Long-Range Interaction in $\bar{K}$-Nucleon and K-Nucleon Elastic Amplitudes*

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A method of calculating, for the K-nucleon interaction, the long-range force arising from the exchange of a pion pair and of a possible three-pion resonant state is formulated. It is shown that the long-range force can be related with the electromagnetic structure parameters of the nucleon and K meson. Finally, relations between K-nucleon and $\bar{K}$-nucleon elastic amplitudes are discussed.

I. INTRODUCTION

IN order to understand the dynamic nature of the K-nucleon interaction, it is important to establish whether or not a long-range force exists. In a previous note we argued that the low-energy (S-wave) behavior of the $\bar{K}$-proton and K-proton scattering may indicate such a force.[1] The criterion for establishing its existence —in the absence of detailed experimental information about any single channel—is that it shall correlate the energy dependence of the various related processes. Relevant are K-nucleon and $\bar{K}$-nucleon elastic and charge-exchange scattering, as well as hyperon production processes. In this paper we formulate a method of calculating the long-range interaction arising from the exchange of a pion pair or a possible three-pion bound state for these processes.[2]

The calculation is made on the basis of partial-wave dispersion relations.[3] The preliminary step of this approach, i.e., the determination of the analytic structure of the various transition amplitudes, has already been performed.[4] In general, the partial-wave amplitudes are analytic in the cut W plane (W is the total barycentric energy of the $\bar{K}$-nucleon system) with physical branch cuts beginning at the thresholds of the lowest energy intermediate states and extending to infinity. In addition, there are unphysical singularities associated with the two "crossed" reactions. The spectral functions of the partial-wave amplitudes, i.e., the discontinuities across the unphysical cuts and the residues of poles, represent—or are defined to be—the "interaction."

The long-range interaction is described by the spectral functions for the singularities that lie close to the physical region.[5] These occur only in the elastic K-nucleon and $\bar{K}$-nucleon amplitudes and depend on the matrix elements for the reactions $\pi+\pi\rightarrow K+\bar{K}$ and $\pi+\pi\rightarrow N+\bar{N}$. A good deal is known about the latter matrix element, but the former is as yet totally unmeasured. In order to estimate the P-wave part, we introduce a charge structure hypothesis for the K meson.[6] This allows a rough guess as to the force that would arise from the exchange of a resonant two-pion state.

It is important to emphasize that pion exchange contributions to the K-nucleon force have a substantially longer range than the Yukawa interactions ($N\Sigma K$) and $N\Lambda K$). The latter may control hyperon production and presumably play an important role for the short-range interaction. Even if the coupling constants $g_{N\Lambda K}$ and $g_{N\Sigma K}$ are relatively small, the effect these have may be enhanced by the presence of the long-range force.

II. AMPLITUDES FOR THE K-NUCLEON SCATTERING

The elastic $\bar{K}$-nucleon and K-nucleon interactions are both represented by a general diagram with four external lines—two K mesons (mass m_K) and two nucleons (mass M).

Let the four-momenta of the initial and final $\bar{K}$ mesons be q_1 and q_2, respectively, those of the incident and outgoing nucleons being p_1 and p_2. If different pairs of the momenta are regarded as the variables of the incoming particles, the diagram describes three distinct processes:

$$\bar{K}+N\rightarrow\bar{K}+N,\qquad (q_1+p_1\rightarrow q_2+p_2),\qquad \text{(II.1a)}$$

$$K+N\rightarrow K+N,\quad (-q_2+p_1\rightarrow -q_1+p_2),\quad \text{(II.1b)}$$

and

$$K+\bar{K}\rightarrow N+\bar{N},\quad (-q_2+q_1\rightarrow p_1-p_2),\qquad \text{(II.1c)}$$

The most striking feature of the problem is that even at threshold the $\bar{K}$-nucleon system initiates hyperon production,

$$\bar{K}+N\rightarrow\pi+Y,\qquad \text{(II.2)}$$

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[1] F. Ferrari, G. Frye, and M. Pusterla, Phys. Rev. Letters 4, **615** (1960).
[2] G. F. Chew, Phys. Rev. Letters 4, 142 (1960); and Y. Nambu, Phys. Rev. **106**, 1366 (1957).
[3] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[4] F. Ferrari, M. Nauenberg, and M. Pusterla, University of California Radiation Laboratory Report UCRL-8985 (unpublished); M. Nauenberg, Ph.D. dissertation, Cornell University (unpublished); F. Ferrari, M. Nauenberg, and M. Pusterla (unpublished).
[5] The K-nucleon threshold is at $W_0=m_K+M\approx10.24m_\pi$.
[6] The $\langle\pi\pi|K\bar{K}\rangle$ matrix element is discussed in detail by G. Frye, thesis, University of California, Berkeley (unpublished).

where Y denotes either Λ or Σ. It is therefore necessary to use a many-channel S-matrix formalism. The unitarity of S establishes a host of conditions among the amplitudes for the above reactions and also relates them to the amplitudes for pion-hyperon processes,

$$\pi+Y\rightarrow\pi+Y'. \tag{II.3}$$

All the preceding reactions, as well as those obtained by applying the substitution rule to Eqs. (II.2) and (II.3), enter into the determination of the amplitude for any one of them.

