Structure of Radiative Decay Amplitudes*

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It is shown that the matrix element for decay processes involving the emission of a single photon may be obtained from the matrix element for the corresponding nonradiative decay and the magnetic moments of the particles involved, up to terms that vanish as the photon frequency $K\rightarrow 0$. Detailed discussions are given for decays involving three and four spinless particles, as well as for four spin $\frac{1}{2}$ particles. The results are similar to those obtained by Low for bremsstrahlung in scattering processes, but some novel features arise when the nonradiative decay is forbidden by selection rules.

1. INTRODUCTION

In this note we wish to discuss briefly the structure of the matrix element for decay processes involving the emission of a single photon. In order to be as explicit as possible, we shall consider first (Sec. 2) the specific case of radiative decay of a K^+ meson giving two or three pions. The more complicated case of fermion decays is then discussed in Sec. 3.

2. DECAYS INVOLVING BOSONS ONLY

Consider the nonradiative decay $K^+\rightarrow 3\pi$. We denote the matrix element by

$$T(\alpha=Q^2,\beta=P_1^2,\gamma=P_2^2,\delta=P_3^2;\; X=(P_2+P_3)^2,\, Y=P_1\cdot(P_2-P_3)),\quad (1)$$

where Q_μ, $P_{1\mu}$ are the four-momenta of the K^+ meson (mass m) and a final π^+ meson (mass μ), respectively, and $P_{2\mu}$, $P_{3\mu}$ are the four-momenta of the other two final pions (charges e', $-e'$). For the physical matrix element, we have $\alpha=-m^2$ and $\beta=\gamma=\delta=-\mu^2$. The variables X, Y are independent internal variables which characterize the sharing of the decay energy among the pions.

The matrix element for the emission of a photon with momentum K and polarization vector ϵ_μ consists of two kinds of contributions: (a) those arising from the emission of the photon by an ingoing or outgoing charged particle (the process of inner bremsstrahlung), and (b) those arising from "direct emission," which reflect the internal structure of the interactions responsible for the nonradiative decay, and of which some (but not all) may be deduced from gauge invariance.

As pointed out by Low,[1] the use of renormalized currents for the ingoing and outgoing bosons gives the same factors as occur in perturbation theory. As a result, the contributions (a) lead to a matrix element

$$\begin{aligned}
&e\Big[-\frac{Q\cdot\epsilon}{Q\cdot K}T(-m^2-2Q\cdot K,-\mu^2,-\mu^2,-\mu^2;X,Y)+\frac{P_1\cdot\epsilon}{P_1\cdot K}T(-m^2,-\mu^2+2P_1\cdot K,-\mu^2,-\mu^2;X,Y+K\cdot(P_2-P_3))\Big]\\
&\quad+e'\Big[\frac{P_2\cdot\epsilon}{P_2\cdot K}T(-m^2,-\mu^2,-\mu^2+2P_2\cdot K,-\mu^2;X+2K\cdot(P_2+P_3),Y+K\cdot P_1)\\
&\qquad-\frac{P_3\cdot\epsilon}{P_3\cdot K}T(-m^2,-\mu^2,-\mu^2,-\mu^2+2P_3\cdot K;x+2K\cdot(P_2+P_3),Y-K\cdot P_1)\Big],\quad (2)
\end{aligned}$$

where the variables X, Y still refer to the internal motion in the three-pion system. The case $e'=e$ corresponds to τ decay, and $e'=0$ to τ' decay. The matrix element (2) is not gauge invariant and needs to be supplemented by the terms linear in ϵ_μ obtained by making the replacements $Q\rightarrow Q-e\epsilon$; $P_1\rightarrow P_1-e\epsilon$; $P_2\rightarrow P_2-e'\epsilon$; $P_3\rightarrow P_3+e'\epsilon$ in (1). These terms are as follows:

$$\begin{aligned}
&e[-2Q\cdot\epsilon T_\alpha(-m^2,-\mu^2,-\mu^2,-\mu^2;X,Y)\\
&\quad-2P_1\cdot\epsilon T_\beta(-m^2,-\mu^2,-\mu^2,-\mu^2;X,Y)\\
&\quad-(P_2-P_3)\cdot\epsilon T_Y(-m^2,-\mu^2,-\mu^2,-\mu^2,X,Y)]\\
&\quad+e'[-2P_2\cdot\epsilon T_\gamma(-m^2,-\mu^2,-\mu^2,-\mu^2;X,Y)\\
&\quad+2P_3\cdot\epsilon T_\delta(-m^2,-\mu^2,-\mu^2\cdot-\mu^2;X,Y)\\
&\quad-2P_1\cdot\epsilon T_Y(-m^2,-\mu^2,-\mu^2,-\mu^2;X,Y)]\\
&\qquad+\epsilon\cdot V(K),\quad (3)
\end{aligned}$$

where the notation T_Z denotes the partial derivative $\partial T/\partial Z$ with respect to the subscript Z. The last term

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[1] F. E. Low, Phys. Rev. **110**, 974 (1958).

