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Space Dispersive Properties of Plasma

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The "space dispersion," i.e., the occurrence of the term $\mathbf{k}$ in the dielectric constant $\epsilon(\omega,\mathbf{k})$ can be attributed either to the Doppler effect or to the magnitude of the term ak that may appear in the formulation of the problem. (a is a characteristic distance such as the Debye length.) Using an approach based on the Doppler effect, the macroscopic parameters of a plasma have been represented in the form of four-dimensional tensors of the fourth order (similar to those introduced by Mandelstam and Tamm). The phenomenological description of plasma has also been formulated in a three-dimensional space by means of two macroscopic parameters: the electric susceptibility χ_e and the "proper magnetic susceptibility" χ_μ/μ. Expressions for these parameters have been given for the general case of a plasma having an electron velocity distribution $f(\mathbf{v})d\mathbf{v}$ and for a few typical specific cases. Both parameters are functions of the frequency and of the wave vector. This formulation brings into evidence the fact that a plasma is a magnetically polarizable medium and the term χ_μ/μ vanishes only if the electron velocity distribution is isotropic. In the current literature on the subject the existence of the term χ_μ/μ has not been taken into account, since, by using a "modified representation" of the dielectric constant, the magnetic properties of plasma have not been brought into evidence. In the "modified representation" the dielectric constant ϵ_M is defined by the relationship $\mathbf{k}\times\mathbf{B}=-(\omega/c)\epsilon_M\mathbf{E}$, whereas in the conventional representation the same relationship has the form $\mathbf{k}\times\mathbf{B}=-(\omega/c)\epsilon\mathbf{E}+4\pi(\chi_\mu/\mu)\mathbf{k}\times\mathbf{B}$ (where $\epsilon=1+4\pi\chi_e$). A general formalism has been developed for deriving the electric and magnetic plasma parameters directly from the Boltzmann-Vlasov equations.

INTRODUCTION

CONSIDERABLE attention has been given in the past to the phenomenological description of plasma as represented by its "dielectric constant" $\epsilon(\omega,\mathbf{k})$. This representation brings into evidence a distinctive property of a plasma defined as "space dispersion" which is characterized by the appearance of the term $\mathbf{k}$ in the expression for the dielectric constant. Plasma does not represent the only type of a space dispersive medium since there are other, particularly molecular media, that exhibit space dispersion. We shall evaluate critically the concept $\epsilon(\omega,\mathbf{k})$ in order to point out that the physical factors that are responsible for space dispersion in a plasma are essentially different from those that produce space dispersion in molecular media.

The terminology that is currently applied to plasma may possibly lead to a certain amount of confusion. In describing the phenomenological properties of plasma it is customary to use the term "dielectric constant" and to attribute to this term a meaning that is different from the one conventionally used. In that connection a comparison will be made between the "conventional" macroscopic parameters that characterize any dispersive medium and the "modified parameters" that have been specifically used to describe the properties of plasma. The "conventional parameters" and the corresponding "modified parameters" have the same identifying names, although the meaning applied to these concepts is not the same.

A phenomenological description of plasma will be given in a four-dimensional covariant form and a generalized "dielectric-magnetic tensor" for a plasma shall be formulated. By reducing the four-dimensional representation to the customary three-dimensional form, we shall bring into evidence the "conventional" macroscopic parameters such as the electric and magnetic susceptibility. The results are significant since they point out the fact that an anisotropic plasma is a magnetically polarizable medium and is space dispersive both in its electric and magnetic properties.

A general discussion will point out certain characteristic anisotropies in a plasma, that are not present in other nonisotropic substances. Macroscopic parameters of a plasma will be expressed in a very general form derived from the Boltzmann-Vlasov equation.

I. GENERAL PROPERTIES OF SPACE-DISPERSIVE MEDIA

A. Terminology

In dealing with dispersive media one often refers to "frequency dispersion" which is usually associated with

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the "difference between the indices of refraction of any substance for any two wavelengths."[1] The reference to the "wavelengths" is anachronistic and possibly inappropriate since the frequency and not the wavelength is the factor that directly determines the characteristics of the medium. A frequency dispersive medium is defined by a relationship

$$D_\alpha(\omega)=\epsilon_{\alpha\beta}(\omega)E_\beta(\omega), \tag{1}$$

where $\mathbf{D}(\omega)$ is the displacement, $\mathbf{E}(\omega)$ the electric field strength, the tensor $\epsilon_{\alpha\beta}(\omega)$ represents the dielectric constant, and ω is the frequency. At this point, and in the rest of the paper, repeated indexes indicate summation. Greek letters represent the coordinates in three-dimensional space, and italic letters represent the coordinates in the space of four dimensions. A relation analogous to (1) can also be expressed in the form[2]

$$D_\alpha(t)=\int_{-\infty}^{t} dt'\, K_{\alpha\beta}(t-t')E_\beta(t'). \tag{2}$$

The displacement and the electric field strength are represented, respectively, by the functions $\mathbf{D}(t)$ and $\mathbf{E}(t)$ varying with time t, and the tensor $K_{\alpha\beta}(t)$ defines the characteristic properties of the medium. Using the formulation (1) one can designate the medium as frequency dispersive, and, similarly, on the basis of the formulation (2) one can refer to the same medium as "time dispersive." Expression (2) states that there is no "instantaneous" dependence between $\mathbf{D}(t)$ and $\mathbf{E}(t)$, i.e., the value $\mathbf{D}(t)$ at any instant depends upon the distribution in time or "time dispersion" of $\mathbf{E}(t)$.

A "time-dispersive" medium may be considered as a particular case of a "time and space dispersive medium." This latter medium shall be abbreviated as "space dispersive" or "spacially dispersive." It is characterized not only by the lack of the "instantaneous dependence" as expressed by (2) but also by a nonlocal relationship between the polarization and the electric field.[3] Thus if $\mathbf{E}(\mathbf{r},t)$ is the applied field strength, the corresponding displacement $\mathbf{D}(\mathbf{r},t)$ is such that

$$D_\alpha(\mathbf{r},t)=\int d\mathbf{r}'dt'\, K_{\alpha\beta}(\mathbf{r}-\mathbf{r}',\, t-t')E_\beta(\mathbf{r}',t'), \tag{3}$$

where $K_{\alpha\beta}(\mathbf{r}-\mathbf{r}')$ defines the characteristic properties of the medium. Using Fourier spectrum representation, the relationship (3) is expressed as

$$D_\alpha(\omega,\mathbf{k})=\epsilon_{\alpha\beta}(\omega,\mathbf{k})E_\beta(\omega,\mathbf{k}), \tag{4}$$

where the dielectric constant $\epsilon_{\alpha\beta}(\omega,\mathbf{k})$ is an explicit function of the frequency and the wave vector. This formulation was applied to optically active substances, and more recently to various "moving media" such as plasma.

The term "plasma" will be used in a generalized sense, i.e., it will designate media comprising charged particles in motion such as ionized gases, moving beams of electrons or protons, etc. We shall exclude from our definition the trivial case of the "stationary" plasma. In the latter case the dielectric constant is represented by the expression $\epsilon=1-\omega_0^2/\omega^2$ (ω_0 is the Langmuir frequency) and there is no space dispersion.