The primary concern of this paper is the amplitude for processes (II.1a–c), but for emphasis and future applications it is convenient to proceed with the many-channel formalism. The S matrix is defined in the same way as that for pion-nucleon scattering,[7]

$$S_{fi}=\delta_{fi}-i(2\pi)^4\delta(q_i+p_i-q_f-p_f)\times\left(\frac{M_iM_f}{4\omega_i\omega_fE_iE_f}\right)^{\frac{1}{2}}\tau_{fi}, \tag{II.4}$$

where $M_i(M_f)$ is the mass of the initial (final) baryon, E is the total energy of the baryon, and ω that of the meson. For the case of even (Σ-Λ) parity and pseudoscalar K meson,[8] the decomposition into spin-independent amplitudes is

$$\tau_{fi}=\bar{u}_f[-A+\tfrac{1}{2}\gamma\cdot(q_i+q_j)B]u_i, \tag{II.5}$$

where A and B are matrices in channel space, the indices having been suppressed.

The relations between A and B and the barycentric differential cross section is established as follows:

$$\frac{d\sigma_{fi}}{d\Omega}=\sum\frac{q_f}{q_i}\left|\left\langle f\left|f_1+\frac{(\boldsymbol{\sigma}\cdot\mathbf{q}_f)(\boldsymbol{\sigma}\cdot\mathbf{q}_i)}{|\mathbf{q}_i|\cdot|\mathbf{q}_f|}f_2\right|i\right\rangle\right|^2, \tag{II.6}$$

where $\sum$ indicates the appropriate sum and average over spin states and $\boldsymbol{\sigma}$ is the three-vector of 2-by-2 Pauli spin matrices. The channel matrices f_1 and f_2 are related to the previous amplitudes by

$$f_1=(1/8\pi W)(E+M)^{\frac{1}{2}}[A+(W-\bar{M})B](E+M)^{\frac{1}{2}},$$

and (II.7)

$$f_2=(1/8\pi W)(E-M)^{\frac{1}{2}}[-A+(W+\bar{M})B](E-M)^{\frac{1}{2}},$$

where M and E are diagonal matrices with components equal to the baryon mass and the c.m. total baryon energy, respectively, and

$$\bar{M}_{ij}=\tfrac{1}{2}(M_i+M_f). \tag{II.8}$$

Each element of the matrix $f_{1,2}$ is a function of the barycentric total energy W and the appropriate scattering or production angle θ. A straightforward method for performing the decomposition into partial waves is available in the work of Jacob and Wick.[9] One easily shows that

$$f_1=\sum_{l=0}^{\infty}(f_{l+}P_{l+1}'-f_{l-}P_{l-1}'), \tag{II.9}$$

and

$$f_2=\sum_{l=1}^{\infty}(f_{l-}-f_{l+})P_l', \tag{II.10}$$

where

$$f_{l\pm}(W)=\tfrac{1}{2}\int_{-1}^{1}d\cos\theta(f_1P_l+f_2P_{l\pm1}). \tag{II.11}$$

The latter amplitudes are useful for the application of partial-wave dispersion relations because they satisfy a simple unitarity condition. For a given isotopic spin, total angular momentum, and parity state, the condition is

$$\operatorname{Im}f_{l\pm}{}^{fi}=\textstyle\sum_j f_{l\pm}{}^{fj}k_j\theta_j(f_{l\pm}{}^*)^{ij}, \tag{II.12}$$

where k_j is the magnitude of the barycentric three-momentum of the intermediate state j, and where

$$\begin{aligned}\theta_j&=1 \quad\text{if}\quad W>W_j,\\ &=0 \quad\text{if}\quad W<W_j,\end{aligned}$$

W_j, finally, is the threshold energy of the state j. It follows from Eq. (II.12) that [10]

$$\operatorname{Im}(f_{l\pm}{}^{-1})^{fi}=-k_i\theta_i\delta_{if}. \tag{II.13}$$

Another interesting property of $f_{l\pm}$, as first pointed out by MacDowell,[11] follows from Eqs. (II.7), (II.8), and (II.11). It is

$$f_{(l+1)-}(W)=-f_{l+}(-W). \tag{II.14}$$

In the sequel, therefore, we need consider only f_{l+}, and not f_{l-}. Finally, if each element of A and B satisfies the Mandelstam representation, it follows that $f_{l\pm}$ has the following "threshold" dependence:

$$f_{l\pm}{}^{if}(W)\approx(k_ik_f)^l, \tag{II.15}$$

for $W\approx M_i+m_i$, M_i-m_i, M_f+m_f, and M_f-m_f; also

$$f_{l\pm}{}^{if}(W)\approx(k_ik_f)^{l\pm1}, \tag{II.16}$$

for $W\approx-M_i+m_i$, $-M_i+m_i$, $-M_f+m_f$, and $-M_f-m_i$, where $M_i(M_f)$ and $m_i(m_f)$ are the respective masses of the initial (final) baryon and meson.

Let us now go on to discuss the analytic properties of $f_{l\pm}(W)$. First of all, it is necessary to remove the branch points that arise from the "kinematic factors" $(E\pm M)^{\frac{1}{2}}$ in Eqs. (II.7) and (II.8). At the same time, let us take into account the above-mentioned threshold behavior. To this end we define the matrix G_J, where

[7] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[8] Formulas for the case of odd (Σ-Λ) parity are given in reference 4.

[9] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959).

[10] We are indebted to Professor S. Mandelstam for discussions clarifying this point.

[11] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).

