic equation for $\mathcal{T}_{\frac{1}{2}1}^2$. The requirement that $\mathcal{T}_{\frac{1}{2}1}$ be real and positive excludes negative and complex values of $\mathcal{T}_{\frac{1}{2}1}^2$, although *a priori* it appears that one cannot exclude multiple solutions. In a similar way one can find values of $\mathcal{T}_{\frac{1}{2}0}$ and ξ which satisfy the relations

$$\begin{aligned} &\tfrac{1}{2}(d\sigma_{+-}{}^A+d\sigma_{+-}{}^B)/d\rho \\ &= (1/9)\{2\mathcal{T}_{\frac{1}{2}0}^2+\mathcal{T}_{\frac{1}{2}1}^2+\tfrac{1}{5}\mathcal{T}_{\frac{3}{2}2}^2+\mathcal{T}_{\frac{3}{2}1}^2-2\mathcal{T}_{\frac{1}{2}1}\mathcal{T}_{\frac{3}{2}1}\cos\eta \\ &\qquad -2(\tfrac{2}{5})^{\frac{1}{2}}\mathcal{T}_{\frac{1}{2}0}\mathcal{T}_{\frac{3}{2}2}\cos(\xi+\eta+\zeta)\}, \end{aligned} \tag{42}$$

$$\tfrac{1}{2}(d\sigma_{+-}{}^{A}-d\sigma_{+-}{}^{B})/d\rho$$
$$=(2/9)\{-(2)^{\frac{1}{2}}\mathcal{T}_{\frac{1}{2}0}\mathcal{T}_{\frac{1}{2}1}\cos\xi+(2)^{\frac{1}{2}}\mathcal{T}_{\frac{1}{2}0}\mathcal{T}_{\frac{3}{2}1}\cos(\xi+\eta)$$
$$+(5)^{-\frac{1}{2}}\mathcal{T}_{\frac{1}{2}1}\mathcal{T}_{\frac{3}{2}2}\cos(\eta+\zeta)-(5)^{-\frac{1}{2}}\mathcal{T}_{\frac{3}{2}1}\mathcal{T}_{\frac{3}{2}2}\cos\zeta\}. \quad (43)$$

The set of solutions (in general nonunique) generated by Eqs. (37)–(43) could be tested for consistency by the experimentally difficult process $\pi^-+p\rightarrow\pi^0+\pi^0+n$:

$$9d\sigma_{00}/d\rho=\mathcal{T}_{\frac{1}{2}0}{}^2+\tfrac{2}{5}\mathcal{T}_{\frac{3}{2}2}{}^2+2(\tfrac{2}{5})^{\frac{1}{2}}\mathcal{T}_{\frac{1}{2}0}\mathcal{T}_{\frac{3}{2}2}\cos(\xi+\eta+\zeta). \quad (44)$$

Recently it has proved possible to measure the *total* cross section for reaction (44).[6]

Despite the nonuniqueness of our solutions, the latter should provide interesting information about the dynamics of the production process.

The *total* cross sections for single pion production are

$$\sigma_{\mathrm{tot}}(\pi^- - p)=\tfrac{1}{3}(\sigma_{\frac{1}{2}0}+\sigma_{\frac{1}{2}1})+\tfrac{1}{6}(\sigma_{\frac{3}{2}1}+\sigma_{\frac{3}{2}2}), \quad (45)$$

$$\sigma_{\mathrm{tot}}(\pi^+ - p)=\tfrac{1}{2}(\sigma_{\frac{3}{2}1}+\sigma_{\frac{3}{2}2}), \quad (46)$$

where the partial cross sections are given by

$$\sigma_{Tt}\equiv(2\pi/v)\int\mathcal{T}_{Tt}{}^2d\rho. \quad (47)$$

IV. REMARKS

The isospin amplitudes are of considerable intrinsic interest. There is another possible application of our results, however. It will be recalled that the quantity $(d\sigma_A-d\sigma_B)$ is particularly sensitive to the phase difference between the even and odd $\pi-\pi$ isospin states [cf. Eqs. (27), (39), (41), and (43)]. Now, if the formation of an "isobar" (either $\pi\pi$ or πN) in the final state is predominant, the phase of the total amplitude varies rapidly as the c.m. energy of the isobaric constituents passes through the resonance energy. This variation is reflected in the behavior of $(d\sigma_A-d\sigma_B)$. For example, in the case of the (πN) isobar, we have found that the total amplitude contains as a factor the elastic scattering ($3-3$ state) amplitude.[3] Further, the πN isobar does not especially favor either even or odd $\pi-\pi$ isospin insofar as the occurrence (but not the amplitude) of resonances is concerned. On the other hand, our coupling scheme emphasizes the dynamical features of the $\pi-\pi$ system [besides yielding the simple symmetry of Eq. (30)].