B. Unified Theory of Space Dispersive Media

Space dispersive media are essentially of two types: "molecular media" and a "plasma." Space dispersion in molecular media has been investigated by Landau.[4] The concepts introduced by Landau have been subsequently broadened by Ginzburg[5] in the form of a unified theory that covers all space dispersive media, both molecular and plasma.

According to Landau, a molecular (or atomic) medium can be either time dispersive or space dispersive, and the character of the dispersion depends essentially upon the factor $a/\lambda=ak$, where a is the "size" of the molecule or atom and λ is the wavelength of the applied field. When $ak\ll1$ and can be neglected, the medium is time dispersive and the relationship between $\mathbf{D}(\omega)$ and $\mathbf{E}(\omega)$ is of the type given by the expression (1). On the other hand, if ak is small but not negligible the relationship between $\mathbf{D}(\omega)$ and $\mathbf{E}(\omega)$ has been formally expressed by Landau in the form of a power series in k, i.e.,

$$D_\alpha=[\epsilon_{\alpha\beta}(\omega)-ik_\tau\gamma_{\alpha\beta\tau}(\omega)-k_\tau k_\rho\delta_{\alpha\beta\tau\rho}(\omega)+\cdots]E_\beta, \tag{5}$$

where $\gamma_{\alpha\beta\tau}(\omega)$, $\delta_{\alpha\beta\tau\rho}(\omega)$ are tensors of third and fourth rank, respectively, and are functions of the frequency only. The relationship (5) is of the type given by the expression (4).

The generalization of Ginzburg is based on the extension of the meaning attributed to the fundamental length a. In such a generalized interpretation the term a designates not only the size of a molecule or an atom but any other suitable "characteristic length" applicable to a material medium, such as the lattice spacing if the substance is a solid, or the Debye constant in case of a plasma. By using formal considerations similar to those of Landau, Ginzburg presented a unified theory that accounts for the occurrences of the vector $\mathbf{k}$ in the expression $\epsilon(\omega,\mathbf{k})$ in such apparently different substances as optically active media, crystalline structures, and plasma.

[1] See for instance (a) *Handbook of Chemistry and Physics,* edited by C. D. Hodgman (Chemical Rubber Publishing Company, Cleveland, Ohio, 1958–1959), 40th ed., p. 3085 or (b) *American Institute of Physics Handbook,* edited by D. E. Gray (McGraw-Hill Book Company, Inc., New York, 1957), 6–4.

[2] See for instance L. D. Landau and E. M. Lifshitz, *Elektrodinamika Sploshnykh Sred* (Gos. Izd. Tekhn-Teor Literatury, Moscow, 1957), p. 315.

[3] See, for instance, V. D. Shafranov, Zhur. Eksptl. i Teoret. Fiz. **34**, 1475 (1958), and a paper on related subject by J. E. Drummond, Phys. Rev. **110** 293, 1958.

[4] Reference 2, p. 425.

[5] V. L. Ginzburg, Zhur. Eksptl. i Teoret. Fiz. **34**, 1593 (1958).

C. Diverse Theories of Space Dispersive Media

In contrast to the formal and unifying approach of Ginzburg we shall base our discussion on more direct physical factors that are responsible for the occurrence of space dispersion in molecular media and in plasma. The properties of these two media shall be considered separately.

1. Space Dispersion in Molecular Media

The phenomenological description of matter involves two fundamental lengths: the molecular radius a_m and the intermolecular distance a_d. The term a_m is used for defining various microscopic concepts, and the term a_d serves to make a transition from the microscopic to the macroscopic point of view. This transition is effective in "smearing out" the granular structure of matter by averaging the microscopic quantities over the "physically infinitesimal elements of volume."[6] The term a_m characterizes the Lorentz theory and is used because of our lack of detailed knowledge regarding the structure of matter within the molecule. Thus all the microscopic concepts based on the Lorentz theory are subject to a restrictive condition $(a_m/\lambda)\ll 1$. By eliminating this condition, a time dispersive medium becomes space dispersive.

Assume that a perturbation

$$\mathbf{E}=-(\omega/c)\ \mathrm{Re}(i\mathbf{A}), \tag{6}$$

applied to a molecule induces transitions from an initially unperturbed state ψ_{i0} to excited states ψ_i. The vector potential $\mathbf{A}$ can be expressed in the form

$$\begin{aligned}A_i&=A_{0i}e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\\&=A_{0i}e^{i\omega t}[1-ik_\alpha r_\alpha+\tfrac{1}{2}k_\alpha k_\beta r_\alpha r_\beta-\cdots].\end{aligned} \tag{7}$$

Some of the microscopic quantities such as the molecular polarization P_{micr} or the electrical susceptibility of the molecule $(\chi_{\mathrm{micr}})_{\alpha\beta}$ can be expressed by means of the relation

$$\mathbf{P}_{\mathrm{micr}}=e\langle i|\mathbf{p}|i\rangle=\kappa_{\mathrm{micr}}\mathbf{E}, \tag{8}$$

where $\mathbf{p}=(e\hbar/m\omega)$ grad. We consider only the diagonal terms in the matrix (8) since these terms correspond to the state ψ_i having the same energy as the initial state ψ_{i0}. The nondiagonal terms represent emission and absorption of light, and are excluded from our considerations. The macroscopic formulation for the dielectric constant can be directly determined from the expression

$$\epsilon_{\alpha\beta}=1+\frac{4\pi}{V}\int\kappa_{\mathrm{micr}}dV, \tag{9}$$

which represents the averaging of the microscopic parameter κ_{micr} over a physically infinitesimal volume of space V.

[6] H. A. Lorentz, *The Theory of Electrons* (B. G. Teubner, Leipzig, Germany, 1916), 2nd ed., p. 132.

The expression given in (9) may be either of the type $\epsilon_{\alpha\beta}(\omega)$ or of the type $\epsilon_{\alpha\beta}(\omega,\mathbf{k})$, depending on the number of terms that are retained in the series within the brackets in the expression (7). If we limit ourselves to the first term, we obtain time dispersion as represented by (1). On the other hand, by including the successive terms in this series, we obtain an expression that is formally similar to the one postulated by Landau in the form (5). Using the expressions (6)–(8) and averaging the microscopic quantities over the physically infinitesimal volumes of space we can express various coefficients of Landau, such as $\epsilon_{\alpha\beta}(\omega)$, $\gamma_{\alpha\beta\tau}(\omega)$, $\delta_{\alpha\beta\tau\rho}(\omega)$, as functions of matrix elements of molecular transitions for electric and magnetic moments. This procedure has been applied to the term $\gamma_{\alpha\beta\tau}(\omega)$ in order to account for the optical activity of various substances.[7] A similar procedure applied to the succeeding terms such as $\delta_{\alpha\beta\tau\rho}(\omega)$ would give physical meaning to the corresponding terms in the Landau expression (5). It would explain in terms of molecular structure the space dispersion in such media as cubic crystals.

2. Space Dispersion in Plasma[8]

The space dispersion in molecular media represents a quantum mechanical refinement which takes into account the effect of a perturbing potential that varies over distances comparable to the "molecular diameter." These considerations do not apply to plasma since there is no "molecular diameter" in a plasma, and the space dispersion results from classical ("orbital") and not "quantum" representation.