$\epsilon\cdot V(K)$ is nonsingular in $K\cdot Q$, $K\cdot P_1$, $K\cdot P_2$, and $K\cdot P_3$ and must vanish as $K\to 0$.

In order to relate the complete radiative decay matrix element to the physical matrix element $T_0 = T(-m^2, -\mu^2, -\mu^2, -\mu^2; X, Y)$ for the nonradiative decay, we make an expansion of (2) about the point $K=0$ and add the direct emission terms (3). The terms proportional to T_α, T_β, T_γ, T_δ cancel, and the sum takes the form

$$\begin{aligned} &e\left(\frac{P_1\cdot\epsilon}{P_1\cdot K}-\frac{Q\cdot\epsilon}{Q\cdot K}\right)T_0 \\ &\quad -e\left[(P_2-P_3)\cdot\epsilon-\frac{P_1\cdot\epsilon}{P_1\cdot K}K\cdot(P_2-P_3)\right]T_Y \\ &\quad +e'\left(\frac{P_2\cdot\epsilon}{P_2\cdot K}-\frac{P_3\cdot\epsilon}{P_3\cdot K}\right)[T_0+2K\cdot(P_2+P_3)T_X] \\ &\quad +e'\left[\left(\frac{P_2\cdot\epsilon}{P_2\cdot K}+\frac{P_3\cdot\epsilon}{P_3\cdot K}\right)P_1\cdot K-2P_1\cdot\epsilon\right]T_Y \\ &\quad\quad +eM_{\mu\nu}(K)(K_\mu\epsilon_\nu-K_\nu\epsilon_\mu). \qquad (4) \end{aligned}$$

In this expression the terms of order $(1/K)$ and (1) are given explicitly and the terms of higher order in K are collected together in the last term. This last term must be gauge invariant and vanish for $K=0$, and may always be written in the form shown above where $M_{\mu\nu}(0)$ is finite. $M_{\mu\nu}$ is an antisymmetric tensor, and is referred to as the structure-dependent term.

The important point to be noted here is that the first two terms of this matrix element are obtainable from a knowledge of the physical matrix element T_0 for the nonradiative decay. The only derivatives T_X and T_Y that survive are with respect to internal variables which are not subject to constraint. In the nonradiative τ' decay, for example, X is linearly related to the π^+ energy in the τ' rest frame and Y has the form $F(X,\cos\theta)$ where θ is the angle between the momenta of the positive pion and one of the neutral pions. The values of (X,Y) that can be reached in the radiative decay are always within the range of the possible values of (X,Y) for the nonradiative decay. The derivatives T_X and T_Y are therefore computable from the nonradiative decay amplitude on the mass and energy shell. The photon spectrum therefore has the low-frequency form,

$$dP(K)/dK=(1/K)(A+BK+\cdots), \qquad (5)$$

where A and B may be deduced from the amplitude for the nonradiative decay. This result is the analog of Low's theorem[1] for low-energy bremsstrahlung in the scattering of a charged particle by a neutral particle. If it happens that $T_0=0$ on the mass shell owing to some selection rule, it follows at once that both these terms A and B vanish and that the only contribution to the spectrum comes from the $M_{\mu\nu}$ term in (4). In this case, the spectrum will have the form K^3dK at low frequencies.

A case of particular simplicity is radiative $K_{\pi 2}$ decay. Here, the nonradiative matrix element T is a function of only three variables, $\alpha=Q^2$, $\beta=P_1^2$, and $\gamma=P_2^2$, where P_1, P_2 refer to the positive and neutral pions, respectively. The inner bremsstrahlung term corresponding to (2) is simply

$$e\left[-\frac{Q\cdot\epsilon}{Q\cdot K}T(-m^2-2Q\cdot K, -\mu_1^2, -\mu_2^2) + \frac{P_1\cdot\epsilon}{P_1\cdot K}T(-m^2, -\mu_1^2+2P_1\cdot K, -\mu_2^2)\right]. \qquad (6)$$

These are the only terms which are singular in $P_1\cdot K$ and $Q\cdot K$. When we add to this the direct emission contributions corresponding to (3) and expand (6) in ascending powers of K, we find that, since the terms T_α, T_β, and T_γ cancel precisely, *the constant term in the expansion is identically zero.* The total matrix element becomes

$$e\left(\frac{P_1\cdot\epsilon}{P_1\cdot K}-\frac{Q\cdot\epsilon}{Q\cdot K}\right)T(-m^2, -\mu_1^2, -\mu_2^2) + eN_{\mu\nu}(K)(K_\mu\epsilon_\nu-K_\nu\epsilon_\mu), \qquad (7)$$

where $N_{\mu\nu}$ is not singular in $P_1\cdot K$ and $Q\cdot K$ and approaches a finite limit as $K\to 0$. The form of this limit will generally be

$$N_{\mu\nu}(K\to 0)=F(Q_\mu P_{1\nu}-Q_\nu P_{1\mu}) + G\epsilon_{\mu\nu\rho\sigma}(Q_\rho P_{1\sigma}-Q_\sigma P_{1\rho}), \qquad (8)$$

as used by Good[2] in his discussion of the radiative $K_{\pi 2}$ decay spectrum. An important difference between (7) and the corresponding matrix element (4) for $K\to 3\pi+\gamma$ is that no term of order constant appears in (7).