$J=l+\frac{1}{2}$, as follows:

$$f_{l+}(W)=(1/W)(E+M)^{\frac{1}{2}}k^lG_J(W)k^l(E+M)^{\frac{1}{2}}. \quad \text{(II.17)}$$

Using Eqs. (II.7), (II.8), and (II.11), one easily shows that

$$G_J(W)=\frac{1}{16\pi}\left\{\frac{1}{k^l}[A_l+(W-\bar{M})B_l]\frac{1}{k^l}\right.$$
$$\left.+\frac{E-M}{k^{l+1}}[-A_{l+1}+(W+\bar{M})B_{l+1}]\frac{E-M}{k^{l+1}}\right\}, \quad \text{(II.18)}$$

where

$$A_l(W)=\int_{-1}^{+1}d\cos\theta P_l(\cos\theta)A(W,\cos\theta),$$
$$B_l(W)=\int_{-1}^{+1}d\cos\theta P_l(\cos\theta)B(W,\cos\theta). \quad \text{(II.19)}$$

The modified amplitudes $G_J(W)$ were defined by Eq. (II.18) to be analytic in the W plane except for the singularities of A_l and B_l. Their location and nature have been studied on the basis of the Mandelstam representation.[4,11] Let us continue the discussion by treating only the amplitudes for elastic $\bar{K}$-nucleon scattering. Then, assuming I and $\tau_N\cdot\tau_{\bar{K}}$ as the basis for the isotopic spin space, the decomposition into isotopic-spin-independent functions is

$$A=A^{(+)}I+\tau_N\cdot\tau_{\bar{K}}A^{(-)}, \quad \text{(II.20)}$$

with a similar relation for B.

The amplitudes for states of definite isotopic spin are therefore

$$A^{(0)}=A^{(+)}-3A^{(-)}, \quad \text{(II.21a)}$$
$$A^{(1)}=A^{(+)}+A^{(-)}. \quad \text{(II.21b)}$$

Similar equations apply for the other amplitudes: $B^{(\pm)}$, $f_{l\pm}{}^{(\pm)}$, and $G_J{}^{(\pm)}$.

The amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ are scalar functions of the invariants

$$s=(q_1+p_1)^2=W^2, \quad \text{(II.22a)}$$
$$u=(p_1-q_2)^2, \quad \text{(II.22b)}$$

and

$$t=(q_1-q_2)^2, \quad \text{(II.22c)}$$

which in turn are related to the $\bar{K}$-nucleon barycentric variables by

$$t=-2k^2(1-\cos\theta), \quad \text{(II.23)}$$
$$4sk^2=[s-(m_K+M)^2][s-(m_K-M)^2], \quad \text{(II.24)}$$

and

$$E=(W^2+M^2-m_K^2)/2W. \quad \text{(II.25)}$$

The location of the singularities of G_J is discussed in reference 4. The singularities of the dynamically coupled $G_J{}^{\alpha\beta}$ elements also influence the behavior of G_J. It is clear that the dynamic singularities arising from the exchange of two and three pions [the cut due to the two-pion exchange extends to $W_{\pi\pi}=(M^2-m_\pi^2)^{\frac{1}{2}}+(m_K^2-m_\pi^2)^{\frac{1}{2}}$] are closer to the physical region than all others and may be expected to produce the strongest energy dependence of the physical amplitudes and, in fact, to dominate those of sufficiently high l. The discontinuity across this part of the cut $\mathrm{Im}G_J(W)$ is defined to be the "long-range interaction." A general expression for the discontinuity follows from Eqs. (II.18) and (II.19). Changing the variable of integration from $\cos\theta$ to t, one finds

$$\mathrm{Im}G_J{}^{(\pm)}(W)=\frac{\epsilon(W)\epsilon(W^2-r^2)}{32\pi k^{2l+2}}\int_0^{-4k^2}dt\,P_l\left(1+\frac{t}{2k^2}\right)$$
$$\times\left[\mathrm{Im}A^{(\pm)}\left(W,1+\frac{t}{2k^2}\right)\right.$$
$$\left.+(W-M)\,\mathrm{Im}B^{(\pm)}\left(W,1+\frac{t}{2k^2}\right)\right]$$
$$+\frac{(E-M)^2\epsilon(W)\epsilon(W^2-r^2)}{32\pi k^{2l+4}}\int_0^{-4k^2}dtP_{l+1}\left(1+\frac{t}{2k^2}\right)$$
$$\times\left[-\mathrm{Im}A^{(\pm)}\left(W,1+\frac{t}{2k^2}\right)\right.$$
$$\left.+(W+M)\,\mathrm{Im}B^{(\pm)}\left(W,1+\frac{t}{2k^2}\right)\right], \quad \text{(II.26)}$$

where $r=M-m_k$ and $\epsilon(z)=z/|z|$. ImA and ImB contribute to the spectral function $\mathrm{Im}G_J(W)$, for real W, in the interval

$$(M^2-m_\pi^2)^{\frac{1}{2}}-(m_K^2-m_\pi^2)^{\frac{1}{2}}<|W|<$$
$$<(M^2-m_\pi^2)^{\frac{1}{2}}+(m_K^2-m_\pi^2)^{\frac{1}{2}}. \quad \text{(II.27)}$$

III. TWO- AND THREE-PION INTERACTIONS

The strength of the long-range interaction depends upon the imaginary part of the invariant amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ when they are evaluated in the region of variables $\sim[(M^2-4m_\pi^2)^{\frac{1}{2}}+(m_K^2-4m_\pi^2)^{\frac{1}{2}}]^2\lesssim s<W_{\pi\pi}{}^2$ and $0<t<-4k^2$. This region of the invariants has a simple interpretation in terms of the barycentric variables of the reaction $K+\bar{K}\rightarrow N+\bar{N}$. Let q and p be the magnitudes of the three momenta of the $\bar{K}$ meson and the nucleons, respectively, and let θ_3 be the production angle:

$$\cos\theta_3=\mathbf{p}_1\cdot\mathbf{q}_1/|\mathbf{p}_1|\,|\mathbf{q}_1|,$$

all in the $K\bar{K}$ barycentric system. These variables are related to the invariants by