The concept of "plasma" is associated with a macroscopically defined medium comprising charged particles in motion. Therefore, in order to determine the response of such a medium to an external harmonic force one has to take into account the Doppler effect. Thus if ω' indicates a frequency in a system of coordinates S' moving with velocity $\mathbf{v}$ relative to S the corresponding frequency in the system S can be expressed as[9]

$$\omega'=(\omega-\mathbf{kv})\gamma,\quad\text{where}\quad\gamma=(1-v^2/c^2)^{-\frac{1}{2}}. \tag{10}$$

The appearance of the vector $\mathbf{k}$ in the expression (10) is associated with space dispersion. The importance of the

[7] L. Rosenfeld, Z. Physik **52**, 161 (1928).

[8] The literature using the explicit formulation $\epsilon(\omega,\mathbf{k})$ for the dielectric constant in a plasma is quite extensive. A number of papers have been published in addition to those given in references 3, 5, 10, 11, 14, and 15. Some of these are as follows: M. E. Gertzenshtein, Zhur. Eksptl. i Teoret. Fiz. **23**, 678 (1952); **27**, 180 (1954). R. Z. Sagdeev and V. D. Shafranov, *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958* (United Nations, Geneva, 1958), Vol. 31, pp. 118–124 (P/2215 U.S.S.R.); Zhur. Eksptl. i Teoret. Fiz. **39**, 180 (1960). A. G. Sitenko and K. N. Stepanov, *ibid.* **31**, 642 (1956). K. N. Stepanov, *ibid.* **34**, 1292 (1958); **36**, 1457 (1959). V. P. Silin, *ibid.* **37**, 273 (1959). Hans Wilhelmsson, *Fourth International Conference on Ionization Phenomena in Gases, Uppsala, Sweden* (North-Holland Publishing Company, Amsterdam, 1960).

[9] See for instance C. Moller, *The Theory of Relativity* (Clarendon Press, Oxford, 1952).

Doppler effect has been pointed out by Gertzenshtein[10] in his derivation of the dielectric constant of a plasma from the Boltzmann-Vlasov equation. In our derivation as outlined in Sec. III, the Doppler effect will be introduced more explicitly since our formulation takes directly into account the effects of the motion of an electron beam in the laboratory system.

3. General Remarks

It appears, therefore, that there are two very distinct factors that are responsible for space dispersion: the magnitude of the terms *ak* and the Doppler effect. These factors occur independently and under different circumstances. However, in some dispersive media such as partly ionized gases, the dispersion is of "mixed type," i.e., the appearance of $\mathbf{k}$ in the expression $\epsilon(\omega,\mathbf{k})$ is due to both factors and the contribution of each factor should be considered separately. We wish to point out, however, that an approach based on the distinction between these two factors may not always be fruitful and in various instances the unifying point of view of Ginzburg may be particularly desirable. Thus a similar mathematical formalism when applied to very different physical situations may reveal significant and valuable analogies.

II. MACROSCOPIC PROPERTIES OF PLASMA

A. General Remarks

Consider the interaction of an electromagnetic field with an assembly of particles having charges e_i and velocities $\mathbf{V}_i(i=1, 2\cdots n)$. The response of such an assembly to an applied field can be described in terms of an electric polarization $\mathbf{P}^0$ and a magnetic polarization $\mathbf{M}^0$ which are as follows:

$$\mathbf{P}^0=\sum_i e_i\mathbf{r}_i;\quad \mathbf{M}^0=\frac{1}{c}\sum_i e_i\mathbf{r}_i\times\mathbf{v}_i, \tag{11}$$

where $\mathbf{r}_i$ is the radius vector connecting a particle having charge e_i and velocity $\mathbf{v}_i$ with a certain arbitrary reference point, and the summation extends to all particles in a unit of volume. In the presence of a perturbing electromagnetic field of the type $\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$, both vectors $\mathbf{P}^0$ and $\mathbf{M}^0$ acquire small alternating components $\mathbf{P}$ and $\mathbf{M}$ (assuming that a linear approximation is justified).

We use the customary relationships:

$$\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}=(1+4\pi\chi_e)\mathbf{E}=\epsilon\mathbf{E}, \tag{12}$$

$$\mathbf{H}=\mathbf{B}-4\pi\mathbf{M}=\left(1-4\pi\frac{\chi_\mu}{\mu}\right)\mathbf{B}=\frac{1}{\mu}\mathbf{B}, \tag{13}$$

where $\mathbf{D}$ is the displacement, $\mathbf{E}$ the electrical intensity, $\mathbf{B}$ the magnetic induction, $\mathbf{H}$ the magnetic intensity, χ_e the electric susceptibility, χ_μ the magnetic susceptibility, ϵ the dielectric "constant," and μ the magnetic permeability. Although $\mathbf{B}$ represents the mean intensity of the microscopic magnetic field, we shall continue to designate it as magnetic "induction" so as to comply faithfully with the anachronistic formulation that is currently used. The relationships (12) and (13) are represented in a somewhat asymmetric form in order to maintain the distinction between the macroscopic values $\mathbf{D}$, and $\mathbf{H}$, and the corresponding microscopic values $\mathbf{E}$, and $\mathbf{B}$. Since $\mathbf{B}$ and not $\mathbf{H}$ is the mean intensity of the microscopic field, the coefficient χ_μ/μ appearing in (13) has a more direct physical meaning than the coefficient χ_μ. The term χ_μ/μ (and not χ_μ) is the magnetic equivalent to the electric susceptibility of the medium. Unfortunately, the present terminology has not provided a specific name identifying the term χ_μ/μ. In order to facilitate our further discussion we shall designate this term as the "proper magnetic susceptibility" of the medium so as to differentiate it from the "magnetic susceptibility" that is designated as χ_μ.

The expressions (11)–(13) describe conventional magnetizable media. We shall apply a similar formulation to a plasma and use these expressions as a basis for defining various macroscopic parameters that describe the phenomenological behavior of a plasma.

B. Conventional and Modified Macroscopic Parameters

Various terms such as the "dielectric constant," the "electric displacement," etc., are often used to describe the macroscopic behavior of plasma. In various instances, however, the same terms designate entirely different concepts when applied to a plasma and when applied to other media. In order to be consistent in our presentation, we shall refer to the usually adopted definitions of the macroscopic parameters as expressed by (11)–(13) as the "conventional" definitions and we shall continue to designate the corresponding symbols such as ϵ, $\mathbf{D}$, etc. without any subscripts. On the other hand, the new "modified plasma parameters" that are still identified as "dielectric constant" or "displacement," etc. (although their meaning is different) will be designated by the subscript M such as ϵ_M, D_M, etc.

The "modified representation" does not bring into evidence the fact that a plasma is a magnetically polarizable medium. In the published literature on this subject there appears to be no discussion on magnetic polarization and its dependence on the structural characteristics of a plasma. It is believed, however, that the magnetic susceptibility is a very significant and important parameter characterizing nonisotropic electron velocity distributions. It will be shown in subsequent paragraphs that plasma is nonmagnetic only if the velocity distribution is isotropic. Therefore in a strict sense the "modified representation" cannot be considered as applicable to a nonisotropic plasma. The re-

[10] M. E. Gertzenshtein, Zhur. Eksptl. Teoret. Fiz. 22, 303 (1952).

strictive condition regarding the applicability of the modified representation was apparently not formulated in the literature. These conditions should be kept in mind, however, in those instances in which the "modified" macroscopic parameters have been specifically used to describe a nonisotropic plasma.

