

where

$$m_{\alpha\beta} = \frac{\omega - k_\alpha v_\beta}{\omega(\omega - \mathbf{k}\mathbf{v})} \quad \text{for } \alpha = \beta, \quad (53)$$

$$m_{\alpha\beta} = \frac{k_\beta v_\alpha}{\omega(\omega - \mathbf{k}\mathbf{v})} \quad \text{for } \alpha \neq \beta.$$

Consequently, the electric susceptibility can be expressed as

$$(\chi_e)_{\gamma\beta} = -\frac{i\omega}{4\pi} \int m_{\gamma\alpha} K_{\alpha\beta} d\mathbf{p}. \quad (54)$$

In order to obtain the magnetic polarization, we put $M = \int M^{(p)} d\mathbf{p}$. The term $M^{(p)}$ can be determined from

(11) and represented in the form

$$M_\alpha^{(p)} = -\frac{1}{c} (P_\beta^{(p)} v_\gamma - P_\gamma^{(p)} v_\beta), \quad (55)$$

where α, β, γ are in a cyclic succession.

Substituting (52) in (55) and utilizing the relationship $M_\alpha = (\chi_{e\mu})_{\alpha\beta} E_\beta$, we can express the "electromagnetic susceptibility" as follows:

$$(\chi_{e\mu})_{\alpha\beta} = -\frac{i\omega}{4\pi} \int v_\gamma m_{\beta\delta} K_{\delta\rho} - v_\beta m_{\gamma\delta} K_{\delta\rho} d\mathbf{p}, \quad (56)$$

where α, β, γ are in a cyclic succession.

Mobility of Ions in a System of Interacting Bose Particles

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The transport property of an ion in a dilute Bose-Einstein gas subject to an external electric field is investigated by means of the Boltzmann equation. The interaction Hamiltonian which describes the ion-phonon scattering processes is obtained by the use of the Bogoliubov transformation and the cross section for the scattering of the ion by phonons is calculated. The solution of the Boltzmann equation is obtained by applying a variation principle and the temperature dependence of the ion mobility is shown to be T^{-4} at very low temperatures. A comparison of the results with the experimental data in liquid helium and the Khalatnikov and Zharkov theory is given and also the ion mobility in a Fermi system is briefly discussed.

1. INTRODUCTION

THE purpose of this paper is to study the mobility of an ion in a dilute Bose gas in connection with the recent experiments in which ions in liquid helium are used as microscopic probe particles to investigate the properties of the superfluid.¹⁻⁴ Because of the superfluid property, one expects that the ions move through liquid helium without encountering any resistance in its ground state at the absolute zero. On the other hand, at finite temperatures the ions suffer the scattering processes which arise from the thermal excitations present in the fluid. Thus, one may expect that by studying the motion of ions, useful information can be deduced about the nature of the elementary

excitations and possible interactions between a particle and the underlying quantum fluid.

The simplest experimental situation is to apply an electric field E and to measure the drift velocity u of the ion. If E is kept sufficiently small, u is expected to be proportional to E and one may define the field-independent mobility $\mu = u/E$. In fact, Meyer and Reif¹ have shown experimentally that μ is independent of E when $E \ll 1$ volt/cm and that its temperature dependence is of the form $\mu = \mu_0 \exp(\Delta/kT)$ in the range below the λ point down to 0.8°K. A possible interpretation of this behavior was proposed by Meyer and Reif¹ based on the scattering of the ion by rotons. At temperatures below 0.6°K, they obtained a temperature dependence of the form $\mu \propto T^{-k}$, where $k = 3.3 \pm 0.3$ for a positive ion and $k = 2.4 \pm 0.4$ for a negative ion, and they pointed out that this behavior disagrees with the prediction by Khalatnikov and Zharkov⁵ obtained on the basis of ion-phonon interactions derived from quantum hydrodynamics. So it is of considerable interest to re-examine the temperature dependence of the ion mobility due to phonon excitations from first principles.

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¹ L. Meyer and F. Reif, Phys. Rev. **110**, 279 (1958) and **119**, 1164 (1960); also Phys. Rev. Letters **5**, 1 (1960).

² R. L. Williams, Can. J. Phys. **35**, 134 (1957).

³ G. Careri, F. Scaramuzzi, and J. O. Thomson, Nuovo cimento **13**, 186 (1959); G. Careri, U. Fasoli, and F. S. Gaeta, *ibid.* **15**, 774 (1960).

⁴ K. R. Atkins, Phys. Rev. **116**, 1339 (1959); D. N. Swan, Proc. Phys. Soc. (London) **76**, 36 (1960); and C. G. Kuper (to be published).

⁵ I. M. Khalatnikov and V. N. Zharkov, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 1108 (1957) [translation: Soviet Phys.—JETP **5**, 905 (1957)].

However, the actual calculation of the ion mobility is very difficult if one deals with a system of moderately high density such as liquid helium and takes into account the actual potential of interaction between particles. On the other hand, if one starts instead from a simple model, for instance a Bose gas with hard-sphere interactions which has been studied by various authors,⁶⁻⁸ one may carry out the actual calculations to some extent and deduce some qualitative features of liquid helium. For example, as Lee and Yang⁸ have shown, the dilute hard-sphere Bose gas undergoes a second-order phase transition which is qualitatively similar to that observed in liquid helium. One may hope, therefore, that the study of the ion mobility in a dilute Bose gas enables one to get some understandings of the qualitative aspect of the observed ion mobility and its relationship with the superfluidity of quantum fluid.

In Sec. 2 the explicit form for the interaction Hamiltonian between the ion and phonons is derived by means of the Bogoliubov transformation.⁹ It gives rise to two different processes, i.e., the scattering of an ion by phonons and the emission and absorption of phonons by an ion. In Sec. 3 the cross section for the ion-phonon scattering is calculated, together with the relaxation time of an ion for emission and absorption processes. An application of simple kinetic theory yields the results that the contribution from emission and absorption processes can be neglected compared to the scattering process. Also, the temperature dependence of the ion mobility due to phonon excitations is shown to be T^{-3} at low temperatures. In Sec. 4 we set up the Boltzmann equation describing the motion of the ion; the solution is obtained by applying the variation principle. The temperature dependence of the ion mobility due to phonon excitations is shown to be T^{-4} . Section 5 is devoted to a discussion of the discrepancy between the result derived by simple kinetic theory and that calculated from the Boltzmann equation. A comparison with the experimental data in liquid helium and the Khalatnikov and Zharkov theory⁵ is given. The Bogoliubov Hamiltonian is discussed in connection with the recent work by Wentzel,¹⁰ and the ion-phonon scattering is studied by the use of Feynman's¹¹ wave function. The ion mobility in a Fermi system is briefly mentioned and its behavior at low temperatures is discussed. Finally, the details of calculations are given in the Appendixes.