If the $\Delta I=\frac{1}{2}$ selection rule held strictly for $K_{\pi 2}$ decay, the matrix element $T(-m^2, -\mu_1^2, -\mu_2^2)$ would be zero (assuming $\mu_1^2=\mu_2^2$). In this case the leading terms of the amplitude for radiative $K_{\pi 2}$ decay would be of the form (8) and the low-frequency spectrum would be proportional to K^3dK. In fact, the $\Delta I=\frac{1}{2}$ rule does not hold precisely and the matrix element $T_0=T(-m^2, -\mu_1^2, -\mu_2^2)$ has a reduced value relative to the matrix element for the K_1^0 decay. As we have shown, all contributions from (off the mass shell) decay amplitudes consistent with the $\Delta I=\frac{1}{2}$ rule, however, can only have the form (8), and must vanish as $K\to 0$. Recently Cabibbo and Gatto[3] have suggested using

$$T(Q^2,P_1^2,P_2^2)=T_0+f(P_1^2-P_2^2). \qquad (9)$$

In the approximation $\mu_1^2=\mu_2^2$ the latter term vanishes when both pions are on the mass shell and therefore does not contribute to the $K_{\pi 2}$ decay. Our discussion

[2] J. Good, Phys. Rev. **113**, 352 (1959).
[3] N. Cabibbo and R. Gatto, Phys. Rev. Letters **5**, 382 (1960).

shows that the f term cannot contribute to the radiative decay either, and that the only additional contributions are those arising from structure-dependent terms (8) discussed by Good.[2] No terms of the structure discussed by Cabbibo and Gatto[3] can arise.

3. FERMION DECAYS AND RELATED PROCESSES

These remarks can readily be extended to decay processes involving fermions. Consider for definiteness the decay of a spin $\frac{1}{2}$ particle of mass m, four-momentum Q, charge e, and anomalous magnetic moment λ, into a photon (momentum K and polarization vector ϵ_μ) and three spin $\frac{1}{2}$ particles with masses m_1, m_2, m_3, momenta P_1, P_2, P_3, charges e_1, e_2, e_3, and anomalous magnetic moments λ_1, λ_2, λ_3, respectively. We take the fermion with momentum P_3 to be an antiparticle described by a spinor $V(P_3)$, and denote the matrix element for the nonradiative decay by $\langle P_1P_2P_3|T|Q\rangle$. Then the inner bremsstrahlung terms may be written

$$\begin{aligned} &e\langle P_1P_2P_3|T|Q-K\rangle\epsilon_\mu J_\mu(Q) \\ &\quad+e_1\epsilon_\mu J_\mu(P_1)\langle P_1+K, P_2, P_3|T|Q\rangle \\ &\quad+e_2\epsilon_\mu J_\mu(P_2)\langle P_1, P_2+K, P_3|T|Q\rangle \\ &\quad\quad+e_3\epsilon_\mu J_\mu(P_3)\langle P_1, P_2, P_3+K|T|Q\rangle, \end{aligned} \tag{10}$$

where $J_\mu(Q)$, etc. denote the renormalized currents of the incoming and outgoing particles. It is shown in the Appendix that these currents are given by

$$\begin{aligned} \epsilon_\mu J_\mu(P_1) &= \bar{U}(P_1)(ie_1\epsilon\cdot\gamma+i\lambda_1\sigma_{\mu\nu}\epsilon_\mu K_\nu)[i\gamma\cdot(P_1+K)+m_1]^{-1}+O(K) \\ &= \bar{U}(P_1)(ie_1\epsilon\cdot\gamma+i\lambda_1\sigma_{\mu\nu}\epsilon_\mu K_\nu)[m_1-i\gamma\cdot(P_1+K)]/(2P_1\cdot K)+O(K) \\ &= \bar{U}(P_1)[2e_1P_1\cdot\epsilon+e_2\gamma\cdot\epsilon\gamma\cdot K+i\lambda_1\sigma_{\mu\nu}K_\nu\epsilon_\mu(m_1-i\gamma\cdot P_1)]/(2P_1\cdot K)+O(K), \end{aligned} \tag{11a}$$