In order to compare the conventional and modified representation we shall point out the difference in the formulation of Maxwell's equation in both cases. One of these equations is represented in the conventional notation as follows:

$$\mathbf{k}\times\mathbf{B}=-\frac{\omega}{c}\epsilon\mathbf{E}+4\pi\mathbf{k}\times\frac{\chi_\mu}{\mu}\mathbf{B}, \tag{14}$$

whereas the same equation in the modified representation appears as

$$\mathbf{k}\times\mathbf{B}=-(\omega/c)\epsilon_M\mathbf{E}. \tag{15}$$

The propagation of an electromagnetic wave through plasma is represented in the conventional form by the expression

$$\text{grad div}\mathbf{E}-\nabla^2\mathbf{E}+\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}+4\pi\,\text{curl}\frac{\partial\mathbf{M}}{\partial t}=0, \tag{16}$$

whereas the corresponding modified expression is

$$\text{grad div}\mathbf{E}+\nabla^2\mathbf{E}+\frac{1}{c^2}\frac{\partial^2\mathbf{D}_M}{\partial t^2}. \tag{17}$$

In some instances the forms (15) and (17) may be more convenient. However, there is no physical equivalence between these two formulations. The "modified electric displacement defined by (17) is not an entirely electrical concept, since it is related to both electric and magnetic properties of the medium.

III. FOUR-DIMENSIONAL COVARIANT FORMULATION

A. General Remarks

We shall treat the problem relativistically and express the macroscopic parameters of a plasma in a form that is invariant under Lorentz transformation.

Following the procedure used by Neufeld and Ritchie,[11] we consider a plasma, characterized by an electron velocity distribution $f(\mathbf{v})d\mathbf{v}$, as an assembly of "component beams." In order to form this assembly, the velocity space is subdivided into equal volume elements $\Delta\mathbf{v}_i$ ($i=1, 2, 3, \cdots$). A vector $\mathbf{v}_i$ connecting the origin of coordinates with any point within a cell $\Delta\mathbf{v}_i$ represents the velocity of an extended and uniform electron beam having density $f(\mathbf{v})\Delta\mathbf{v}_i$ and the density of the entire assembly is $\sum f(\mathbf{v}_i)\Delta\mathbf{v}_i$. The continuous distribution represents a limit when each volume element $\Delta\mathbf{v}_i$ tends to zero.

We shall determine the macroscopic parameters for a single "component beam" having velocity $\mathbf{v}$ and then generalize our results to cover any distribution of the type $f(\mathbf{v})d\mathbf{v}$. The parameters for a single component beam will be obtained in two steps. In the first step we formulate the properties of the beam in four-dimensional space for the case of $v=0$ in the laboratory system. Such a beam appears, therefore, as a stationary electron gas. In the second step we apply the Lorentz transformation for an observer moving with the velocity $\mathbf{v}$ with respect to the laboratory system, and we determine the macroscopic parameters of the beam with respect to the moving observer. The result displays a certain symmetry with respect to both systems. Therefore, changing $\mathbf{v}$ into $-\mathbf{v}$ we obtain for the laboratory system the macroscopic parameters of a beam moving with the velocity $\mathbf{v}$.

B. Stationary Electron Gas

We shall formulate a four-dimensional generalization of Eqs. (12) and (13) as applied to a stationary electron gas. The usual vectors defining various field quantities shall be replaced by antisymmetric field tensors, F_{ij}, M^{ij}, and H^{ij} defined as follows:

$$\begin{aligned} c\mathbf{B}&=(F_{32},F_{13},F_{21}); & \mathbf{E}&=(F_{41},F_{42},F_{43});\\ (1/c)\mathbf{M}&=(M^{32},M^{13},M^{21}); & \mathbf{P}&=(M_{41},M_{42},M_{43});\\ (1/c)\mathbf{H}&=(H^{32},H^{13},H^{21}); & \mathbf{D}&=(H^{41},H^{42},H^{43}). \end{aligned} \tag{18}$$

There are two different formulations of equations of electrodynamics of moving matter: the commonly used formulation by Minkowski[12] and a relatively less used formulation by Mandelstam and Tamm.[13] Minkowski expressed the relationship between the field values and the corresponding macroscopic quantities in the form

$$H_{ik}u_k=\epsilon F_{ik}u_k \quad\text{and}\quad H_{ik}{}^*u_k=(1/\mu)F_{ik}{}^*u_k$$

(where u_k is the velocity four-vector and $F_{ik}{}^*$, $H_{ik}{}^*$ are pseudotensors dual to F_{ik} and H_{ik}). It is noted that these expressions did not retain the same form as (12), (13) since they contain the velocity terms u_k and no corresponding velocity terms appear in (12) and (13). It is, however, our objective to obtain an expression that is formally analogous to the relationships as expressed in (12), (13). It should be kept in mind that we wish not only to obtain a relativistically invariant formulation but to derive relativistically covariant concepts that may be considered as the generalization of the macroscopic plasma parameters such as ϵ and μ. Such concepts have been introduced by Mandelstam and Tamm and expressed in the form of a four-dimensional tensor of the fourth order identified by the symbol ϵ^{lmik}. This tensor is defined by the relationship $H^{lm}=\epsilon^{lmik}F_{ik}$, and is designated as D-M, i.e., "dielectric-magnetic" tensor. We shall use a similar formulation and express the rela-

[11] J. Neufeld and R. H. Ritchie, Phys. Rev. **98**, 1631 (1955).

[12] H. Minkowski and M. Born, Math. Ann. **68**, 526 (1910).

[13] L. E. Mandelstam and I. E. Tamm, Math. Ann. **95**, 154 (1925).

tion between M^{lm} and F_{ik} in the form

$$M^{lm}=\chi^{lmik}F_{ik}, \tag{19}$$

where χ^{lmik} will be designated as the "D-M susceptibility tensor."

We shall consider the expression (19) in the framework of two observers designated as A and B. For the observer A the electron gas is stationary and various physical quantities relating to the system A shall be designated by a subscript A. The observer B moves with respect to A with velocity $-\mathbf{v}$ and the corresponding physical quantities shall be designated by a subscript B. The tensor χ^{lmik} has a particularly simple form in the system A. In this system it is so formulated that the relationship (19) reduces to the form $\mathbf{P}=\chi_e\mathbf{E}$; $\mathbf{M}=(\chi_\mu/\mu)\mathbf{B}$. It can be shown that out of 256 components of the tensor $\chi_A{}^{lmik}$ only 12 components are different from zero, and these are as follows:

$$\chi_A{}^{lmik}=\chi_e,\quad\text{when}\begin{cases} l=i=4; & m=k=1,2,3;\\ m=k=4; & l=i=1,2,3;\end{cases}$$

$$\chi_A{}^{lmik}=\frac{1}{c^2}\frac{\chi_\mu}{\mu},\quad\text{when}\begin{cases} l=i=0; & m=k=2,3;\\ l=i=2; & m=k=3,1;\\ l=i=3; & m=k=1,2.\end{cases} \tag{20}$$

The magnetic polarization produced by a perturbing electric field is in a stationary gas by an order of magnitude smaller than the electric polarization. Therefore

$$\chi_\mu/\mu=0. \tag{21a}$$

Furthermore we have

$$\chi_e=-(\omega_0)_A{}^2/\omega_A{}^2. \tag{21b}$$

The term ω_A represents the frequency (in the system A) and the term $(\omega_0)_A$ represents the Langmuir frequency which is defined as

$$(\omega_0)_A=(4\pi n_A e^2/m)^{\frac{1}{2}}, \tag{22}$$

where n_A is the electron density in the reference system A.