Two cases are of especial interest. First of all, suppose the final state is dominated by a $\pi-\pi$ isobar (e.g., a $t=1$ resonance[7]). In this case a straightforward Q-value plot should detect the resonance. The methods of the present paper could then be used to identify the resonant isospin state, and also to verify the rapid change of phase associated with the resonance. Secondly it may be desirable to segregate the single pion exchange contribution[8,9] to the reaction (2) from the total amplitude. This separation is expected to be difficult in the resonance region ($0.5<E_\pi<1.5$ Bev) because of the frequency of πN isobar formation (in this energy range the energy of at least one of the final pions can be near the $3-3$ resonance energy). Therefore it is necessary to choose kinematic conditions carefully. Note that in the measurement of $d\sigma_A-d\sigma_B$ the relative $\pi-N$ energy for a pion *of given charge* is not the same for conditions A and B. Therefore the conditions specified by Drell and Zachariasen[10] are not satisfied, so that one must attempt to avoid the $\pi-N$ resonance region for both the final pions.

TABLE II. The coefficients a_{Tt} of the isospin amplitudes T_{Tt} for the $\pi+p\rightarrow2\pi+N$ reactions.

Reaction	$a_{\frac{1}{2}0}$	$a_{\frac{1}{2}1}$	$a_{\frac{3}{2}1}$	$a_{\frac{3}{2}2}$
$\pi^-+p\rightarrow\pi^-+\pi^++n$	$-\frac{1}{3}(2)^{\frac{1}{2}}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}(5)^{-\frac{1}{2}}$
$\pi^-+p\rightarrow\pi^-+\pi^0+p$	0	$-\frac{1}{3}(2)^{\frac{1}{2}}$	$-\frac{1}{3}(2)^{-\frac{1}{2}}$	$-(10)^{-\frac{1}{2}}$
$\pi^-+p\rightarrow\pi^0+\pi^0+n$	$\frac{1}{3}(2)^{\frac{1}{2}}$	0	0	$(2/3)(5)^{-\frac{1}{2}}$
$\pi^++p\rightarrow\pi^++\pi^0+p$	0	0	$(2)^{-\frac{1}{2}}$	$-(10)^{-\frac{1}{2}}$
$\pi^++p\rightarrow\pi^++\pi^++n$	0	0	0	$2/(5)^{\frac{1}{2}}$

Finally we remark on the interesting features of the reaction $\pi^-+p\rightarrow\pi^-+\pi^0+p$ near a Bev incident pion lab energy. Recently Pickup, Ayer and Salant,[11] and Rushbrooke and Radojicic[12] reported the existence of a sharp peak in the $Q_{\pi\pi}$ spectrum. However the reaction $\pi^-+p\rightarrow\pi^-+\pi^++n$ shows no such peak,[11] but behaves as expected from the $(\pi-N)$ isobar model. This behavior can be understood easily if the formation of a $(\pi-\pi)$ $t=1$ isobar occurs in the final state. In this case one expects the amplitudes $T_{\frac{1}{2}1}$ and $T_{\frac{3}{2}1}$ to have roughly equal amplitude and phase. From Table II it is then seen that for $\pi^-+p\rightarrow\pi^-+\pi^++n$ the $t=1$ amplitudes cancel, while for $\pi^-+p\rightarrow\pi^-+\pi^0+p$ the interference is constructive in the $t=1$ states. Therefore the apparent fact that the $(\pi-\pi)$ isobar dominates the final state in the reaction $\pi^-+p\rightarrow\pi^++\pi^0+p$ while the $(\pi-N)$ isobar dominates $\pi^-+p\rightarrow\pi^-+\pi^++n$ is simply understood in terms of interference between the $(Tt)=(\frac{1}{2}1)$, $(\frac{3}{2}1)$ states.

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