with a similar expression for

$$\epsilon_\mu J_\mu(P_2), \tag{11b}$$

and

$$\epsilon_\mu J_\mu(Q) = -(2Q\cdot K)^{-1}[2eQ\cdot\epsilon-e\gamma\cdot K\gamma\cdot\epsilon+i\lambda(m-i\gamma\cdot Q)\sigma_{\mu\nu}K_\nu\epsilon_\mu]U(Q)+O(K), \tag{11c}$$

$$\epsilon_\mu J_\mu(P_3) = [2P_3\cdot K][2e_3P_3\cdot\epsilon+e_3\gamma\cdot K\gamma\cdot\epsilon+i\lambda_3(m_3+i\gamma\cdot P_3)\sigma_{\mu\nu}K_\nu\epsilon_\mu]V(P_3)+O(K). \tag{11d}$$

The most general form of the matrix element $\langle P_1P_2P_2|T|Q\rangle$ consistent with Lorentz invariance is, when all particles are real,

$$\langle P_1P_2P_3|T|Q\rangle = \sum_i G_i(\alpha,\beta,\gamma,\delta;X,Y)\bar{U}(P_1)\Gamma_iU(Q)\bar{U}(P_2)\Gamma_i'V(P_3), \tag{12}$$

where Γ_i, Γ_i' are some basic set of γ matrices contracted with appropriate covariants formed from the momenta P_1, P_2, P_3, and Q. The invariant functions G_i are assumed to be known from the nonradiative decay. If one of the particles, say, the one with momentum P_1, is off the mass shell, the spinor $\bar{U}(P_1)$ goes over into a propagator $S_F(P_1)$ and the matrix element may contain additional terms of the form

$$\begin{aligned} &\sum_i S_F(P_1)R'U(Q)\bar{U}(P_2) \\ &\quad\times\Gamma_i'V(P_3)H_i^{(1)}(\alpha,\beta,\gamma,\delta;X,Y), \end{aligned} \tag{13}$$

where R' is a matrix which gives zero when operating on $\bar{U}(P_1)$ from the right, and where the functions $H_i^{(1)}$ are no longer obtainable from the amplitude of the real nonradiative decay. Since, however, $\bar{U}(P_1)R'=0$, we may factor out the operator $(i\gamma\cdot P_1+m_1)$ from R' and write $R'=(i\gamma\cdot P_1+m_1)R_1$. Similarly, when one of the other particles is off the mass shell, the additional unknown terms may be written

$$\sum_i \bar{U}(P_1)R\cdot[i\gamma\cdot Q+m]S_F(Q)B_iH_i(\alpha,\beta,\gamma,\delta;X,Y)\quad(\text{when } Q^2\neq -m^2), \tag{14a}$$

$$\sum_i S_F(P_2)[i\gamma\cdot P_2+m_2]\cdot R_2V(P_3)B_iH_i^{(2)}(\alpha,\beta,\gamma,\delta;X,Y)\quad(\text{when } P_2^2\neq -m_2^2), \tag{14b}$$

$$\sum_i \bar{U}(P_2)R_3\cdot[-i\gamma\cdot P_3+m_3]S_F(P_3)A_iH_i^{(3)}(\alpha,\beta,\gamma,\delta;X,Y)\quad(\text{when } P_3^2\neq -m_3^2), \tag{14c}$$

with $A_i=\bar{U}(P_1)\Gamma_iU(Q)$, $B_i=\bar{U}(P_2)\Gamma_i'V(P_3)$. It is convenient to denote the expressions obtained from Γ_i, Γ_i', R, etc. by the replacement $P_j\to P_j\pm K$ by $\Gamma_i(P_j\pm K)$, $\Gamma_i'(P_j\pm K)$, $R(P_j\pm K)$, etc. With these notations, the inner bremsstrahlung term (9) becomes