$$s=-p^2-q^2+2pq\cos\theta_3, \quad \text{(III.1a)}$$
$$u=-p^2-q^2-2pq\cos\theta_3, \quad \text{(III.1b)}$$

and

$$t=4(q^2+m_K^2)=4(p^2+M)^2. \quad \text{(III.1c)}$$

Now, for $4m_\pi^2<t<4m_K^2$, Eq. (II.27) may be rewritten as

$$-(p-q)^2<s<-(p+q)^2,$$

or (III.2)

$$-1<\cos\theta_3<1,$$

showing that the amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ are to be evaluated for physical values of the production angle. On the other hand, the total energy $\sqrt{t}$ has unphysical values corresponding to the two-pion intermediate state. The imaginary parts of $A^{(\pm)}$ and $B^{(\pm)}$ thus obtained are the so-called "absorptive parts" for the process $K+\bar{K}\to N+\bar{N}$ and are denoted by $A_{\rm III}{}^{\pm}(t,s)$ and $B_{\rm III}{}^{\pm}(t,s)$.

The amplitude for the production process is obtained by applying the substitution rule to the S-matrix element of $\bar{K}$-nucleon scattering Eq. (II.4). Frazer and Fulco have shown that the helicity amplitudes $f(\lambda,\bar{\lambda})$ of Wick and Jacob provide[11] a simple way of expressing the result.[12] From their work one easily finds that, for each value of the isotopic spin index $(\pm)$,

$$f(\lambda,\bar{\lambda})=\left(\frac{2\pi}{q}\right)\langle\lambda,\bar{\lambda}(N,\bar{N})|T|(K,\bar{K})\rangle$$
$$=\left(\frac{2\pi}{q}\right)\frac{(pq)^{\frac{1}{2}}M}{8\pi^2t^{\frac{1}{2}}}\tau(\lambda,\bar{\lambda}),\quad\text{(III.3)}$$

where

$$\tau_{++}=\tau_{--}=-(p/M)A+q\cos\theta_3 B,\quad\text{(III.4)}$$

and

$$\tau_{+-}=-\tau_{-+}=(Eq/M)\sin\theta_3 Be^{i\phi_3}.\quad\text{(III.5)}$$

The absorptive parts $A_{\rm III}$ and $B_{\rm III}$ are then expressed, for physical values of $t>4M^2$, in terms of the τ amplitudes

$$A_{\rm III}=-\frac{M}{p}\frac{1}{2i}[\tau_{++}-\tau_{++}{}^\dagger]+\frac{M^2\cot\theta_3}{pE}\frac{1}{2i}[\tau_{+-}-\tau_{+-}{}^\dagger],\quad\text{(III.6)}$$

and

$$B_{\rm III}=\frac{M}{qE\sin\theta_3}\frac{1}{2i}[\tau_{+-}-\tau_{+-}{}^\dagger].\quad\text{(III.7)}$$

The central physical condition upon which the calculation is based, is the unitarity of the S matrix for the production channel. It is expressed in terms of the amplitudes of Jacob and Wick by

$$\frac{1}{2i}\langle\lambda,\bar{\lambda}(N,\bar{N})|T-T^\dagger|(K,\bar{K})\rangle=\tfrac{1}{2}\sum_n\sum_J\frac{2J+1}{4\pi}\times d_{0\mu}{}^J(\theta)\langle\lambda,\bar{\lambda}(N,\bar{N})|T^J|n\rangle\langle n|T^{\dagger J}|(K\bar{K})\rangle,\quad\text{(III.8)}$$

where the first summation extends over all the energetically accessible intermediate states that connect a $K\bar{K}$ pair of a $N\bar{N}$ pair. The $d_{0\mu}{}^J$ are the wave functions for a symmetrical top and $\mu=\lambda-\bar{\lambda}$. Finally, the matrix elements appearing under the double sum are the amplitudes for the production of a state (n) by $N\bar{N}$ or $K\bar{K}$ systems of definite total angular momentum J. Both sides of Eq. (III.8) may be expressed in terms of analytic functions and the equation may be continued to unphysical values of t, where only the two-pion state contributes. The interesting feature of the present application is that Eq. (III.8) relates the absorptive parts to other amplitudes which (in principle) may be calculated independently. Let us now discuss the two-pion intermediate state. The absorptive parts then depend only upon the matrix elements for the reactions $\pi+\pi\to K+\bar{K}$ and $\pi+\pi\to N+\bar{N}$. The Jacob-Wick amplitudes for the latter are expressed in terms of the Frazer-Fulco amplitudes $f_\pm{}^J$ as follows:

$$\langle\lambda,\bar{\lambda}(N,\bar{N})|T^J|(\pi\pi)\rangle=(q_\pi p/2\pi)\mathfrak{F}^J(\lambda\bar{\lambda})\quad\text{(III.9)}$$

where q_π is the magnitude of the pion three-momentum in the barycentric system,

$$\mathfrak{F}_{++}{}^J(t)=(4\pi/p\sqrt{t})(pq_\pi)^Jf_+{}^J(t),\quad\text{(III.10)}$$

and

$$\mathfrak{F}_{+-}{}^J(t)=(2\pi/p)(pq_\pi)^Jf_-{}^J(t).\quad\text{(III.11)}$$

The amplitudes $f_\pm{}^J$ were defined so that they are real analytic functions in the cut t plane, with a physical branch extending from $4m_\pi^2$ to $t=+\infty$, and a left-hand cut whose discontinuity is purely dynamical. The corresponding expression for the process $\pi+\pi\to K+\bar{K}$ is