2. INTERACTION OF AN ION WITH PHONONS

Let us start with a brief review of the method for dealing with the ground state and low-lying excited states of interacting Bose particles.¹² We consider a system of N identical particles with mass M in a box of volume V . Let us denote the interaction potential between particles by $u(\mathbf{x})$ and assume that it has a finite positive Fourier component u_q defined by

$$u_q = \int u(\mathbf{x}) \exp(-i\mathbf{q} \cdot \mathbf{x}) d\mathbf{x}, \quad (1)$$

where the integration is over the volume V .

Then the Hamiltonian of the system is written in the second-quantized form in terms of creation and annihilation operators, η^* and η , satisfying the commutation relations characteristic of Bose statistics. If one restricts oneself to the discussion of the ground state and low-lying energy levels, it is permissible to assume that nearly all the particles are in a single-particle state with momentum zero and to set η_0^* or η_0 equal to $N^{1/2}$. This simplification leads to the following reduced Hamiltonian:

$$H_{\text{red}} = \frac{1}{2} N u_0 + \sum_{q \neq 0} \{ (E_q + \rho u_q) \eta_q^* \eta_q + \frac{1}{2} \rho u_q (\eta_q^* \eta_{-q}^* + \eta_q \eta_{-q}) \}, \quad (2)$$

with $\rho = N/V$ and

$$E_q = q^2/2M. \quad (3)$$

Equation (2) is a simple quadratic form, and can be diagonalized by means of the Bogoliubov transformation⁹:

$$\eta_q = \lambda_q a_q - \mu_q a_{-q}^*, \quad \eta_q^* = \lambda_q a_q^* - \mu_q a_{-q}, \quad (4)$$

with

$$\lambda_q = (\omega_q + E_q)(4E_q \omega_q)^{-1/2}, \quad \mu_q = (\omega_q - E_q)(4E_q \omega_q)^{-1/2}. \quad (5)$$

Substitution of Eqs. (4) and (5) into Eq. (2) leads to

$$H_{\text{red}} = \sum_{q \neq 0} \omega_q a_q^* a_q + E_0, \quad (6)$$

where ω_q is the energy of elementary excitation (phonon or roton) given by

$$\omega_q = (E_q^2 + 2\rho E_q u_q)^{1/2}, \quad (7)$$

and E_0 is the ground-state energy given by

$$E_0 = \frac{1}{2} N \rho u_0 + \frac{1}{2} \sum_{q \neq 0} (\omega_q - E_q - \rho u_q).$$

So far we have assumed the existence of u_q , but even if it diverges to infinity as in the case of hard spheres, exactly the same argument can be applied if one replaces the potential function by the pseudopotential or scattering matrix.^{6,7} In this way the single formula given by Eq. (7) reproduces the energy spectrum of

⁶ K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957); T. D. Lee, K. Huang, and C. N. Yang, *ibid.* **106**, 1135 (1957).

⁷ K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117, 1128 (1957); R. Abe, Progr. Theoret. Phys. (Kyoto) **19**, 699 (1958); and A. J. F. Siegert, Phys. Rev. **116**, 1057 (1959).

⁸ T. D. Lee and C. N. Yang, Phys. Rev. **112**, 1419 (1958); **113**, 1406 (1959).

⁹ N. Bogoliubov, J. Phys. (U.S.S.R.) **11**, 23 (1947).

¹⁰ G. Wentzel, Phys. Rev. **120**, 1572 (1960).

¹¹ R. P. Feynman, Phys. Rev. **94**, 262 (1954).

¹² Throughout this paper we shall use a system of units such as $\hbar=1$.

elementary excitations quite similar to that observed in liquid helium. However, the validity of the Bogoliubov transformation may be rather suspect in the roton region, so that in the following discussions we restrict ourselves to the phonon excitations and assume that ω_q takes the form

$$\omega_q = cq, \quad (8)$$

where c is sound velocity.

Now if an ion with mass M_i is put in the system under consideration, it gives rise to the interaction Hamiltonian:

$$H_{\text{int}} = \int v(\mathbf{x} - \mathbf{x}') \Psi^*(\mathbf{x}') \Psi(\mathbf{x}') d\mathbf{x}', \quad (9)$$

where v is the interaction potential between ion and Bose particle, \mathbf{x} is the position vector of the ion and Ψ is the field operator for the Bose particles. If we expand Ψ and Ψ^* in a form of Fourier series:

$$\begin{aligned} \Psi(\mathbf{x}) &= V^{-\frac{1}{2}} \sum \eta_q \exp(i\mathbf{q} \cdot \mathbf{x}), \\ \Psi^*(\mathbf{x}) &= V^{-\frac{1}{2}} \sum \eta_q^* \exp(-i\mathbf{q} \cdot \mathbf{x}), \end{aligned} \quad (10)$$

we can express H_{int} in terms of η and η^* and the substitution of Eq. (4) leads to the explicit form for the interaction Hamiltonian between the ion and phonons. In this way

$$H_{\text{int}} = \mathcal{H} + \mathcal{H}', \quad (11)$$

where \mathcal{H} describes the scattering of the ion by phonons, i.e.,

$$\mathcal{H} = V^{-1} \sum_{\mathbf{q}, \mathbf{q}' \neq 0} \gamma_{\mathbf{q}, \mathbf{q}'} a_{\mathbf{q}'}^* a_{\mathbf{q}} \exp[i(\mathbf{q} - \mathbf{q}') \cdot \mathbf{x}], \quad (12)$$

with

$$\gamma_{\mathbf{q}, \mathbf{q}'} = \frac{1}{2} v_{\mathbf{q}-\mathbf{q}'} (\omega_q \omega_{q'} + E_q E_{q'}) (\omega_q \omega_{q'} E_q E_{q'})^{-\frac{1}{2}}. \quad (13)$$

Here v_q is the Fourier component of $v(\mathbf{x})$, defined in the same way as in Eq. (1).

On the other hand, \mathcal{H}' describes the processes in which phonons are created or annihilated by the ion and contains terms such as a^* , a , $a^* a^*$, and aa . If the terms are restricted to a^* and a , the explicit form for \mathcal{H}' reads

$$\mathcal{H}' = N^{\frac{1}{2}} V^{-1} \sum_{\mathbf{q} \neq 0} \gamma_q [a_q^* \exp(-i\mathbf{q} \cdot \mathbf{x}) + a_q \exp(i\mathbf{q} \cdot \mathbf{x})], \quad (14)$$

where γ_q is given by

$$\gamma_q = v_q (E_q / \omega_q)^{\frac{1}{2}}. \quad (15)$$

In the following section and Appendix 2, it is shown that the contribution of \mathcal{H}' to the ion mobility is negligibly small in the temperature range in which we are interested, and it may be expected that the contribution of terms such as $a^* a^*$ and aa can be neglected as well. We refer to this point later on.