$$\begin{aligned} &-(2Q\cdot K)^{-1}\sum_i G_i(\alpha-2Q\cdot K,\beta,\gamma,\delta;X,Y)\bar{U}(P_1)\Gamma_i(Q-K)[2eQ\cdot\epsilon-e\gamma\cdot K\gamma\cdot\epsilon+i\lambda(m-i\gamma\cdot Q)\sigma_{\mu\nu}K_\nu\epsilon_\mu]U(Q) \\ &\qquad\times\bar{U}(P_2)\Gamma_i'(Q-K)V(P_3) \\ &+(2P_1\cdot K)^{-1}\sum_i G_i(\alpha,\beta+2P_1\cdot K,\gamma,\delta;X,Y+K\cdot(P_2-P_3))\bar{U}(P_1)[2e_1P_1\cdot\epsilon+e_1\gamma\cdot\epsilon\gamma\cdot K+i\lambda_1\sigma_{\mu\nu}K_\nu\epsilon_\mu(m_1-i\gamma\cdot P_1)] \\ &\qquad\times\Gamma_i(P_1+K)U(Q)\bar{U}(P_2)\Gamma_i'(P_1+K)V(P_3) \\ &+(2P_2\cdot K)^{-1}\sum_i G_i(\alpha,\beta,\gamma+2P_2\cdot K,\delta;X+2K\cdot(P_2+P_3),Y+P_1\cdot K)\bar{U}(P_1)\Gamma_i(P_2+K)U(Q) \\ &\qquad\times\bar{U}(P_2)[2e_2P_2\cdot\epsilon+e_2\gamma\cdot\epsilon\gamma\cdot K+i\lambda_2\sigma_{\mu\nu}K_\nu\epsilon_\mu(m_2-i\gamma\cdot P_2)]\Gamma_i'(P_2+K)V(P_3) \\ &+(2P_3\cdot K)^{-1}\sum_i G_i(\alpha,\beta,\delta+2P_3\cdot K;X+2K\cdot(P_2+P_3),Y-K\cdot P_1)\bar{U}(P_1)\Gamma_i(P_3+K)U(Q) \\ &\qquad\times\bar{U}(P_2)\Gamma_i'(P_3+K)[2e_3P_3\cdot\epsilon+e_3\gamma\cdot K\gamma\cdot\epsilon+i\lambda_3(m_3+i\gamma\cdot P_3)\sigma_{\mu\nu}K_\nu\epsilon_\mu]V(P_3) \end{aligned}$$

$$
\begin{aligned}
&-(2Q\cdot K)^{-1}\textstyle\sum_i \bar U(P_1)R(Q-K)[i\gamma\cdot(Q-K)-m][2eQ\cdot\epsilon-e\gamma\cdot K\gamma\cdot\epsilon+i\lambda(m-i\gamma\cdot Q)\sigma_{\mu\nu}K_\nu\epsilon_\mu]U(Q)\\
&\qquad\times\bar U(P_2)\Gamma_i'(Q-K)V(P_3)H_i(\alpha-2Q\cdot K,\beta,\gamma,\delta;X,Y)\\
&+(2P_1\cdot K)^{-1}\textstyle\sum_i\bar U(P_1)[2e_1P_1\cdot\epsilon+e_1\gamma\cdot\epsilon\gamma\cdot K+i\lambda_1\sigma_{\mu\nu}K_\nu\epsilon_\mu(m_1-i\gamma\cdot P_1)][i\gamma\cdot(P_1+K)+m_1]R_1(P_1+K)U(Q)\\
&\qquad\times\bar U(P_2)\Gamma_i'(P_1+K)V(P_3)H_i^{(1)}(\alpha,\beta+2K\cdot P_1,\gamma,\delta;X,Y+K\cdot(P_2-P_3))\\
&+(2P_2\cdot K)^{-1}\textstyle\sum_i H_i^{(2)}(\alpha,\beta,\gamma+2P_2\cdot K,\delta;X+2K\cdot(P_2+P_3),Y+P_1\cdot K)\bar U(P_1)\Gamma_i(P_2+K)U(Q)\\
&\qquad\times\bar U(P_2)[2e_2P_2\cdot\epsilon+e_2\gamma\cdot\epsilon\gamma\cdot K+i\lambda_2\sigma_{\mu\nu}K_\nu\epsilon_\mu(m_2-i\gamma\cdot P_2)][i\gamma\cdot(P_2+K)+m_2]R_2(P_2+K)V(P_3)\\
&+(2P_3\cdot K)^{-1}\textstyle\sum_i H_i^{(3)}(\alpha,\beta,\gamma,\delta+2P_3\cdot K;X+2K\cdot(P_2+P_3),Y-P_1\cdot K)\bar U(P_1)\Gamma_i(P_3+K)U(Q)\\
&\qquad\times\bar U(P_2)R_3(P_3+K)[m_3-i\gamma\cdot(P_3+K)][2e_3P_3\cdot\epsilon+e_3\gamma\cdot K\gamma\cdot\epsilon+i\lambda_3(m_3+i\gamma\cdot P_3)\sigma_{\mu\nu}K_\nu\epsilon_\mu]V(P_3)+O(K).\quad(15)
\end{aligned}
$$