$$\langle(K\bar{K})|T^J|(\pi\pi)\rangle=(qq_\pi/t)^{\frac{1}{2}}(qq_\pi)^Jf_J(t),\quad\text{(III.12)}$$

where the f_J have the same analytic structure as the $f_\pm{}^J$. Their relation to the invariant amplitudes of π-K scattering will be given in the following section. Expressions for the absorptive parts are finally obtained by substituting Eqs. (III.3) and (III.8) into Eqs. (III.6) and (III.7). Using Eqs. (III.9) through (III.12), we have

$$A_{\rm III}{}^{(\pm)}(t,s)=-\frac{2\pi}{p^2\sqrt{t}}\sum_J(2J+1)q_\pi{}^{2J+1}(pq)^J\times\left\{f_+{}^{J(\pm)}(t)P_J(\cos\theta_3)-f_-{}^{J(\pm)}(t)\frac{M\cos\theta_3P_J'(\cos\theta_3)}{[J(J+1)]^{\frac{1}{2}}}\right\}f_J{}^{*(\pm)}(t),\quad\text{(III.13)}$$

and

$$B_{\rm III}{}^{(\pm)}(t,s)=\bar{M}\sum_J\frac{(2J+1)}{[J(J+1)]^{\frac{1}{2}}}P_J'(\cos\theta_3)\times q_\pi{}^{2J+1}(pq)^{J-1}f_-{}^{J(\pm)}(t)f_J{}^{*(\pm)}(t),\quad\text{(III.14)}$$

where $\cos\theta_3$ is related to s and t by Eqs. (III.1a) and (III.1c). The symmetry of the two-pion state implies that terms with even J contribute only to $B_{\rm III}{}^{(+)}$ and $A_{\rm III}{}^{(+)}$, while those of odd J contribute only to $B_{\rm III}{}^{(-)}$ and $A_{\rm III}{}^{(-)}$. Taking into account Eqs. (III.13) and (III.14) we have the following symmetry properties

[12] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).

for the two-pion approximation to the absorptive parts:

$$\pi\pi:\ A_{\mathrm{III}}^{(\pm)}(t,s)=\pm A_{\mathrm{III}}^{(\pm)}(t,u), \tag{III.15}$$

and

$$\pi\pi:\ B_{\mathrm{III}}^{(\pm)}(t,s)=\mp B_{\mathrm{III}}^{(\pm)}(t,u). \tag{III.16}$$

A convenient parametrization of the three-pion state is not yet available. However, if there is a $J=1$, $I=0$ three-pion bound state (mass m_B), it gives rise to poles in the invariant amplitudes $A^{(+)}$ and $B^{(+)}$ which, in principle, can be handled. If there is no bound state but a sharp resonance, the pole approximation is still reasonable. Note that this three-pion state has to give a substantial contribution to the scalar charge structure of the nucleon, whereas the magnitude of the scalar part of the anomalous magnetic moment is experimentally small. Now the pole in the $A^{(+)}$ amplitude is proportional to the nucleon three-pion matrix element present in the scalar anomalous magnetic moment; thus it is plausible to neglect the three-pion contribution to $A^{(+)}$ and to regard only the pole in the $B^{(+)}$ amplitude as important for the $I=0$, $J=1$ three-pion contribution to the long-range K-nucleon force. The consequent absorptive part is

$$B_{\mathrm{III}}^{(+)}(t,s)=-\beta\pi\delta(t-m_B^2). \tag{III.17}$$

IV. $\langle\pi\pi|K\bar{K}\rangle$ MATRIX ELEMENT

In this section, we shall estimate the strength of the matrix element for the process $\pi+\pi\rightarrow K+\bar{K}$. Let $-q_1$ and q_2 be the momenta of the outgoing K and $\bar{K}$ mesons, respectively [see Eq. (II.1c)], while the four momenta and isotopic spin indices of the pions are $(-q_{\pi1},\alpha)$ and $(q_{\pi2},\beta)$. The decomposition of the S matrix into scalar amplitudes $A^{(\pm)}$ is then

$$S_{fi}=-i\delta(-q_{\pi1}+q_{\pi2}+q_1-q_2)\frac{16\pi^2}{(16\omega_{\pi1}\omega_{\pi2}\omega_1\omega_2)^{\frac{1}{2}}}\times\{A^{(+)}\delta_{\alpha\beta}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]A^{(-)}\}, \tag{IV.1}$$

where $\omega_{\pi1}$ and $\omega_{\pi2}$ are the pion energies. The $A^{(\pm)}$ are assumed to satisfy the Mandelstam representation as functions of the two independent invariants $t=-(q_1-q_2)^2$ and $v=-(q_1+q_{\pi1})^2$. The latter is related to the barycentric production angle θ' and magnitudes of the pion and K-meson three-momenta by $v=-q^2-q_\pi^2+2qq_\pi\cos\theta'$. The previously introduced analytical partial wave amplitudes $f_l^{(\pm)}$ are defined in terms of the invariant amplitudes by

$$f_l^{(\pm)}(t)=\frac{1}{(qq_\pi)^l}\int_{-1}^{+1}d\cos\theta' P_l(\cos\theta')A^{(\pm)}(t,\cos\theta'). \tag{IV.2}$$

The symmetry of the two-pion state implies that $f_l^{(+)}=0$ for odd l, and $f_l^{(-)}=0$ for even l.

Dispersion theory in its present form does not attempt to establish the magnitude of $A^{(\pm)}$, but only provides a way of relating them to π-K scattering. Even in the absence of experimental information, such an analysis does lead to some restrictions on the parameters, providing there are no π-K bound states and that "ghosts" are avoided in the calculable part of the unphysical region.[13] A similar treatment of the π-nucleon interaction, of course, provides a determination of the $\pi+\pi\rightarrow N+\bar{N}$ amplitudes $f_\pm^{J(\pm)}(t)$.