C. Beam Moving with Velocity v

We shall describe the beam in the framework of an observer moving with the beam. In this case the moving reference system is labeled as A since the beam appears in this system as a stationary electron gas. We have

$$M_A{}^{lm}=\chi_A{}^{lmik}F_{ik,A}, \tag{23}$$

where $\chi_A{}^{lmik}$ is defined by (20), (21a; b), and (22). Our problem consists in representing the relationship (23) in the laboratory system, i.e., in the framework of the observer B. This representation is expressed symbolically as

$$M_B{}^{lm}=\chi_B{}^{lmik}F_{ik,B}. \tag{24}$$

In making the transformation from the system A to the system B, the following two conditions should be taken into account: (1) The expression $\chi_A{}^{lmik}$ represents a tensor. (2) Various components of this tensor contain the expression χ_e given by (21b) which is not relativistically invariant.

One of the terms contained in χ_e is the frequency ω_A that transforms in accordance with (10). The other term is the Langmuir frequency which in the system B is represented as $(\omega_0)_B=(4\pi n_B e^2/m)^{\frac{1}{2}}$. Therefore the Langmuir frequency is transformed from the system A to the system B as the square root of electron density. Since $n_B=n_A\gamma$, we have:

$$(\omega_0)_B=(\omega_0)_A\gamma^{\frac{1}{2}}. \tag{25}$$

Consequently, the transformation from the system A to the system B is effected in two steps. In the first step we apply the covariant properties of the tensor $\chi_A{}^{lmik}$ and in the second step we take into account the transformation properties of the terms ω and ω_0 as given by (10) and (25), respectively. Therefore

$$(\chi_e)_A=-\frac{(\omega_0)_B{}^2}{(\omega_B-kv_B)^2\gamma^3}. \tag{26}$$

In order to represent symbolically these two steps we shall introduce a term $\chi_{\text{int}}{}^{lmik}$ representing an "intermediate state." Thus the first step leads from $\chi_A{}^{lmik}$ to $\chi_{\text{int}}{}^{lmik}$ and the second step leads from $\chi_{\text{int}}{}^{lmik}$ to $\chi_B{}^{lmik}$. More specifically, in the first step the susceptibility tensor is transformed as

$$\chi_{\text{int}}{}^{prst}=Q_l{}^pQ_m{}^rQ_i{}^sQ_k{}^t\chi_A{}^{lmik}, \tag{27}$$

where the Q's represent the parameters of the Lorentz transformation.[9]

The second step is represented symbolically as

$$\chi_B{}^{prst}=(\chi_{\text{int}}{}^{prst})_{A\to B}. \tag{28}$$

This transformation exhibits the Doppler effect and it accounts for the occurrence of the vector $\mathbf{k}$ in the expression for $\chi_B{}^{lmik}$. Thus the medium that was frequency dispersive in the system A became space dispersive in the system B.

D. Continuous Velocity Distribution

The expression (28) represents the D-M susceptibility tensor for a beam having velocity $\mathbf{v}$ in $d\mathbf{v}$ and density $f(\mathbf{v})d\mathbf{v}$. Our further description will be in the laboratory system, i.e., in the framework of the observer B. Therefore, the subscript B shall no longer be necessary. In order to show explicitly the dependence of this tensor on the velocity of the beam we shall write

$$\chi_B{}^{prst}\equiv\chi_{\mathbf{v}}{}^{prst}d\mathbf{v}. \tag{29}$$

The generalized susceptibility of a plasma considered as a limiting case of an assembly of component beams can,

therefore, be represented as

$$\chi_{\text{plasma}}{}^{lmik} = \int \chi_{\mathbf{v}}{}^{lmik} d\mathbf{v}. \tag{30}$$

IV. THREE-DIMENSIONAL FORMULATION

For practical applications it may be appropriate to replace the covariant four-dimensional representation by an equivalent description in three dimensions. We shall, therefore, reformulate our expressions in a relativistic three-dimensional representation and derive the macroscopic parameters for two typical cases: a beam having velocity $\mathbf{v}$ and an assembly of beams having a spherically symmetrical velocity distribution $f(\mathbf{v})d\mathbf{v}$.

A. Beam Having Velocity v

We shall designate as $\mathbf{P}^{(\mathbf{v})}d\mathbf{v}$ and $\mathbf{M}^{(\mathbf{v})}d\mathbf{v}$ the electric and magnetic polarization produced in a beam moving with velocity $\mathbf{v}$. The relationship between the polarization and the corresponding field quantities can be expressed as follows:

$$P_\alpha{}^{(\mathbf{v})} = (\chi_e{}^{(\mathbf{v})})_{\alpha\beta} E_\beta, \tag{31}$$

$$M_\alpha{}^{(\mathbf{v})} = (\chi_\mu/\mu)_{\alpha\beta}{}^{(\mathbf{v})} B_\beta. \tag{32}$$

Thus the electric susceptibility of the beam is $(\chi_e)_{\alpha\beta}{}^{(\mathbf{v})}d\mathbf{v}$ and the "proper magnetic susceptibility" is $(\chi_\mu/\mu)_{\alpha\beta}{}^{(\mathbf{v})}d\mathbf{v}$.

We shall rewrite the expression (24) in three-dimensional space. Instead of field tensors M^{lm} and F_{ik} we shall use the corresponding field vectors $\mathbf{P}$, $\mathbf{M}$ and $\mathbf{E}$, $\mathbf{B}$. Taking into account (20), (23), (24), and (27)–(29), the relationship as expressed by (24) can be represented in the following form:

$$\mathbf{P}^{(\mathbf{v})}(1-\beta^2)+\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{P}^{(\mathbf{v})}) = -\frac{\omega_0{}^2(1-\beta^2)^{\frac{3}{2}}}{4\pi(\omega-c\mathbf{k}\cdot\boldsymbol{\beta})^2}[\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}], \tag{33}$$

$$\mathbf{M}^{(\mathbf{v})}(1-\beta^2)+\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{M}^{(\mathbf{v})}) = -\frac{\omega_0{}^2(1-\beta^2)^{\frac{3}{2}}}{4\pi(\omega-c\mathbf{k}\cdot\boldsymbol{\beta})^2}[\beta^2\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}-\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})], \tag{34}$$

where $\boldsymbol{\beta}=\mathbf{v}/c$.