As in the boson case, the direct emission term is obtained by making the replacements $P_1\to P_1-e_1\epsilon$, etc. in the matrix element for the nonradiative decay and keeping only the terms linear in ϵ_μ. It should be noted that although $\bar U(P_1)(i\gamma\cdot P_1+m_1)=0$, $\bar U(P_1)\times(\partial/\partial P_{1\mu})(i\gamma\cdot P_1+m_1)\epsilon_\mu=\bar U(P_1)(i\gamma\cdot\epsilon)$ is different from zero, so that the functions H_i in (13)–(14c) will also contribute. The result is

$$
\begin{aligned}
&\{-2\textstyle\sum_i A_iB_i(eQ\cdot\epsilon G_{i,\alpha}+e_1P_1\cdot\epsilon G_{i,\beta}\\
&\quad+e_2P_2\cdot\epsilon G_{i,\gamma}+e_3P_3\cdot\epsilon G_{i,\delta})\\
&\quad-i\textstyle\sum_i\bar U(P_1)(eH_iR\gamma\cdot\epsilon+e_1H_i^{(1)}\gamma\cdot\epsilon R_1)U(Q)B_i\\
&\quad+A_i\bar U(P_2)(e_2H_i^{(2)}\gamma\cdot\epsilon R_2-e_3H_i^{(3)}R_3\gamma\cdot\epsilon)V(P_3)\}\\
&\quad-\textstyle\sum_i A_iB_i(2(e_2+e_3)(P_2+P_3)\cdot\epsilon G_{i,X}\\
&\quad+[e_1(P_2-P_3)\cdot\epsilon+(e_2-e_3)P_1\cdot\epsilon]G_{i,Y})\\
&\quad-\sum_i\sum_{j=1}^{4}e_j\epsilon_\mu\Bigg[A_i\bar U(P_2)\frac{\partial\Gamma_i'}{\partial P_{j\mu}}V(P_3)\\
&\qquad\qquad+\bar U(P_1)\frac{\partial\Gamma_i}{\partial P_{j\mu}}U(Q)B_i\Bigg]G_i+O(K),\quad(16)
\end{aligned}
$$

where $P_4=Q$, $e_4=e$, $G_{i,\alpha}=\partial G_i/\partial\alpha$, $\cdots$ and all invariant functions are evaluated at $K=0$, i.e., for $(\alpha,\beta,\gamma,\delta;X,Y)=(-m^2,-m_1^2,-m_2^2,-m_3^2,(P_2+P_3)^2,P_1\cdot(P_2-P_3))$. We have collected all the terms involving derivatives or functions unobtainable from the real nonradiative decay amplitude within the braces. The total matrix element for the radiative decay is given by the sum of (15) and (16). It will be seen that up to order $O(K)$, all unphysical terms cancel out and the remaining terms involve only the magnetic moments λ_i, the nonradiative decay amplitude, and its physical derivatives. Note first that expansion of the terms involving the functions G_i in (15) in powers of K reproduces the first summation within the braces in (16) with the opposite sign, plus some physical terms and terms of order K. To show that the unknown functions $H_i^{(n)}$ also drop out, it suffices to consider the terms involving H_i in (15). Since $(i\gamma\cdot Q+m)U(Q)=0$ and $(i\gamma\cdot Q+m)\times(m-i\gamma\cdot Q)=m^2+Q^2=0$, these terms reduce to

$$
\begin{aligned}
-(2Q\cdot K)^{-1}\textstyle\sum_i B_iH_i\bar U(P_1)R\cdot[-2ieQ\cdot\epsilon\gamma\cdot K-ie\gamma\cdot Q\gamma\cdot K\gamma\cdot\epsilon-em\gamma\cdot K\gamma\cdot\epsilon]U(Q)+O(K)&\\
=+ie\textstyle\sum_i B_iH_i\bar U(P_1)R\gamma\cdot\epsilon U(Q)+O(K),&\quad(17)
\end{aligned}
$$

which cancel the H_i terms in (16) up to order K. The other H functions drop out in precisely the same way. The final expression for the matrix element of the radiative decay is then (recall that $e=e_1+e_2+e_3$)