In order to estimate the probable physical values of $f_1^{(-)}(t)$, we assume that a K meson emits pion pairs with essentially the same strength as does a nucleon. This approach is based on the hypothesis that it is characteristic of strong interactions always to be "about as strong as is consistent with the requirements of unitarity."[14] The most easily treated two-pion state is that which contributes to the electromagnetic structure (i.e., $J=1$, $I=1$). The S-matrix element for the production of a $K-\bar{K}$ pair by a virtual photon (four-momentum q_μ) is

$$\langle -q_1,q_2|S|q_\mu\rangle=-i(2\pi)^{-\frac{1}{2}}\delta(-q_1+q_2-q_\mu)\times\frac{e(q_1+q_2)_\mu}{(8\mathcal{H}\omega_1\omega_2)^{\frac{1}{2}}}\{F_K{}^S+\tau_3F_K{}^V\}, \tag{IV.3}$$

where $\mathcal{H}$ is the photon energy and $F_K{}^S$ and $F_K{}^V$ are the form factors for the isotopic scalar and vector parts of the K-meson charge structure. Following a suggestion made by Chew,[15] we assume that $F_K{}^V$ satisfies an unsubtracted dispersion relation[16]:

$$F_K{}^V(q_\mu{}^2)=\frac{1}{\pi}\int_{4m_\pi{}^2}^{\infty}\frac{dt\,g(t)}{t+q_\mu{}^2}, \tag{IV.4}$$

where the spectral function $g(t)=\mathrm{Im}F_K{}^V(-t)$. The form factors are normalized so that

$$F_K{}^{S,V}(0)=\tfrac{1}{2}. \tag{IV.5}$$

The expression corresponding to Eq. (IV.3) for the production of a pion pair is

$$\langle -q_{\pi1},\alpha;q_{\pi2},\beta|S|q_\mu\rangle=\delta(-q_{\pi1}+q_{\pi2}-q_\mu)\times(2\pi)^{-\frac{1}{2}}\epsilon_{\alpha\beta3}\frac{e(q_{\pi1}+q_{\pi2})_\mu}{(8\mathcal{H}\omega_{\pi1}\omega_{\pi2})^{\frac{1}{2}}}F_\pi(t), \tag{IV.6}$$

where F_π is the pion form factor. The S-matrix elements Eqs. (IV.1), (IV.2), and (IV.6) are related by the unitarity condition. Keeping only the two-pion intermediate-state (the so-called "two-pion approximation"), the expression for the spectral function g follows:

$$[g(t)]_{\pi\pi}=q_\pi{}^3F_\pi{}^*(t)f_1{}^{(-)}(t)/\sqrt{t}. \tag{IV.7}$$

[13] G. Frye, Ph.D. dissertation, Department of Physics, University of California, Berkeley (unpublished).
[14] G. F. Chew, University of California Radiation Laboratory Report UCRL-9289 (unpublished).
[15] G. F. Chew, University of California Radiation Laboratory Report UCRL-8194 (unpublished).
[16] G. F. Chew *et al.*, Phys. Rev. **110**, 265 (1958).

The question arises as to what value is obtained for $F_K{}^V(0)$ if the integral in Eq. (IV.4) is evaluated in the two-pion approximation. To avoid ambiguity arising from the unknown behavior of $g(t)$ for large value of t, we define $f_K{}^V$ to be the value of $2F_K{}^V(0)$, obtained when the integral is cut off at the arbitrary value $t=30m_\pi{}^2$. For this limited-energy interval we use the following approximation for $f_1{}^{(-)}(t)$.

According to the unitarity condition $f_1{}^{(-)}(t)$ has the same phase as $F_\pi(t)$ for $4m_\pi{}^2<t<16m_\pi{}^2$. Thus the function $f_1{}^{(-)}(t)/F_\pi(t)$ is analytic in the cut t plane with the right-hand cut extending from $16m_\pi{}^2$ to $t=+\infty$. We will approximate this function by its average value over the interval $4m_\pi{}^2<t<30m_\pi{}^2$. Then, using the Frazer and Fulco expression for F_π in order to evaluate the integral, one obtains for $f_1{}^{(-)}$

$$f_1{}^{(-)}(t)\approx F_\pi(t)f_K{}^V\times\tfrac{1}{3}. \qquad \text{(IV.8)}$$

V. RELATION OF THE LONG-RANGE INTERACTION TO ELECTROMAGNETIC STRUCTURE

Two possible sources of the long-range interaction are treated in detail. Each arises from the exchange of a system of pions which are in a definite symmetry state. They are (A) the $I=1$, $J=1$ two-pion state, denoted by $\pi\pi(-)$, and (B) a possible $I=0$, $J=1$ three-pion bound state.