By using instead of $\mathbf{B}$ its equivalent expression $(c/\omega)\mathbf{k}\times\mathbf{E}$ and substituting this expression in (33), (34) we obtain a formulation in which the electric and magnetic polarizations appear as explicit functions of the electrical force. Thus the expression (33) assumes a form (31) and the expression (34) assumes a form

$$M_\alpha{}^{(\mathbf{v})} = (\chi_{e\mu})_{\alpha\beta}{}^{(\mathbf{v})} E_\beta. \tag{35}$$

The tensor $(\chi_{e\mu})_{\alpha\beta}$ shall be designated as the "electromagnetic susceptibility tensor." Using x, y, z coordinates and assuming $k_x=k_y=0$; $k_z=k$; $v_x=v\sin\theta$; $v_y=0$; $v_z=v\cos\theta$, we can express the terms $(\chi_e)_{\alpha\beta}{}^{(\mathbf{v})}$ and $(\chi_{e\mu})_{\alpha\beta}{}^{(\mathbf{v})}$ as follows:

$$\begin{aligned}
(\chi_e)_{11}{}^{(\mathbf{v})} &= -\frac{\omega_0{}^2}{4\pi\gamma}\left[\frac{1}{\omega(\omega-kv\cos\theta)}-\frac{v^2}{c^2}\frac{\sin^2\theta}{(\omega-kv\cos\theta)^2}\right];\\
(\chi_e)_{13}{}^{(\mathbf{v})} &= \frac{\omega_0{}^2}{4\pi c^2\gamma}\frac{v^2\sin\theta\cos\theta}{(\omega-kv\cos\theta)^2};\\
(\chi_e)_{22}{}^{(\mathbf{v})} &= -\frac{\omega_0{}^2}{4\pi\gamma}\frac{1}{\omega(\omega-kv\cos\theta)};\\
(\chi_e)_{31}{}^{(\mathbf{v})} &= -\frac{\omega_0{}^2}{4\pi\gamma}\left[\frac{kv\sin\theta}{\omega(\omega-kv\cos\theta)^2}-\frac{v^2\sin\theta\cos\theta}{c^2(\omega-kv\cos\theta)^2}\right];\\
(\chi_e)_{33}{}^{(\mathbf{v})} &= -\frac{\omega_0{}^2}{4\pi\gamma}\left[\frac{1}{(\omega-kv\cos\theta)^2}-\frac{v^2\cos^2\theta}{c^2(\omega-kv\cos\theta)^2}\right];\\
(\chi_e)_{12}{}^{(\mathbf{v})} &= (\chi_e)_{21}{}^{(\mathbf{v})} = (\chi_e)_{23}{}^{(\mathbf{v})} = (\chi_e)_{32}{}^{(\mathbf{v})} = 0;
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
(\chi_{e\mu})_{12}{}^{(\mathbf{v})} &= -\frac{\omega_0{}^2}{4\pi\gamma}\left[\frac{v\cos\theta}{c(\omega-kv\cos\theta)^2}-\frac{kv^2\cos^2\theta}{\omega c(\omega-kv\cos\theta)^2}\right];\\
(\chi_{e\mu})_{21}{}^{(\mathbf{v})} &= -\frac{\omega_0{}^2}{4\pi\gamma}\left[\frac{kv^2}{\omega c(\omega-kv\cos\theta)^2}-\frac{v\cos\theta}{c(\omega-kv\cos\theta)^2}\right];\\
(\chi_{e\mu})_{23}{}^{(\mathbf{v})} &= -\frac{\omega_0{}^2}{4\pi\gamma}\frac{v\sin\theta}{c(\omega-kv\cos\theta)^2};\\
(\chi_{e\mu})_{11}{}^{(\mathbf{v})} &= (\chi_{e\mu})_{13}{}^{\mathbf{v}} = (\chi_{e\mu})_{22}{}^{\mathbf{v}} = (\chi_{e\mu})_{31}{}^{\mathbf{v}} = (\chi_{e\mu})_{33}{}^{\mathbf{v}} = 0;\\
(\chi_{e\mu})_{32}{}^{(\mathbf{v})} &= -\frac{\omega_0{}^2}{4\pi\gamma}\left[\frac{kv^2\sin\theta\cos\theta}{\omega c(\omega-kv\cos\theta)^2}-\frac{v\sin\theta}{c(\omega-kv\cos\theta)^2}\right].
\end{aligned} \tag{37}$$

The dispersion equation for a beam can be expressed in the relativistic formulation as follows:

$$|a_{\alpha\beta}-\epsilon_M{}^{(\mathbf{v})}|=0, \tag{38}$$

where

$$\begin{aligned}
a_{11}&=a_{22}=c^2k^2/\omega^2;\\
a_{12}&=a_{13}=a_{21}=a_{23}=a_{31}=a_{32}=a_{33}=0,
\end{aligned} \tag{39}$$

and $\epsilon_M{}^{(\mathbf{v})}$ is the "modified dielectric constant" expressed as follows:

$$\begin{aligned}
(\epsilon_M)_{11}{}^{(\mathbf{v})} &= 1-\frac{\omega_0{}^2}{\omega^2\gamma}-\frac{\omega_0{}^2k^2v^2\sin^2\theta}{\omega^2\gamma(\omega-kv\cos\theta)^2}+\frac{\omega_0{}^2v^2\sin^2\theta}{\gamma c^2(\omega-kv\cos\theta)^2};\\
(\epsilon_M)_{13}{}^{(\mathbf{v})} &= (\epsilon_M)_{31}{}^{(\mathbf{v})} = -\frac{\omega_0{}^2kv\sin\theta}{\gamma\omega(\omega-kv\cos\theta)^2}+\frac{\omega_0{}^2v^2\sin\theta\cos\theta}{\gamma c^2(\omega-kv\cos\theta)^2};\\
(\epsilon_M)_{22}{}^{(\mathbf{v})} &= 1-\frac{\omega_0{}^2}{\omega^2\gamma};\\
(\epsilon_M)_{33}{}^{(\mathbf{v})} &= 1-\frac{\omega_0{}^2}{\gamma(\omega-kv\cos\theta)^2}+\frac{\omega_0{}^2v^2\cos^2\theta}{c^2\gamma(\omega-kv\cos\theta)^2};\\
(\epsilon_M)_{12}{}^{(\mathbf{v})} &= (\epsilon_M)_{21}{}^{(\mathbf{v})} = (\epsilon_M)_{23}{}^{(\mathbf{v})} = (\epsilon_M)_{32}{}^{(\mathbf{v})} = 0.
\end{aligned} \tag{40}$$

The expression (40) is known in the literature and is referred to as the "dielectric constant." There are also analogous expressions published in the literature that describe a plasma having different "longitudinal" and "transverse" temperatures and these expressions are also referred to as "dielectric" constants. Furthermore, one can find expressions of the type $D=\epsilon E$, where ϵ is as defined in (40) and D is referred to as the "electric displacement." We wish to emphasize that these expressions are inappropriate since they do not represent the physical reality.

In order to illustrate our point, we may consider the accuracy of the following statement: "A static electric charge produces an electromagnetic field." Such a statement is obviously untrue since it conflicts with the meaning of the well-defined concepts of electrostatic and electromagnetic fields. Similarly such concepts as the "electric displacement" and "dielectric constant" as currently applied to an anisotropic plasma are in conflict with these concepts as defined by Maxwell and Lorentz, respectively. The "electrical displacement" as used in the literature is actually intrinsically related to magnetic properties of plasma. Therefore it does not designate an entity that is entirely electrical. A similar situation applies to the term "dielectric constant." There is, therefore, a certain amount of confusion in the macroscopic description of plasma. This confusion can be clarified by recognizing that anisotropic plasma is a magnetizable medium and introducing the macroscopic concepts as expressed by (36) and (37).