$$
\begin{aligned}
&\langle P_1,P_2,P_3|T|Q\rangle\left(-e\frac{Q\cdot\epsilon}{Q\cdot K}+e_1\frac{P_1\cdot\epsilon}{P_1\cdot K}+e_2\frac{P_2\cdot\epsilon}{P_2\cdot K}+e_3\frac{P_3\cdot\epsilon}{P_3\cdot K}\right)+\sum_i A_iB_i\Bigg\{2\Bigg[e_2\left(\frac{P_2\cdot\epsilon}{P_2\cdot K}P_3\cdot K-P_3\cdot\epsilon\right)\\
&\quad+e_3\left(\frac{P_3\cdot\epsilon}{P_3\cdot K}-P_2\cdot K\right)\Bigg]G_{i,X}+\Bigg[e_1\left(\frac{P_1\cdot\epsilon}{P_1\cdot K}(P_2-P_3)\cdot K-(P_2-P_3)\cdot\epsilon\right)+e_2\left(\frac{P_2\cdot\epsilon}{P_2\cdot K}P_1\cdot K-P_1\cdot\epsilon\right)\\
&\quad+e_3\left(P_1\cdot\epsilon-\frac{P_3\cdot\epsilon}{P_3\cdot K}P_1\cdot K\right)\Bigg]G_{i,Y}\Bigg\}+\sum_i\sum_{j=1}^{4}e_j\left(\frac{P_j\cdot\epsilon}{P_j\cdot K}K_\mu-\epsilon_\mu\right)\left[A_i\bar U(P_2)\frac{\partial\Gamma_i'}{\partial P_{j\mu}}V(P_3)+\bar U(P_1)\frac{\partial\Gamma_i}{\partial P_{j\mu}}U(Q)B_i\right]G_i\\
&\quad-(2Q\cdot K)^{-1}\textstyle\sum_i G_iB_i\bar U(P_1)\Gamma_i[-e\gamma\cdot K\gamma\cdot\epsilon+i\lambda(m-i\gamma\cdot Q)\sigma_{\mu\nu}\epsilon_\mu K_\nu]U(Q)\\
&\quad+(2P_3\cdot K)^{-1}\textstyle\sum_i G_iA_i\bar U(P_2)\Gamma_i'[e_3\gamma\cdot K\gamma\cdot\epsilon+i\lambda_3(m_3+i\gamma\cdot P_3)\sigma_{\mu\nu}\epsilon_\mu K_\nu]V(P_3)\\
&\quad+(2P_2\cdot K)^{-1}\textstyle\sum_i G_iA_i\bar U(P_2)[e_2\gamma\cdot\epsilon\gamma\cdot K+i\lambda_2\sigma_{\mu\nu}\epsilon_\mu K_\nu(m_2-i\gamma\cdot P_2)]\Gamma_i'V(P_3)\\
&\qquad\qquad+(2P_1\cdot K)^{-1}\textstyle\sum_i G_iB_i\bar U(P_1)[e_1\gamma\cdot\epsilon\gamma\cdot K+i\lambda_1\sigma_{\mu\nu}\epsilon_\mu K_\nu(m_1-i\gamma\cdot P_1)]\Gamma_iU(Q)+O(K),\quad(18)
\end{aligned}
$$

where all invariant functions are evaluated at $(\alpha,\beta,\gamma,\delta;X,Y)=(-m^2,-m_1^2,-m_2^2,-m_3^2;(P_2+P_3)^2,P_1(P_2-P_3))$.

It is clear that the above considerations are completely general and are applicable to processes involving any number of bosons and fermions. Up to terms of

order K, the unphysical functions and derivatives always cancel and the radiative matrix element is expressible in terms of the magnetic moments of the particles and quantities known from the nonradiative process. In particular, if the nonradiative decay is allowed, the photon spectrum has the form $(1/K)(A+BK+\cdots)dK$ with A and B predictable. If, however, the allowed nonradiative reaction involves no moving charged particles (such as μ capture at rest in hydrogen), the low-energy photon spectrum becomes $CKdK$, where C is determined by the magnetic moments and the nonradiative matrix element.[4] This may be seen by noting that the only terms of order $1/K$ in the radiative matrix element are proportional to $e_iP_i\cdot\epsilon/P_i\cdot K$,[5] where the index i refers to the ith particle. These singular terms vanish if either $e_i=0$ or $\mathbf{P}_i=0$, in which latter case $P_i\cdot\epsilon=0$, since we may choose a gauge with $\epsilon_4=0$. The matrix element contains, therefore, only terms which are either finite or zero as $K\to 0$, and our conclusion follows. Finally, if the nonradiative process is forbidden, the photon spectrum begins with DK^3dK.

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APPENDIX

We derive here an approximate expression for the renormalized current operator $J_\mu(P',P)U(P)=S_F(P')\times\Gamma_\mu(P'P)U(P)$ for an ingoing fermion with momentum P. Here $S_F(P')$ denotes the renormalized fermion propagator and $\Gamma_\mu(P',P)$ denotes the irreducible electromagnetic vertex operator for the fermion. The latter may be written in the form

$$\Gamma_\mu(P',P)U(P)=\{[A(P'^2)+(i\gamma\cdot P'+m)B(P'^2)]\gamma_\mu +[C(P'^2)+(i\gamma\cdot P'+m)D(P'^2)]\sigma_{\mu\nu}K_\nu\}U(P), \quad \text{(A.1)}$$

where $K_\nu=(P'-P)_\nu$. The functions A, B, C, D are nonsingular at the point $P'^2=-m^2$ (i.e., $K=0$). This follows from the fact that the only Lorentz-invariant singularity that can arise in the operator $J_\mu(P',P)U(P)$ may be expressed in the form $[1/(P\cdot K)]F_\mu(P',P)U(P)$, with $F_\mu(P',P)U(P)$ finite.[6] Since, however, $P\cdot K=\frac{1}{2}[(P+K)^2-P^2]=\frac{1}{2}(P'^2+m^2)$, this singularity is contained in the propagator $S_F(P')$. [Note that $S_F(P')\to(i\gamma\cdot P'+m)^{-1}$ as $P'-P=K\to 0$.] The remaining factor $\Gamma_\mu(P',P)U(P)$ is, therefore, finite as $K\to 0$.