A. $\pi\pi(-)$ Exchange

Let the spectral functions for the nucleon charge and anomalous magnetic moment form factors be denoted by g_1 and g_2. The two-pion approximation for these then gives

$$[g_2(t)]_{\pi\pi}=q_\pi{}^3F_\pi{}^*(t)\frac{1}{2p^2\sqrt{t}}\times\left[f_+{}^{1(-)}(t)-Mf_-{}^{1(-)}(t)\frac{1}{\sqrt{2}}\right], \qquad \text{(V.1)}$$

and

$$[g_2(t)+g_1(t)/2M]_{\pi\pi}=q_\pi{}^3F_\pi{}^*(t)f_-{}^{1(-)}(t)/[2M(2t)^{\frac{1}{2}}], \qquad \text{(V.2)}$$

where $f_\pm{}^{J(-)}$ are the $J=1$ nucleon amplitudes introduced in Eq. (III.13) of Frazer and Fulco.[12]

Simple expressions for the absorptive parts $A_{\text{III}}{}^{(-)}$ and $B_{\text{III}}{}^{(-)}$ are obtained by reëxpressing the partial-wave amplitudes $f_\pm{}^{1(-)}$ and $f_1{}^{(-)}$ in terms of the electromagnetic structure quantities $g_{1,2}$ and f_K. Substituting Eqs. (IV.7), (IV.8), and (V.1) into Eqs. (III.13) and (III.14), one easily obtains

$$A_{\text{III}}{}^{(-)}(t,s)=-2\pi(s+\tfrac{1}{2}t-m_K{}^2-M^2)f_K{}^Vg_2(t), \qquad \text{(V.3)a}$$

and

$$B_{\text{III}}{}^{(-)}(t,s)=4\pi Mf_K{}^V(g_2+g_1/2M). \qquad \text{(V.4)}$$

The interaction due to the $\pi\pi(-)$ exchange contribution to the several K-nucleon amplitudes, is obtained by substituting Eqs. (V.3) and (V.4) into Eq. (II.26). According to Eqs. (II.21a) and (II.21b), the $\pi\pi(-)$ contributions to the $I=0$ and $I=1$ states are in the ratio -3.

B. Bound-State Exchange

According to the arguments of Sec. IV, the bound- or resonant-state exchange contributes only to the absorptive part $B_{\text{III}}{}^{(+)}$. An explicit expression for the consequent spectral function $\text{Im}G_J{}^{(+)}$ is obtained by substituting Eq. (III.17) into Eq. (II.26). The result is

$$[\text{Im}q_J{}^{(+)}(W)]_B=-\beta\theta(-m_B{}^2-4k^2)\frac{1}{32k^{2l+2}}\left[(W-M)P_l(z_0)+\frac{(E-M)^2}{k^2}(W+M)P_{l+1}(z_0)\right], \qquad \text{(V.5)}$$

where

$$z_0=1+(m_B{}^2/2k^2).$$

The parameter β is given in terms of the scalar charge fractions $f_K{}^S$ for the K meson and $f_N{}^S$ for the nucleon, due to the three-pion bound state, by the equation:

$$\beta=4\pi^2e^2m_B{}^4f_K{}^S(0)f_N{}^S(0)/|\lambda(m_B{}^2)|^2, \qquad \text{(V.6)}$$

where

$$\langle\gamma|S|B\rangle=i\delta^{(4)}(p_i-p_f)\epsilon\cdot\eta\lambda(t)/(4k\omega_B)^{\frac{1}{2}}.$$

ϵ and η are the polarization vectors for the photon and for the B particle; furthermore

$$F_K{}^S(t)f_K{}^S=\Gamma_K{}^S/(t-m_B{}^2),$$
$$F_N{}^S(t)f_K{}^N=\Gamma_N{}^S/(t-m_B{}^2),$$

where $F_K{}^S$ is defined by Eqs. (IV.3) and (IV.5); $F_N{}^S$ by Eqs. (II.7) and (IV.3) of reference 11.

We emphasize here that the constant $\lambda(m_B{}^2)$ enters in photoproduction processes[17] and in π^0 decay[18] and, therefore, may be calculated independently.

VI. RELATION BETWEEN THE $\bar{K}$-NUCLEON AND K-NUCLEON LONG-RANGE INTERACTIONS

In Sec. III, dependence of the long-range interactions on the matrix element for the production process $K+\bar{K}\rightarrow N+\bar{N}$ has been discussed. It is then easy to see that the $\bar{K}$-nucleon and K-nucleon interactions, as far as the long-range potentials are concerned, are related by charge conjugation on the $K\bar{K}$ state (crossing relation).

To be more specific, let us consider for the $\bar{K}$-nucleon elastic scattering the Jacob-Wick matrix element T_{fi}:

$$T_{fi}=-(9M/8\pi^2W)\tau_{fi}, \qquad \text{(VI.1)}$$

where τ_{fi} is defined by Eq. (II.4), M is the nucleon mass, and q is the center-of-mass momentum of the

[17] H. Wong, University of California Radiation Laboratory Report UCRL-9333 Rev. (unpublished).

[18] J. Ball, University of California Radiation Laboratory Report UCRL-9172 (unpublished).

K-nucleon system. T_{fi} is related to the phase shifts by

$$T_{fi}=\frac{1}{2\pi}\sum_J (J+\tfrac{1}{2})\langle\lambda_f|T^J(E)|\lambda_i\rangle \times\exp(\lambda_i-\lambda_f)d_{\lambda_i\lambda_f}{}^J(\theta), \quad \text{(VI.2)}$$

and

$$\langle+|T^J(E)|\pm\rangle=e^{i\delta_{l+}}\sin\delta_{l+}\pm e^{i\delta_{(l+1)-}}\sin\delta_{(l+1)-}. \quad \text{(VI.3)}$$

T_{fi} is given in terms of the invariant amplitudes A and B which, in the region $t>4m_\pi^2$ and $s<s_0$, s_0 being the physical threshold, are defined by Eqs. (III.4) and (III.5).

Let us consider now the K-nucleon elastic scattering. The process is described by the same diagram inverting the arrows on the meson line, i.e., exchanging $K\rightarrow\bar{K}$ in the matrix element $\langle N\bar{N}|T|K\bar{K}\rangle$, Eq. (III.3). This implies that the long-range part of the $\bar{K}$-nucleon and K-nucleon potentials are related by charge conjugation on the $K\bar{K}$ system.