As is seen in Eq. (12), \mathcal{H} represents a process in which the ion with momentum \mathbf{k} collides with the phonon of momentum \mathbf{q} and is scattered to a state with momentum $\mathbf{k} + \mathbf{q} - \mathbf{q}'$, while the phonon is scattered to a state with momentum \mathbf{q}' . By applying the standard method of time-dependent perturbation theory, the transition probability for the process $(\mathbf{k}, N_q, N_{q'}) \rightleftharpoons (\mathbf{k} + \mathbf{q} - \mathbf{q}', N_q - 1, N_{q'} + 1)$, where N_q is the occupation number of a phonon of momentum \mathbf{q} , is shown to be

$$2\pi V^{-2} \gamma_{\mathbf{q}, \mathbf{q}'}^2 N_q (N_{q'} + 1) \delta(\epsilon_k + \omega_q - \epsilon_{\mathbf{k} + \mathbf{q} - \mathbf{q}'} - \omega_{q'}), \quad (16)$$

where ϵ_k is the kinetic energy of the ion, i.e.,

$$\epsilon_k = \mathbf{k}^2 / 2M_i, \quad (17)$$

and δ is the Dirac delta function representing the energy conservation.

Similarly, the transition probabilities for emission and absorption of phonons are easily calculated by the use of Eq. (14) and are shown to be

$$2\pi N V^{-2} \gamma_q^2 N_q \delta(\epsilon_{\mathbf{k} + \mathbf{q}} - \epsilon_k - \omega_q) \quad (18)$$

for the process $(\mathbf{k}, N_q) \rightleftharpoons (\mathbf{k} + \mathbf{q}, N_q - 1)$, and

$$2\pi N V^{-2} \gamma_q^2 (N_q + 1) \delta(\epsilon_{\mathbf{k} - \mathbf{q}} - \epsilon_k + \omega_q) \quad (19)$$

for the process $(\mathbf{k}, N_q) \rightleftharpoons (\mathbf{k} - \mathbf{q}, N_q + 1)$, respectively.

3. ELEMENTARY KINETIC THEORY

Before entering into the main problem of calculating the ion mobility based on the Boltzmann equation, we discuss the order of magnitude and the temperature dependence by applying elementary kinetic theory in order to select the most important process among the various ones contributing to the ion mobility.

If we let the relaxation time of the ion be τ , elementary kinetic theory yields the relation $M_i u = eE\tau$ for the ion drift velocity u , E being the strength of applied electric field and e the charge of the ion. Then the ion mobility μ is expressed as

$$\mu = (e/M_i)\tau. \quad (20)$$

If we neglect the processes in which two phonons are emitted or absorbed, there are two processes in the present case which lead to two relaxation times: τ_p for the scattering by phonons and τ_p' for the emission and absorption processes of phonons. Then the over-all relaxation time τ is given by

$$\tau^{-1} = \tau_p^{-1} + \tau_p'^{-1}. \quad (21)$$

The relaxation time or collision time for scattering processes can be estimated by a simple kinetic relation,¹

$$\tau_p^{-1} = n_p v_p \sigma_p, \quad (22)$$

where n_p is the number density of phonons given by

$$n_p = (2.4/2\pi^2) (kT/c)^3; \quad (23)$$

k equals Boltzmann's constant and T the absolute

temperature. Furthermore, v_p is the average speed of the ion relative to the phonon and is given approximately by

$$v_p = c. \quad (24)$$

Finally, σ_p is the average cross section for the ion-phonon scattering. We calculate this cross section on the basis of the interaction Hamiltonian [Eq. (12)]. As for the relaxation time τ_p' , we identify it with the average lifetime due to emission and absorption processes given by the Hamiltonian [Eq. (14)].

In calculating the cross section σ_p by the use of Eq. (16), we notice that since we are interested in excitations of long wavelength, the contribution of small \mathbf{q} and \mathbf{q}' is most important. Therefore, from Eq. (13), neglecting higher order terms, we have

$$\gamma_{\mathbf{q}, \mathbf{q}'}^2 = g^2 M^2 c^2 / q q', \quad (25)$$

where $g = v_0$. If we set the scattering length for the interaction between the ion and the background Bose particle to be a , g is given by^{6,7}

$$g = 2\pi a / m, \quad (26)$$

with m the reduced mass defined by

$$m^{-1} = M^{-1} + M_i^{-1}. \quad (27)$$

Under the above approximations, σ_p is calculated in the usual way. In the actual calculation, the relative velocity between ion and phonon is replaced by its average value given by Eq. (24). Furthermore, in evaluating the phase integral appearing in the expression for σ_p , we note that the effective momenta of the ion and phonon contributing to ion-phonon scattering are small compared to $M_i c$, so that the phonon momentum does not change appreciably in the scattering process. In this way we find approximately

$$\sigma_p = 4\pi a^2. \quad (28)$$

Substituting Eqs. (23), (24), and (28) into Eq. (22), we have

$$\tau_p^{-1} = (4.8/\pi) a^2 c (kT/c)^3. \quad (29)$$

Next we estimate the relaxation time τ_p' for emission and absorption processes. We calculate it as the mean lifetime of the ion by the use of Eqs. (18) and (19). After making appropriate average over ion and phonon states, we get

$$\tau_p'^{-1} = \frac{\rho}{(2\pi)^2} \int d\mathbf{k} d\mathbf{k}' d\mathbf{q} f_0(\mathbf{k}) \gamma_{\mathbf{q}}^2 \delta(\mathbf{k} - \mathbf{k}' - \mathbf{q}) \{ (1 + n_{\mathbf{q}}) \times \delta(\epsilon_{\mathbf{k}} - \omega_{\mathbf{q}} - \epsilon_{\mathbf{k}'}) + n_{\mathbf{q}} \delta(\epsilon_{\mathbf{k}} + \omega_{\mathbf{q}} - \epsilon_{\mathbf{k}'}) \}, \quad (30)$$

where $f_0(\mathbf{k})$ is the Boltzmann distribution function for ions in thermal equilibrium and is given by

$$f_0(\mathbf{k}) = \exp(-\beta \epsilon_{\mathbf{k}}) \left[\int \exp(-\beta \epsilon_{\mathbf{k}'}) d\mathbf{k}' \right]^{-1}, \quad (31)$$

with $\beta = (kT)^{-1}$ and $n_{\mathbf{q}}$, the distribution function for phonons in thermal equilibrium, i.e.,

$$n_{\mathbf{q}} = [\exp(\beta \omega_{\mathbf{q}}) - 1]^{-1}, \quad (32)$$

$\omega_{\mathbf{q}}$ being given by Eq. (8).