Multiplying (A.1) on the left by $\bar{U}(P')$, we have

$$\begin{aligned}\bar{U}(P')\Gamma_\mu(P',P)U(P)&=\bar{U}(P')(A\gamma_\mu+C\sigma_{\mu\nu}K_\nu)U(P)\\&=\bar{U}(P')(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}K_\nu)U(P)\quad\text{for}\quad K\to 0.\end{aligned}\quad \text{(A.2)}$$

Hence $A(-m^2)=ie$, $C(-m^2)=i\lambda$. The propagator $S_F(P')$ is of the form

$$[i\gamma\cdot P'+m+\Sigma(\gamma\cdot P')]^{-1}=\frac{F(P'^2)+G(P'^2)(i\gamma\cdot P'+m)}{i\gamma\cdot P'+m},$$

with

$$\Sigma(im)=0;\quad [\partial\Sigma/\partial(\gamma\cdot P')]_{\gamma\cdot P'=im}=0, \quad \text{(A.3)}$$

and where F and G are both nonsingular at $P'^2=-m^2$, and $F(-m^2)=1$.

The product $S_F(P')\Gamma_\mu(P',P)U(P)$ is then equal to

$$\begin{aligned}(i\gamma\cdot P'm)^{-1}\{&([AF-BG(m^2+P'^2)]\\&+[BF+AG+2mBG](i\gamma\cdot P'+m))\gamma_\mu\\&+([CF-DG(m^2+P'^2)]+[DF+CG\\&+2mDG](i\gamma\cdot P'+m))\sigma_{\mu\nu}K_\nu\}U(P).\end{aligned}\quad \text{(A.4)}$$

Each of the square brackets represents a function of P'^2 which is nonsingular at $P'^2=-m^2$.

Now we appeal to the generalized Ward's identity[7] which here reduces to the equality

$$K_\mu J_\mu(P',P)U(P)=eU(P) \quad \text{(A.5)}$$

for all P'. Since $K_\mu\sigma_{\mu\nu}K_\nu=0$, and $\gamma\cdot K=\gamma\cdot(P'-P)=-i(i\gamma\cdot P'+m)$, (A.5) becomes, after substituting (A.4),

$$\begin{aligned}-i\{&[AF-BG(m^2+P'^2)]+[BF+AG+2mBG]\\&\times(i\gamma\cdot P'+m)\}U(P)=eU(P).\end{aligned}\quad \text{(A.6)}$$

This implies that for all P',

$$AF-BG(m^2+P'^2)=ie, \quad \text{(A.7a)}$$

$$BF+AG+2mBG=0. \quad \text{(A.7b)}$$

Note that (A.7a) is satisfied at $P'^2=-m^2$ by virtue of (A.2) and (A.3). Ward's identity shows that (A.7a) is valid for all P'. The expression for the current now becomes

$$J_\mu(P',P)U(P)=\left\{\frac{ie}{i\gamma\cdot P'+m}\gamma_\mu+\left[\frac{CF-(m^2+P'^2)DG}{i\gamma\cdot P'+m}+(DF+CG+2mDG)\right]\sigma_{\mu\nu}K_\nu\right\}U(P). \quad \text{(A.8)}$$

Note that we may write

$$CF-(m^2+P'^2)DG=i\lambda+(P'^2+m^2)X(P'^2), \quad \text{(A.9)}$$

$$DF+CG+2mDG=Y(P'^2), \quad \text{(A.10)}$$

where $X(P'^2)$, $Y(P'^2)$ are unknown functions of P'^2 and are finite at $P'^2=-m^2$. Finally,

$$\begin{aligned}J_\mu(P',P)U(P)&=\{(i\gamma\cdot P'+m)^{-1}[ie\gamma_\mu+i\lambda\sigma_{\mu\nu}K_\nu]\\&\quad+[Y(P'^2)+X(P'^2)(m-i\gamma\cdot P')]\sigma_{\mu\nu}K_\nu\}U(P)\\&=(i\gamma\cdot P'+m)^{-1}(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}K_\nu)U(P)+O(K).\end{aligned}\quad \text{(A.11)}$$

This establishes (11c). The derivations for the outgoing currents (11a,b,d) are essentially the same and need not be repeated here.

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[4] See, for example, J. Bernstein, Phys. Rev. **115**, 694 (1959), especially Eq. (30a).

[5] See (18) and (4), for example. (18) can be easily modified to describe the reaction with two incoming and two outgoing fermions by applying the substitution law.

[6] For a real photon, $P'\cdot K=P\cdot K$. Singularities of higher order in $1/(P\cdot K)$ would involve ghost states and may be excluded. We thank Professor Y. Nambu for helpful comments on this point.

[7] Y. Takahashi, Nuovo cimento **6**, 371 (1957).