These relations lead to interesting consequences if we analyze the intermediate states (n) [see Eq. (III.8)] in terms of even or odd numbers of pions. We have: (a) The $\bar{K}$-nucleon and K-nucleon interactions exhibit equal or opposite contributions according to the isotopic spin $I=0$ or $I=1$ of the intermediate state (n) with even number of pions; (b) The $\bar{K}$-nucleon and K-nucleon interactions exhibit equal or opposite contributions according to the isotopic spin $I=1$ or $I=0$ of the intermediate state (n) with odd number of pions.

In particular, the $J=1$, $I=1$ two-pion and the $J=1$, $I=0$ three-pion systems give opposite contributions to the $\bar{K}$-nucleon and K-nucleon interactions.

Finally, it is of interest to point out that these results depend mainly on the bosonic character of the K meson. In fact an intermediate state with even (odd) number of pions gives the same (opposite) contribution to the nucleon-nucleon and nucleon-antinucleon potential.

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APPENDIX A. ISOTOPIC STATES FOR THE K-$\bar{K}$ SYSTEM

We consider the $K^-\text{-}\bar{K}^0$ system as described by the field operator

$$K(x)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{(\infty)}d^3p\frac{1}{(2\omega_p)^{\frac{1}{2}}}[a_+X_++a_-X_-]e^{-ipx} +[b_-{}^\dagger X_++b_+{}^\dagger X_-]e^{ipx}, \quad \text{(A.1)}$$

$$K^+(x)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{(\infty)}d^3p'\frac{1}{(2\omega^{p\prime})^{\frac{1}{2}}}[a_+{}^\dagger X_++a_-{}^\dagger X_-{}^\dagger]e^{ip'x} +[b_-X_+{}^\dagger+b_+X_-{}^\dagger]e^{-ip'x},$$

where

$$X_+=\begin{vmatrix}1\\0\end{vmatrix},\quad X_-=\begin{vmatrix}0\\1\end{vmatrix},$$

and $a_+{}^\dagger$, $a_-{}^\dagger$, $b_+{}^\dagger$, $b_-{}^\dagger$ are creation operators for $\bar{K}^0$, K^-, K^+, K^0, respectively.

The isotopic spin states for the $K\bar{K}$ system are

$$T=1:\begin{cases}|\bar{K}^0K^+\rangle,\\ (1/\sqrt{2})[|K^-K^+\rangle-|K^0\bar{K}^0\rangle],\\ -|K^-K^0\rangle,\end{cases} \quad \text{(A.2)}$$
$$T=0:\ (1/\sqrt{2})[|K^-K^+\rangle+|K^0\bar{K}^0\rangle].$$

From these definitions, and assuming for the $N\bar{N}$ system those given by Cziffra,[19] the projection operators for the $K\bar{K}\rightarrow N\bar{N}$ amplitudes are

$$\bar{P}_0=\tfrac{1}{2}I,\quad \bar{P}_2=\tfrac{1}{2}\boldsymbol{\tau}_{\bar{K}}\cdot\boldsymbol{\tau}_N.$$

APPENDIX B. CHOICE OF PHASES FOR ELASTIC AND ABSORPTION AMPLITUDES

The discontinuity across the left cut for the elastic $\bar{K}$-nucleon process was obtained by using the unitarity condition for the side reaction of the diagram, $\bar{K}+K\rightarrow N+\bar{N}$, inserting the contributions of the $K+\bar{K}\rightarrow\pi+\pi$ and $\pi+\pi\rightarrow N+\bar{N}$ processes. Defining the $N\bar{N}$ states as Cziffra, and using his helicity spinors, Eqs. (III.2), (III.3), and (III.4) of Frazer-Fulco[12] may be written as

$$\tau_{++}=-(p/M)A+q\cos\theta' B, \quad \text{(A.4)}$$

$$\tau_{+-}=(E/M)qB\sin\theta', \quad \text{(A.5)}$$

where

$$\cos\theta'=\mathbf{p}_2\cdot\mathbf{q}_2/|\mathbf{p}_2||\mathbf{q}_2|.$$

We assumed analogous formulas for the reactions $\bar{K}_1+K_2\rightarrow N_1\bar{N}_2$. Equations (III.4), (III.5), (A.4), and (A.5) establish a correspondence between the $K\bar{K}$ state and the two-pion state. From this convention, we have for the electromagnetic vertex functions:

$$\langle\gamma|S|K_2\bar{K}_1(T=1,T_3=0)\rangle=-i(2\pi)^{-\frac{1}{2}} \times\frac{e^\mu(p_K{}^1+p_K)_\mu F_K{}^V e}{2(\mathcal{H}\omega_K\omega_K')^{\frac{1}{2}}}\delta^{(4)}(p_K'-p_K+\mathcal{H}), \quad \text{(A.6)}$$

and

$$\langle\gamma|S|\pi_2\pi_1(T=1,T_3=0)\rangle=i(2\pi)^{-\frac{1}{2}} \times\frac{e^\mu(p_\pi'+p_\pi)_\mu F_\pi e}{2(\mathcal{H}\omega_\pi\omega_\pi)^{\frac{1}{2}}}\delta^{(4)}(p_\pi-p_\pi'-\mathcal{H}). \quad \text{(A.7)}$$

Here $F_K{}^V(0)=\frac{1}{2}$, $F_\pi(0)=1$, and e is the elementary charge [note the different sign in Eqs. (A.6) and (A.7)].

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[19] P. Cziffra, University of California Radiation Laboratory Report UCRL-9249 (unpublished).