If we assume that $kT \ll M_i c^2/2$, we find approximately

$$\tau_p'^{-1} = 33.6 (2\pi)^{1/2} \rho a^2 c (M_i/M)^{1/2} (\beta M c^2)^{-1/2} \times \exp(-\beta M_i c^2/2). \quad (33)$$

It should be noted that emission and absorption processes in Eq. (30) give rise to the same contribution because we assume thermal equilibrium for the phonon distribution. Also note that Eq. (33) contains a threshold factor $\exp(-\beta M_i c^2/2)$ which arises from the conservation of energy and momentum in emission and absorption processes. Physically, the appearance of this factor is due to the fact that an ion moving with the velocity less than c can absorb but cannot emit a phonon. In the two-phonon processes (neglected up to this point), it is shown that there is the same critical velocity for the ion, so that the inverse of the lifetime is expected to decrease exponentially as the temperature is lowered.

If we take the ratio of (33) to (29) and evaluate it, for example, at $T = 0.5^\circ \text{K}$, we find $\tau_p'^{-1}/\tau_p^{-1} \approx 4 \times 10^{-12} (M_i/M)$. This shows that one can completely neglect the emission and absorption processes. We obtain the same conclusion in Appendix 2 by using the Boltzmann equation.

4. BOLTZMANN EQUATION AND ITS SOLUTION

In the previous section we discussed the ion mobility by applying simple kinetic theory. In this section we derive the Boltzmann equation describing the motion of the ion and solve it by means of a variation principle.

If we assume that the electric field E is applied along the x axis, the Boltzmann equation for the distribution function of the ion, $f(\mathbf{k}, t)$, takes the form:

$$\frac{\partial f}{\partial t} + eE \frac{\partial f}{\partial k_x} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}}, \quad (34)$$

where $(\partial f / \partial t)_{\text{coll}}$ is the change of distribution function due to collisions. Based on the expressions for the transition probabilities derived in Sec. 2, $(\partial f / \partial t)_{\text{coll}}$ is easily calculated in the same way as in the theory of electron conduction in metals.¹³ Since emission and absorption processes of phonons can be neglected as we showed in Sec. 3, if we assume that the ion obeys Boltzmann statistics and that the phonons are in a thermal equilibrium, we have from Eq. (16):

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = \frac{2\pi}{V^2} \sum_{\mathbf{q}, \mathbf{q}'} \gamma_{\mathbf{q}, \mathbf{q}'}^2 \{ (n_{\mathbf{q}} + 1) n_{\mathbf{q}'} f(\mathbf{k} + \mathbf{q} - \mathbf{q}') - n_{\mathbf{q}} (n_{\mathbf{q}'} + 1) f(\mathbf{k}) \} \delta(\epsilon_{\mathbf{k}} + \omega_{\mathbf{q}} - \epsilon_{\mathbf{k} + \mathbf{q} - \mathbf{q}'} - \omega_{\mathbf{q}'}), \quad (35)$$

¹³ See for example, A. H. Wilson, *The Theory of Metals* (Cambridge University Press, New York, 1953), 2nd ed., p. 251.

where n_q is the distribution function of the phonons given by Eq. (32).

For a steady state we have $\partial f / \partial t = 0$ in Eq. (34) and if we assume that E is sufficiently small, we can put

$$f(\mathbf{k}) = f_0(\mathbf{k}) - \beta e E M_i^{-1} k_x c(\epsilon_k) f_0(\mathbf{k}), \quad (36)$$

and retain the terms of the order of E in Eq. (34). Here $f_0(\mathbf{k})$ is the Boltzmann distribution function given by Eq. (31). If we replace the summation over \mathbf{q} and \mathbf{q}' by an integral, then by a straightforward calculation the integral equation for $c(\epsilon_k)$ is shown to be

$$\mathcal{L}(c) = k_x, \quad (37)$$

where $\mathcal{L}(c)$ is an integral operator defined by

$$\begin{aligned} \mathcal{L}(c) = (2\pi)^{-5} \int d\mathbf{q} d\mathbf{q}' \gamma_{\mathbf{q}, \mathbf{q}'}^2 n_{\mathbf{q}}(n_{\mathbf{q}'} + 1) \{ (k_x + q_x - q'_x) \\ \times c(\epsilon_{\mathbf{k}+\mathbf{q}-\mathbf{q}'} - k_x c(\epsilon_k)) \} \delta(\epsilon_k + \omega_{\mathbf{q}} - \epsilon_{\mathbf{k}+\mathbf{q}-\mathbf{q}'} - \omega_{\mathbf{q}'}). \end{aligned} \quad (38)$$

On the other hand, the ion mobility μ is easily shown to be

$$\mu = - \frac{\beta e}{M_i^2} \int k_x^2 c(\epsilon_k) f_0(\mathbf{k}) d\mathbf{k}. \quad (39)$$

In order to solve the integral Eq. (37), let us apply the variation principle which has been used to solve problems of transport phenomena. For the present problem this principle is stated as follows. Let f and g be any two functions and let

$$(f, g) = - \int k_x \exp(-\beta \epsilon_k) f(\epsilon_k) \mathcal{L}(g) d\mathbf{k}. \quad (40)$$

Then the solution of Eq. (37) is such as to make the integral (f, f) a maximum, under the subsidiary condition

$$\begin{aligned} \int k_x \exp(-\beta \epsilon_k) f(\epsilon_k) \mathcal{L}(f) d\mathbf{k} \\ = \int k_x^2 \exp(-\beta \epsilon_k) f(\epsilon_k) d\mathbf{k}. \end{aligned} \quad (41)$$

To prove the principle, it is necessary to establish the relations $(f, g) = (g, f)$ and $(f, f) \geq 0$. The proof of these relations is given in Appendix 1. Once these relations are established, the variation principle is proved as in the theory of metals and the details are not given here.

In the theory of metals, it is known that the choice of a constant trial function gives a good approximation in the whole range of temperatures and that correction terms are small. We may expect the same situation in the present case and anticipate that the variation with constant trial function yields the dominant contribution to the mobility. We show in Appendix 1 that the

correction terms are really small at low-temperature limits.

If we set the trial function $f(\epsilon_k)$ as

$$f(\epsilon_k) = \text{const} = -\tau, \quad (42)$$

then from Eq. (41) the best choice for τ is shown to be

$$\tau^{-1} = d_{00} \left[\int k_x^2 \exp(-\beta \epsilon_k) d\mathbf{k} \right]^{-1}, \quad (43)$$

where d_{00} is given [see Eq. (A1) in Appendix 1] by

$$\begin{aligned} d_{00} = [6(2\pi)^5]^{-1} \int d\mathbf{k} d\mathbf{q} d\mathbf{q}' \exp(-\beta \epsilon_k) \gamma_{\mathbf{q}, \mathbf{q}'}^2 n_{\mathbf{q}}(n_{\mathbf{q}'} + 1) \\ \times (\mathbf{q} - \mathbf{q}')^2 \delta(\epsilon_k + \omega_{\mathbf{q}} - \epsilon_{\mathbf{k}+\mathbf{q}-\mathbf{q}'} - \omega_{\mathbf{q}'}). \end{aligned} \quad (44)$$

Substitution of Eq. (42) into Eq. (39) leads to the result that the ion mobility is given by exactly the same equation as Eq. (20). In other words, the best constant trial function is the relaxation time with the negative sign, as we have taken in Eq. (42).