Using the formulation (15) we can express $\mathbf{E}$ as a function of $\mathbf{B}$ in the form $E=-(c/\omega)\epsilon_M^{-1}(\mathbf{k}\times\mathbf{B})$. Substituting this expression in (35), we obtain a relationship of the type (32) in which the magnetic polarization is expressed as an explicit function of $\mathbf{B}$. Various terms expressing the proper magnetic susceptibility are as follows:

$$\left(\frac{\chi_\mu}{\mu}\right)_{11}^{(\mathbf{v})}=\frac{\omega_0^2 kv\cos\theta}{4\pi\gamma(\omega-kv\cos\theta)(\omega^2-\omega_0^2/\gamma)};\quad \left(\frac{\chi_\mu}{\mu}\right)_{31}^{\mathbf{v}}=\frac{-\omega_0^2 kv\sin\theta}{4\pi(\omega-kv\cos\theta)(\omega^2-\omega_0^2/\gamma)\gamma},$$

$$\left(\frac{\chi_\mu}{\mu}\right)_{22}^{(\mathbf{v})}=\frac{\dfrac{\omega_0^2 k}{4\pi\omega^2\gamma}\left[\dfrac{v\cos\theta}{\omega-kv\cos\theta}-\dfrac{kv^2\sin^2\theta}{(\omega-kv\cos\theta)^2}-\dfrac{\omega_0^2 v\cos\theta(c^2-v^2)}{c^2\gamma(\omega-kv\cos\theta)^3}\right]}{\left(1-\dfrac{\omega_0^2}{\omega^2\gamma}\right)\left(1-\dfrac{\omega_0^2(c^2-v^2\cos^2\theta)}{\gamma c^2(\omega-kv\cos\theta)^2}\right)-\dfrac{\omega_0^2v^2\sin^2\theta}{\gamma(\omega-kv\cos\theta)^2}\,\dfrac{(c^2k^2-\omega^2+\omega_0^2/\gamma)}{c^2\omega^2}};\tag{41}$$

$$\left(\frac{\chi_\mu}{\mu}\right)_{12}^{(\mathbf{v})}=\left(\frac{\chi_\mu}{\mu}\right)_{13}^{(\mathbf{v})}=\left(\frac{\chi_\mu}{\mu}\right)_{21}^{(\mathbf{v})}=\left(\frac{\chi_\mu}{\mu}\right)_{23}^{(\mathbf{v})}=\left(\frac{\chi_\mu}{\mu}\right)_{32}^{(\mathbf{v})}=\left(\frac{\chi_\mu}{\mu}\right)_{33}^{(\mathbf{v})}=0.$$

B. Spherically Symmetrical Distribution

A general expression for the macroscopic parameters of a plasma having any velocity distribution $f(\mathbf{v})d\mathbf{v}$ can be easily obtained. The polarization vectors in such a plasma can be expressed as $\mathbf{P}=\int\mathbf{P}^{(\mathbf{v})}d\mathbf{v}$, $\mathbf{M}=\int\mathbf{M}^{(\mathbf{v})}d\mathbf{v}$ and consequently the electric and magnetic susceptibilities are as follows:

$$(\chi_e)_{\alpha\beta}=\int(\chi_e)_{\alpha\beta}^{(\mathbf{v})}d\mathbf{v};\quad (\chi_{e\mu})_{\alpha\beta}=\int(\chi_{e\mu})_{\alpha\beta}^{(\mathbf{v})}d\mathbf{v};$$
$$(\chi_\mu/\mu)_{\alpha\beta}=\int(\chi_\mu/\mu)_{\alpha\beta}^{(\mathbf{v})}d\mathbf{v}.\tag{42}$$

A particularly interesting case is represented by a spherically symmetrical velocity distribution, i.e., when $f(\mathbf{v})=f(v)$. Substituting (36), (37) and (41) in (42), we obtain the following expression for various macroscopic parameters (assuming $k_2=k$; $k_x=k_y=0$):

$$(\chi_{e\mu})_{\alpha\beta}=0;\quad (\chi_\mu/\mu)_{\alpha\beta}=0;\tag{43}$$

and

$$(\chi_e)_{11}=(\chi_e)_{22}=-\frac{\omega_0^2}{4\pi\omega}\int\frac{f(v)dv}{(\omega-kv\cos\theta)\gamma}+\frac{\omega_0^2}{4\pi kc^2}\times\int\frac{f(v)v\cos\theta}{(\omega-kv\cos\theta)\gamma}dv;$$

$$(\chi_e)_{33}=-\frac{\omega_0^2}{4\pi}\int\frac{f(v)dv}{(\omega-kv\cos\theta)^2\gamma}+\frac{\omega_0^2}{4\pi c^2}\times\int\frac{f(v)v^2\cos^2\theta}{(\omega-kv\cos\theta)^2\gamma}dv;\tag{44}$$

$$(\chi_e)_{12}=(\chi_e)_{13}=(\chi_e)_{21}=(\chi_e)_{23}=(\chi_e)_{31}=(\chi_e)_{32}=0.$$

There are two particularly significant features of an isotropic plasma: (a) the absence of magnetic polarization and (b) the diagonal form of the electric susceptibility tensor.

C. Anisotropy

The functional dependence between the polarization vectors and the field intensities is represented in form

of tensors and, therefore, there is an anisotropy that characterizes the electromagnetic behavior of a plasma. There are certain distinctive features exhibited in the anisotropies of a plasma and these are not present in other nonisotropic media such as crystals. In order to clarify these distinctions we shall classify various non-isotropic media as belonging to type A, type B, and type C.

Type A designates the usual anisotropy occurring in crystals. It is expressed by an explicit and unique dependence between the relationship $\mathbf{P}=f(\mathbf{E})$ and a fixed direction in space. The fixed direction in space is the orientation of one of the principal axes of a crystal.

In the anisotropy of type B there are no fixed "reference" orientations such as the optical axes in a crystal. The substances exhibiting this anisotropy are structurally isotropic and their anisotropy is expressed by an explicit and unique dependence between the relationship $\mathbf{P}=f(\mathbf{E})$ and the direction of the propagation of the wave. Such a dependence exists in the spherically symmetrical plasma discussed above. This plasma is structurally isotropic and thus there is no preferred orientation. However, the relationship between $\mathbf{P}$ and $\mathbf{E}$ as expressed by the susceptibility tensor is of a directional nature and depends upon the direction $\mathbf{k}$. We have here two different values of the electric susceptibility depending upon the direction of $\mathbf{E}$ with reference to the direction of $\mathbf{k}$. Thus if $\mathbf{E}\|\mathbf{k}$, the susceptibility is determined by $(\chi_e)_{11}$ as given by (44) and defined as the "longitudinal susceptibility." On the other hand if $\mathbf{E}\perp\mathbf{k}$ the susceptibility is determined by $(\chi_e)_{11}=(\chi_e)_{22}$ as given by (44) and defined as the "transverse susceptibility."

The anisotropy of type C occurs in a plasma having a structural cylindrical symmetry. Such a plasma is characterized by a certain "fixed reference orientation" which may be the direction of the velocity vector $\mathbf{v}$ as in the case of the "one-beam medium" described by (36), (37), and (41). In such a medium there is a unique dependence between the relationship $\mathbf{P}=f(\mathbf{E})$ and two directions. One of these is the direction of the vector $\mathbf{k}$ and the other is the "reference orientation."