If we further assume that $f(\epsilon_k)$ is expressed as a power series of ϵ_k , i.e., $f(\epsilon_k) = \sum c_n \epsilon_k^n$, then the variation principle leads to an infinite set of linear equations for the c_n 's and we can formally solve these equations by introducing the determinants whose elements are given by

$$d_{rs} = (\epsilon_k^r, \epsilon_k^s) = - \int k_x \exp(-\beta \epsilon_k) \epsilon_k^r \mathcal{L}(\epsilon_k^s) d\mathbf{k}, \quad (45)$$

and

$$\alpha_r = - \int k_x^2 \exp(-\beta \epsilon_k) \epsilon_k^r d\mathbf{k}. \quad (46)$$

Substitution of the solution in Eq. (39) yields an expression in which the mobility is given by a ratio of infinite determinants. This ratio is further simplified with the aid of the theorem on determinants and finally we have¹⁴

$$\mu = - \frac{e \beta^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} M_i^{7/2}} \left\{ \frac{\alpha_0^2}{d_{00}} + \sum_{n=2}^{\infty} \frac{[\mathcal{D}_\alpha^{(n-1)}]^2}{\mathcal{D}^{(n-1)} \mathcal{D}^{(n)}} \right\}, \quad (47)$$

where $\mathcal{D}^{(n)}$ is $n \times n$ determinant of which the r th row ($r=0, 1, \dots, n-1$) is given by $d_{r0}, d_{r1}, \dots, d_{r, n-1}$, and $\mathcal{D}_\alpha^{(n-1)}$ is the determinant formed by replacing the last row (or column) of $\mathcal{D}^{(n)}$ by $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$.

The calculation of α_r is quite easy, and one is led to

$$\alpha_r = - (2\pi/3) (2M_i)^{\frac{1}{2}} \beta^{-r-\frac{1}{2}} \Gamma(r+\frac{5}{2}), \quad (48)$$

where $\Gamma(x)$ is the usual gamma function. On the other hand, the actual calculation of d_{rs} is rather complicated so that the details are given in Appendix 1, and we mention only the results in the following.

¹⁴ Details of variation principle and derivation of Eq. (47) are given in reference 13, p. 300.

By using Eqs. (8), (25), and (44), we have

$$d_{00} = (4/45)\sqrt{2}\pi^{1/2}g^2M^{1/2}M_i^{3/2}\beta^{-13/2}c^{-4}, \quad (49)$$

in the low-temperature limit ($\beta \rightarrow \infty$). It is seen that terms $[\mathfrak{D}_\alpha^{(n-1)}]^2/\mathfrak{D}^{(n-1)}\mathfrak{D}^{(n)}$ in Eq. (47) are of higher orders than the first term α_0^2/d_{00} at low temperatures and thus can be neglected. This shows that the variation with constant trial function yields the correct temperature dependence of the ion mobility.

From Eqs. (47)–(49), μ is calculated to be

$$\mu = (45/2\pi)eg^{-2}\beta^4c^4M^{-2}, \quad (50)$$

in the low-temperature limit.

5. DISCUSSION

We discuss in this section the comparison of the results obtained in the previous section with those obtained by elementary arguments in Sec. 3. Also we compare our results with the experimental data in liquid helium and with the Khalatnikov and Zharkov⁵ theory.

First, we discuss the temperature dependence of the ion mobility. In the low-temperature limit where phonon excitations are expected to be important, the simple theory predicts a T^{-3} dependence [Eq. (29)], whereas the Boltzmann equation yields a T^{-4} dependence [Eq. (50)]. The latter temperature dependence may be understood in a simple way as in the case of the low-temperature behavior of electrical conductivity in metals,¹⁵ which is proportional to T^{-5} . Since the average momentum k of the ion is much larger than the average momentum q of the phonon at low temperatures, i.e.,

$$(q/k)^2 \approx (\beta M_i c^2)^{-1}, \quad (51)$$

the momentum transfer Δk of the ion in the scattering process is much smaller than k . Therefore, the effective cross section is given by

$$\sigma_{\text{eff}} \approx (\Delta k/k)^2 \sigma, \quad (52)$$

since $(1 - \cos\phi)$ is the measure of the loss of forward momentum in a collision with scattering angle ϕ , and this can be approximated by $\phi^2/2$ because ϕ is very small ($\phi \approx \Delta k/k \approx q/k$).

From Eqs. (51) and (52) it follows that the combined effect of the T^3 temperature dependence of the number of phonons and the T dependence of the scattering angle factor makes the mobility proportional to T^{-4} . Thus we can see that simple kinetic theory is not applicable to ion-phonon scattering processes at low temperatures, unless the effective reduction of the cross section due to the small scattering angle is taken into account. On the other hand, the experiment by Meyer and Reif¹ shows that at low temperatures ($T < 0.6^\circ\text{K}$) the temperature dependence is T^{-k} , where $k = 3.3 \pm 0.3$ for a positive ion and $k = 2.4 \pm 0.4$ for a negative ion.

Therefore, as far as the agreement with experimental results is concerned, simple kinetic theory is more favorable than the Boltzmann equation approach. However, it should be noted that these indices are determined only in the temperature interval between 0.5°K and 0.6°K , and hence it seems that a decisive conclusion cannot be drawn from the experimental data available at present. Considering these facts, we may say that our theory reproduces the experimental temperature dependence and that the assumed effective Hamiltonian, Eq. (12), is not in contradiction with experiment.

In connection with this point, we should mention that the theory by Khalatnikov and Zharkov⁵ predicts a T^{-9} dependence. This is because they derive the interaction Hamiltonian from quantum hydrodynamics and the ion-phonon scattering cross section is proportional to $k^2 q^4$, whereas in our case it is constant. Since the temperature dependence of the ion mobility is determined mainly by the assumed form of cross section, their interaction Hamiltonian is in contradiction with experiment.

Next we consider the absolute magnitude of the ion mobility. Since the absolute values of the mobility cannot be measured quite accurately¹ and since our theory is based on the dilute-gas model of liquid helium, the following arguments are of semiquantitative nature. However, we hope to draw some conclusions about the interaction between the ion and phonons present in liquid helium.

If we assume⁴ that $M_i \gg M$, the reduced mass m is given by $m = M$ in Eq. (26), so that the only unknown parameter in Eq. (50) is a , the scattering length between the ion and a helium atom in the liquid. Namely, we have

$$\mu = 45(2\pi)^{-3}e\beta^4c^4a^{-2}. \quad (53)$$

If we assume that the ion is singly ionized, then from the observed values of μ and c , the parameter a is calculated. In fact, if we substitute the experimental value for a positive ion, $\mu = 6.2 \times 10^3 \text{ cm}^2 \text{ sec}^{-1} \text{ volt}^{-1}$ at $T = 0.5^\circ\text{K}$, taken from Fig. 1 in reference 1, we get $a = 2.8 \times 10^{-8} \text{ cm}$, which is a reasonable magnitude for the scattering length.

So far we have been interested mainly in the phonon excitations, but a similar calculation may be extended formally to the roton excitations. However, in this case some difficulties arise which do not exist in the case of phonon excitations. In the first place, the quantity $v_{\mathbf{q}-\mathbf{q}'}$ in Eq. (13) may not be replaced by a constant as we have done in this paper, since $\mathbf{q}-\mathbf{q}'$ is not necessarily small in the ion-roton scattering processes. Second, it is not quite certain whether the Bogoliubov transformation can yield the energy spectrum of the roton excitations.

We have calculated the ion mobility based on the Bogoliubov Hamiltonian given by Eq. (2), but there may arise some question regarding the use of this

¹⁵ See, for example, C. Kittel, *Introduction to Solid-State Physics* (John Wiley & Sons, Inc., New York, 1956), 2nd ed., p. 588.

Hamiltonian. For example, as Wentzel¹⁰ has shown, if one improves it by including some higher order terms, the excitation energy spectrum is no longer of phonon type but exhibits a gap above the ground state. Therefore, one might conjecture that the Bogoliubov Hamiltonian fails to account for the phonon spectrum of actual liquid helium, though the final answer to this question needs further investigation. However, one may look into the ion-phonon scattering processes from a more realistic point of view, using Feynman's¹¹ wave function in configuration space. Then one can see that essentially the same result is obtained for the ion mobility as is yielded by the Bogoliubov Hamiltonian. (Details are discussed in Appendix 3.)

In this paper we have been concerned mainly with the ion in a system of Bose particles, but the ion mobility in a Fermi system at low temperatures is likewise of considerable interest, since it would yield information about the low-lying excited states in a Fermi liquid.¹⁶ If we adopt a simple model of free Fermi particles interacting with an ion, the Boltzmann equation can be set up and solved in exactly the same way as we have done in this paper, and it turns out that ion mobility at low temperatures is given by

$$\mu_F = 3\pi^3 e \beta^2 / 2g^2 M_F^2 M_i^2, \quad (54)$$

where M_F is the mass of Fermi particle and g is given by Eq. (26) with M , the mass of Bose particle, replaced by M_F . If we assume that $M_i \gg M_F$, and that the scattering length a is the same in both systems, we have

$$\mu_F / \mu_B = (\pi^4 / 15) (M_i \beta c^2)^{-2} \quad (55)$$

at low temperatures, where μ_B stands for the ion mobility in a system of Bose particles. At $T = 0.5^\circ\text{K}$ this ratio takes a value 1.8×10^{-5} , if we assume $M_i = 10M$; so the ion mobility in liquid He³ is expected to be extremely small compared to that in liquid He⁴. Furthermore, the temperature dependence is given by $\mu_F \propto T^{-2}$, whereas in the Bose system, $\mu_B \propto T^{-4}$ at low temperatures. At present experimental data in liquid He³ are not available so that our theory cannot be compared with experimental results. The problems connected with the roton excitations and the Fermi system will be discussed in another occasion.

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¹⁶ B. Z. Zharkov and V. P. Silin, J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 143 (1959) [translation: Soviet Phys.—JETP **10**, 102 (1960)].

APPENDIX 1. SYMMETRY OF (f, g) AND CALCULATION OF d_{rs}

From the defining equation for (f, g) , Eq. (40), and by the use of Eq. (38), we have

$$\begin{aligned} (f, g) = [2(2\pi)^5]^{-1} \int d\mathbf{k} d\mathbf{q} d\mathbf{q}' \exp(-\beta \epsilon_{\mathbf{k}}) \gamma_{\mathbf{q}, \mathbf{q}'}^2 \\ \times n_{\mathbf{q}}(n_{\mathbf{q}'} + 1) \{ (k_x + q_x - q'_x) f(\epsilon_{\mathbf{k} + \mathbf{q} - \mathbf{q}'}) \\ - k_x f(\epsilon_{\mathbf{k}}) \} \{ (k_x + q_x - q'_x) g(\epsilon_{\mathbf{k} + \mathbf{q} - \mathbf{q}'}) \\ - k_x g(\epsilon_{\mathbf{k}}) \} \delta(\epsilon_{\mathbf{k}} + \omega_{\mathbf{q}} - \epsilon_{\mathbf{k} + \mathbf{q} - \mathbf{q}'} - \omega_{\mathbf{q}'}). \quad (A1) \end{aligned}$$

This equation clearly shows that $(f, g) = (g, f)$ and $(f, f) \geq 0$.

Using Eqs. (8) and (25), introducing a new integral variable $\mathbf{k}' = \mathbf{k} + \mathbf{q} - \mathbf{q}'$ instead of \mathbf{q}' in Eq. (A1), and integrating over the angular parts of \mathbf{q} , we have

$$\begin{aligned} (f, g) = \frac{g^2 M^2 c^2}{2(2\pi)^4} \int' d\mathbf{k} d\mathbf{k}' \frac{F(\mathbf{k}, \mathbf{k}')}{|\mathbf{k} - \mathbf{k}'|} \{ k'_x f(\epsilon_{\mathbf{k}'}) \\ - k_x f(\epsilon_{\mathbf{k}}) \} \{ k'_x g(\epsilon_{\mathbf{k}'}) - k_x g(\epsilon_{\mathbf{k}}) \}. \quad (A2) \end{aligned}$$

Here the prime on the integral sign means that the integration should be extended over the region $|\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}| \leq c|\mathbf{k} - \mathbf{k}'|$, and $F(\mathbf{k}, \mathbf{k}')$ is given by

$$\begin{aligned} F(\mathbf{k}, \mathbf{k}') \\ = \int_{b_-}^{\infty} dq \frac{\exp(-\beta \epsilon_{\mathbf{k}'} + \beta c q)}{(e^{\beta c q} - 1) \{ \exp[\beta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'} + \beta c q)] - 1 \}}, \quad (A3) \end{aligned}$$

where

$$b_{\pm}(\mathbf{k}, \mathbf{k}') = \frac{|\mathbf{k} - \mathbf{k}'|}{2} \pm \frac{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}}{2c}. \quad (A4)$$

If we introduce a new integration variable $x = \exp(\beta c q)$ in Eq. (A3), the integration can be carried out exactly, and we find

$$\begin{aligned} F(\mathbf{k}, \mathbf{k}') = \frac{(\beta c)^{-1}}{\exp(\beta \epsilon_{\mathbf{k}}) - \exp(\beta \epsilon_{\mathbf{k}'})} \\ \times \ln \left(\frac{1 - \exp[-\beta c b_+(\mathbf{k}, \mathbf{k}')] }{1 - \exp[-\beta c b_-(\mathbf{k}, \mathbf{k}')] } \right). \quad (A5) \end{aligned}$$

At the low-temperature limit, it may be expected that the terms $\exp[-\beta c b_{\pm}(\mathbf{k}, \mathbf{k}')]$ are small, so that one can expand the logarithmic function in powers of these terms and retain only the lowest order terms. Thus we have the following approximate integral kernel:

$$F(\mathbf{k}, \mathbf{k}') = (\beta c)^{-1} \exp[-\frac{1}{2}\beta c |\mathbf{k} - \mathbf{k}'| - \frac{1}{2}\beta(\epsilon_{\mathbf{k}} + \epsilon_{\mathbf{k}'})]. \quad (A6)$$

It turns out that this approximation gives the correct temperature dependence, except for the numerical constant.