D. Determination of the Macroscopic Parameters from Boltzmann-Vlasov Equations

In previous paragraphs we considered the form $f(\mathbf{v})d\mathbf{v}$ as a limiting case of an assembly of discrete electron beams and the macroscopic parameters of a plasma were expressed as a superposition of the parameters relating to the component beams. We shall apply now a somewhat different procedure and derive the macroscopic parameters directly from the Boltzmann-Vlasov equations. This procedure introduced by Gertzenshtein[10] and Lindhard[14] was subsequently used by Shafranov[15] to obtain a general expression for the "modified" dielectric constant ϵ_M. By integrating the Boltzmann-Vlasov equations along the "trajectories" [in Lagrangian coordinates of particles $r(t)$ and $p(t)$], Shafranov expressed the perturbing term f_1 for the distribution function f_0 as follows:

$$f_1=-e\int_{-\infty}^{t}\left\{\mathbf{E}(\mathbf{r}(t'),t')+\frac{1}{c}\mathbf{v}(t')\times\mathbf{H}(\mathbf{r}(t'),t')\right\}\times\frac{\partial f_0}{\partial\mathbf{p}(t')}dt', \quad (45)$$

and obtained ϵ_M from the following relationship:

$$i\frac{(\epsilon_M)_\tau-\delta_{\tau\rho}}{4\pi}\omega E_\rho=j_\tau=\int j_\tau^{(\mathbf{p})}d\mathbf{p}, \quad (46)$$

where j_τ represents the total current density, and $j_\tau^{(\mathbf{p})}d\mathbf{p}$ represents the portion of the current density produced by particles in the volume element $d\mathbf{p}$ of the momentum space. The term $j_\tau^{\mathbf{p}}$ can be represented as

$$j_\tau^{(\mathbf{p})}=v_\tau f_1=-\frac{i\omega}{4\pi}K_{\tau\rho}E_\rho, \quad (47)$$

where

$$K_{\tau\rho}=\frac{4\pi ne^2}{\omega}\int_0^t v_\tau(t)\left\{\left(1-\frac{\mathbf{kv}}{\omega}\right)\frac{\partial f_0}{\partial p_\rho}+\frac{v_\rho}{\omega}\mathbf{k}\frac{\partial f_0}{\partial\mathbf{p}}\right\}\times\exp\left\{-i\left[\omega t-\mathbf{k}\int_0^t\mathbf{v}(t')dt'\right]\right\}dt. \quad (48)$$

We shall now derive the expressions representing the electric and magnetic properties of a plasma in the "conventional" representation. Let $\mathbf{P}^{(\mathbf{p})}d\mathbf{p}$ represent the polarization associated with the particles within the element $d\mathbf{p}$, i.e., $\mathbf{P}=\int\mathbf{P}^{(\mathbf{p})}d\mathbf{p}$. The current $\mathbf{j}^{(p)}$ contains the following parts: (a) the current caused by the rate of change of the polarization and the motion of the polarized medium, i.e.,

$$\frac{D\mathbf{P}^{(\mathbf{p})}}{Dt}=\frac{\partial\mathbf{P}^{(\mathbf{p})}}{\partial t}+(\mathbf{v}\cdot\text{grad})\mathbf{P}^{(\mathbf{p})}, \quad (49)$$

and, (b) the convective current;

$$-\mathbf{v}\,\text{div}\mathbf{P}^{(\mathbf{p})}. \quad (50)$$

Consequently

$$(\omega-\mathbf{k}\cdot\mathbf{v})\mathbf{P}^{(\mathbf{p})}+\mathbf{v}(\mathbf{k}\cdot\mathbf{P}^{(\mathbf{p})})=\mathbf{j}^{(\mathbf{p})}. \quad (51)$$

The equations (47) and (51) give

$$P_\tau^{(\mathbf{p})}=m_{\tau\rho}j_\rho^{(\mathbf{p})}=-\frac{i\omega}{4\pi}m_{\tau\rho}K_{\tau\zeta}E_\zeta, \quad (52)$$

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where

$$m_{\alpha\beta}=\frac{\omega-k_\alpha v_\beta}{\omega(\omega-\mathbf{kv})} \quad \text{for} \quad \alpha=\beta,$$
$$m_{\alpha\beta}=\frac{k_\beta v_\alpha}{\omega(\omega-\mathbf{kv})} \quad \text{for} \quad \alpha\neq\beta. \tag{53}$$

Consequently, the electric susceptibility can be expressed as

$$(\chi_e)_{\gamma\beta}=-\frac{i\omega}{4\pi}\int m_{\gamma\alpha}K_{\alpha\beta}d\mathbf{p}. \tag{54}$$

In order to obtain the magnetic polarization, we put $M=\int M^{(\mathbf{p})}d\mathbf{p}$. The term $M^{(\mathbf{p})}$ can be determined from (11) and represented in the form

$$M_\alpha^{(\mathbf{p})}=\frac{1}{c}(P_\beta^{(\mathbf{p})}v_\gamma-P_\gamma^{(\mathbf{p})}v_\beta), \tag{55}$$

where α, β, γ are in a cyclic succession.

Substituting (52) in (55) and utilizing the relationship $M_\alpha=(\chi_{e\mu})_{\alpha\beta}E_\beta$, we can express the "electromagnetic susceptibility" as follows:

$$(\chi_{e\mu})_{\alpha\beta}=-\frac{i\omega}{4\pi}\int v_\gamma m_{\beta\delta}K_{\delta\rho}-v_\beta m_{\gamma\delta}K_{\delta\rho})d\mathbf{p}, \tag{56}$$

where α, β, γ are in a cyclic succession.

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Mobility of Ions in a System of Interacting Bose Particles

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The transport property of an ion in a dilute Bose-Einstein gas subject to an external electric field is investigated by means of the Boltzmann equation. The interaction Hamiltonian which describes the ion-phonon scattering processes is obtained by the use of the Bogoliubov transformation and the cross section for the scattering of the ion by phonons is calculated. The solution of the Boltzmann equation is obtained by applying a variation principle and the temperature dependence of the ion mobility is shown to be T^{-4} at very low temperatures. A comparison of the results with the experimental data in liquid helium and the Khalatnikov and Zharkov theory is given and also the ion mobility in a Fermi system is briefly discussed.

1. INTRODUCTION

THE purpose of this paper is to study the mobility of an ion in a dilute Bose gas in connection with the recent experiments in which ions in liquid helium are used as microscopic probe particles to investigate the properties of the superfluid.[1–4] Because of the superfluid property, one expects that the ions move through liquid helium without encountering any resistance in its ground state at the absolute zero. On the other hand, at finite temperatures the ions suffer the scattering processes which arise from the thermal excitations present in the fluid. Thus, one may expect that by studying the motion of ions, useful information can be deduced about the nature of the elementary excitations and possible interactions between a particle and the underlying quantum fluid.

The simplest experimental situation is to apply an electric field E and to measure the drift velocity u of the ion. If E is kept sufficiently small, u is expected to be proportional to E and one may define the field-independent mobility $\mu=u/E$. In fact, Meyer and Reif[1] have shown experimentally that μ is independent of E when $E\ll1$ volt/cm and that its temperature dependence is of the form $\mu=\mu_0\exp(\Delta/kT)$ in the range below the λ point down to 0.8°K. A possible interpretation of this behavior was proposed by Meyer and Reif[1] based on the scattering of the ion by rotons. At temperatures below 0.6°K, they obtained a temperature dependence of the form $\mu\propto T^{-k}$, where $k=3.3\pm0.3$ for a positive ion and $k=2.4\pm0.4$ for a negative ion, and they pointed out that this behavior disagrees with the prediction by Khalatnikov and Zharkov[5] obtained on the basis of ion-phonon interactions derived from quantum hydrodynamics. So it is of considerable interest to re-examine the temperature dependence of the ion mobility due to phonon excitations from first principles.

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