If we put $f=g=1$ in Eq. (A2), and introduce new variables \mathbf{x} and \mathbf{y} defined by $\mathbf{k}-\mathbf{k}'=\mathbf{x}$, $(\mathbf{k}+\mathbf{k}')/2=\mathbf{y}$, we have by the use of Eq. (A6),

$$d_{00} = \frac{g^2 M_i^2 c^7}{96\pi^4} \int' \exp\left(-\frac{\alpha x}{2} - \frac{\alpha x^2}{8} - \frac{\alpha y^2}{2}\right) x dx dy, \quad (\text{A7})$$

where the prime on the integral sign means $|\mathbf{x} \cdot \mathbf{y}| \leq x = |\mathbf{x}|$, and α is defined as

$$\alpha = M_i c^2 \beta. \quad (\text{A8})$$

The integral in Eq. (A7) is easily shown to be

$$\begin{aligned} \int' &= 16\pi^2 \int_0^\infty \exp\left(-\frac{\alpha x}{2} - \frac{\alpha x^2}{8}\right) x^3 dx \left\{ \int_0^1 dy y^2 \right. \\ &\quad \times \exp\left(-\frac{\alpha y^2}{2}\right) + \int_1^\infty dy y \exp\left(-\frac{\alpha y^2}{2}\right) \left. \right\}. \quad (\text{A9}) \end{aligned}$$

In the low-temperature limit ($\alpha \rightarrow \infty$), the second term in the curly brackets vanishes as $\exp(-\alpha/2)$ and can be neglected as compared to the first term. On the other hand, in calculating the first term it is permissible to extend the upper limit of integration to ∞ , without any appreciable error. Thus in the limit $\alpha \rightarrow \infty$, we have

$$d_{00} = 8\sqrt{2} g^2 M_i^2 c^7 \pi^{-3/2} \beta^{-1} \alpha^{-11/2}. \quad (\text{A10})$$

If we use the exact form of the integral kernel given by Eq. (A5), it is easily shown that we should multiply Eq. (A10) by $\pi^4/90$ ($=1+2^{-4}+3^{-4}+\dots$), and we get Eq. (49) in the text, by using Eq. (A8).

A similar situation exists in calculating all other d_{rs} . One can first use the approximate kernel and obtain the correct temperature dependence. Then if one calculates d_{rs} based on the exact form of the integral kernel, one should multiply the results by $\pi^4/90$ as in the case of d_{00} . In this way, we have in the low-temperature limit

$$d_{rs} = A \beta^{-(r+s)} \delta_{rs}, \quad (\text{A11})$$

where

$$A = (8/45) \sqrt{2} \pi^2 g^2 M_i^2 c^4 \beta^{-13/2}, \quad (\text{A12})$$

$$\delta_{rs} = \left\{ 1 + \frac{2}{3}(r+s) + \frac{4}{3}rs \right\} \Gamma(r+s+\frac{3}{2}). \quad (\text{A13})$$

On the other hand, from Eq. (48) α_r can be written as

$$\alpha_r = B \beta^{-r} \delta_r, \quad (\text{A14})$$

where

$$B = -(2\pi/3) (2M_i)^{1/2} \beta^{-5/2}, \quad (\text{A15})$$

$$\delta_r = \Gamma(r+\frac{5}{2}). \quad (\text{A16})$$

From Eqs. (A11), (A13), (A14), and (A16), we have

$$\frac{\alpha_0^2}{d_{00}} + \sum \frac{[\mathcal{D}_\alpha^{(n-1)}]^2}{\mathcal{D}^{(n-1)} \mathcal{D}^{(n)}} = \frac{B^2}{A} \left(\frac{\delta_0^2}{\delta_{00}} + \sum \frac{[\Delta_\delta^{(n-1)}]^2}{\Delta^{(n-1)} \Delta^{(n)}} \right), \quad (\text{A17})$$

where $\Delta^{(n)}$ is a determinant formed by replacing d_{rs} of $\mathcal{D}^{(n)}$ by δ_{rs} , and $\Delta_\delta^{(n-1)}$ is a determinant formed by

replacing the last row (or column) of $\Delta^{(n)}$ by $\delta_0, \delta_1, \dots, \delta_{n-1}$.

One can easily see that $\Delta_\delta^{(n-1)} = 0$ ($n \geq 2$), and therefore in the low-temperature limit, only the first term δ_0^2/δ_{00} contributes to the mobility. If one calculates d_{rs} more accurately, it is seen that $[\Delta_\delta^{(1)}]^2/\Delta^{(1)} \Delta^{(2)}$ is of the order of β^{-2} and can be neglected at large β .

APPENDIX 2. CONTRIBUTION OF \mathcal{H}'

If we take account of the process in which the ion emits or absorbs phonons, it is easily seen that the only change in the Boltzmann equation is adding a new integral operator $\mathcal{L}'(c)$ to the left-hand side of Eq. (37). This operator is found to be

$$\begin{aligned} \mathcal{L}'(c) &= \frac{\rho}{(2\pi)^2} \int d\mathbf{q} \gamma_q^2 \{ (n_q+1) [(k_x - q_x) c(\epsilon_{k-q}) \\ &\quad - k_x c(\epsilon_k)] \delta(\epsilon_{k-q} - \epsilon_k + \omega_q) + n_q [(k_x + q_x) c(\epsilon_{k+q}) \\ &\quad - k_x c(\epsilon_k)] \delta(\epsilon_{k+q} - \epsilon_k - \omega_q) \}. \quad (\text{A18}) \end{aligned}$$

If one deals with the sum $\mathcal{L}(c) + \mathcal{L}'(c)$ instead of the operator $\mathcal{L}(c)$, the variation principle can be proved exactly in the same way as we have done in the text, and it turns out that d_{rs} should be replaced by $d_{rs} + d_{rs}'$, where d_{rs}' is given by replacing \mathcal{L} by \mathcal{L}' in Eq. (45).

By a straightforward calculation, d_{00}' is shown to be

$$d_{00}' = \frac{16g^2 M_i^2 \rho c^{-\alpha/2}}{3M c^6 \beta^6} \int_0^\infty \frac{x^4}{\sinh x} \exp(-x^2/2\alpha) dx. \quad (\text{A19})$$

The asymptotic expansion of the integral at large α yields the following equation for d_{00}' :

$$d_{00}' = 256 C M_i^2 g^2 \rho M^{-1} (c\beta)^{-6} \exp(-\alpha/2), \quad (\text{A20})$$

where C is a numerical constant given by

$$C = 1 + 3^{-5} + 5^{-5} + \dots$$

If we assume $M = M_i$, substitution of numerical values for liquid helium in Eqs. (49) and (A20) leads to $d_{00}'/d_{00} \approx 8.4 \times 10^{-12}$, at $T = 0.5^\circ\text{K}$, and therefore the contribution of \mathcal{H}' can be completely neglected. It is easily proved that d_{rs}' ($r, s = 0, 1, 2, \dots$) vanishes as $\exp(-\alpha/2)$ at low temperatures and is negligible as compared to d_{rs} .

APPENDIX 3. USE OF FEYNMAN'S WAVE FUNCTION

According to Feynman,¹¹ the wave function φ_q of the first excited state with momentum \mathbf{q} is written as

$$\begin{aligned} \varphi_q(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) &= [NS(\mathbf{q})]^{-1/2} \phi(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) \\ &\quad \times \sum_i \exp(i\mathbf{q} \cdot \mathbf{x}_i), \quad (\text{A21}) \end{aligned}$$

where ϕ is the wave function of ground state, $S(\mathbf{q})$ is

the structure factor of liquid helium at absolute zero, and the sum is taken over all helium atoms. The wave functions given by Eq. (A21) form an orthonormal set, i.e.,

$$\int \varphi_q^* \varphi_{q'} d\mathbf{x}_1 d\mathbf{x}_2 \cdots d\mathbf{x}_N = \delta_{q,q'}.$$

The excitation energy spectrum corresponding to φ_q is given by

$$\omega_q = E_q / S(q). \quad (\text{A22})$$

The interaction Hamiltonian between the ion and the helium atoms is given in configuration space by

$$H_{\text{int}} = \sum_i v(\mathbf{x} - \mathbf{x}_i). \quad (\text{A23})$$

One can now calculate the matrix element of H_{int} corresponding to the collision $(\mathbf{k}, \mathbf{q}) \rightleftharpoons (\mathbf{k}', \mathbf{q}')$, \mathbf{k}, \mathbf{k}' being the momenta of the ion, \mathbf{q}, \mathbf{q}' those of the phonon. Namely, we have

$$\begin{aligned} \langle \mathbf{k}' \mathbf{q}' | H_{\text{int}} | \mathbf{k} \mathbf{q} \rangle &= V^{-1} \int \exp(-i\mathbf{k}' \cdot \mathbf{x} + i\mathbf{k} \cdot \mathbf{x}) \\ &\times \varphi_q^* H_{\text{int}} \varphi_q d\mathbf{x} d\mathbf{x}_1 d\mathbf{x}_2 \cdots d\mathbf{x}_N. \end{aligned} \quad (\text{A24})$$

Let us here introduce a set of distribution functions defined by

$$\rho = N \int \phi^2 d\mathbf{x}_2 d\mathbf{x}_3 \cdots d\mathbf{x}_N, \quad (\text{A25})$$

$$\rho_2(\mathbf{x}_1, \mathbf{x}_2) = N(N-1) \int \phi^2 d\mathbf{x}_3 \cdots d\mathbf{x}_N, \quad (\text{A26})$$

$$\rho_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = N(N-1)(N-2) \int \phi^2 d\mathbf{x}_4 \cdots d\mathbf{x}_N. \quad (\text{A27})$$

If one writes $\rho_2(\mathbf{x}_1, \mathbf{x}_2) = \rho^2 g(\mathbf{x}_1 - \mathbf{x}_2)$, the function g is the radial distribution function, and $S(q)$ is given by

$$S(q) = 1 + \rho \int g(\mathbf{x}) \exp(-i\mathbf{q} \cdot \mathbf{x}) d\mathbf{x}. \quad (\text{A28})$$

Now from Eqs. (A21), (A23), and (A24), if we expand $v(\mathbf{x})$ in terms of Fourier series, i.e.,

$$v(\mathbf{x}) = V^{-1} \sum v_q \exp(i\mathbf{q} \cdot \mathbf{x}),$$

then after carrying out the integration over the coordinates of the ion, we have

$$\begin{aligned} \langle \mathbf{k}' \mathbf{q}' | H_{\text{int}} | \mathbf{k} \mathbf{q} \rangle &= V^{-1} \{S(q)S(q')\}^{-\frac{1}{2}} \\ &\times v_{\mathbf{k}'-\mathbf{k}} \{ \delta(\mathbf{q} + \mathbf{k} - \mathbf{q}' - \mathbf{k}') [S(\mathbf{q} - \mathbf{q}')] \\ &+ \rho \int g(\mathbf{x}) \exp(i\mathbf{q}' \cdot \mathbf{x}) d\mathbf{x} + \rho \int g(\mathbf{x}) \exp(-i\mathbf{q} \cdot \mathbf{x}) d\mathbf{x} \} \\ &+ \frac{1}{N} \int \exp(-i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{x}_1 + i\mathbf{q} \cdot \mathbf{x}_2 - i\mathbf{q}' \cdot \mathbf{x}_3) \\ &\times \rho_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 \}. \end{aligned} \quad (\text{A29})$$

If we use the approximation¹¹

$$\rho_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \rho^3 g(\mathbf{x}_2 - \mathbf{x}_1) g(\mathbf{x}_3 - \mathbf{x}_1), \quad (\text{A30})$$

we have

$$\begin{aligned} \langle \mathbf{k}' \mathbf{q}' | H_{\text{int}} | \mathbf{k} \mathbf{q} \rangle &= V^{-1} v_{\mathbf{k}'-\mathbf{k}} \{S(q)S(q')\}^{-\frac{1}{2}} \\ &\times \delta(\mathbf{q} + \mathbf{k} - \mathbf{q}' - \mathbf{k}') \{S(\mathbf{q} - \mathbf{q}') + S(q)S(q') - 1\}. \end{aligned} \quad (\text{A31})$$

Since we are interested in the excitation of small \mathbf{q}, \mathbf{q}' , we can approximate this as

$$\begin{aligned} \langle \mathbf{k}' \mathbf{q}' | H_{\text{int}} | \mathbf{k} \mathbf{q} \rangle &\approx -V^{-1} v_0 \{S(q)S(q')\}^{-\frac{1}{2}} \delta(\mathbf{q} + \mathbf{k} - \mathbf{q}' - \mathbf{k}'). \end{aligned} \quad (\text{A32})$$

If one compares Eq. (A32) with the corresponding matrix element derived from Eqs. (12) and (13), one may easily see that Eq. (A32) is two times the previous one. However, the dependence on \mathbf{q} and \mathbf{q}' is the same, so that no change occurs in the temperature dependence of the ion mobility at low temperatures.