Relativistic Electron-Pair Systems and the Structure of Neutral Mesons*

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In an effort to obtain a semiclassical model for the neutral π meson, a Bohr-Sommerfeld type of system with the proton replaced by a positron is investigated in the limit of high velocities. It is found that as a result of the relativistic increase in the electromagnetic field between the two moving charges, a natural minimum approach distance occurs equal to one-half of the classical "shell-electron" radius. At this separation, a new set of quantized states becomes possible which is found to be energetically unstable. The lowest state possesses an energy approximately equal to the π^0 meson energy. The relativistic states are characterized further by the greatly increased importance of perihelion precession, which accounts for one-half of the total angular momentum in the extreme relativistic case. When the effect of precession on the intrinsic magnetic moment is taken into account, the total energy of the system is found to be $263m_0c^2$, in close agreement with the observed π^0 meson mass. The lifetime of the system against annihilation into two gamma rays is calculated on the basis of the close analogy to singlet positronium. Its value is found to be 2.06×10^{-16} sec, in good agreement with the latest value of the observed π^0 meson lifetime. The implications for the structure of other nuclear particles and their interactions are briefly discussed.

I. INTRODUCTION

THE discovery of an increasingly large number of unstable nuclear particles has repeatedly led to the hope that these entities may actually be complex structures composed of only a few truly elementary particles. In particular, it was suggested by Fermi and Yang[1] that the π^0 meson might be explained as a pair of heavy nucleons, for instance a proton-antiproton pair. Even prior to the discovery of the π^0 meson, a somewhat related hypothesis had been advanced by Wentzel,[2] and by Gamow and Teller[3] who proposed that nuclear forces might be described by the exchange of electron-positron pairs. Along the same lines, Marshak[4] subsequently showed that many of the theoretical difficulties of an electron-positron pair theory of nuclear forces might be removed if one assumed the particles to be abnormally heavy, with masses a few hundred times that of the ordinary electron as proposed earlier by Yukawa. However, at that time Marshak concluded that the necessity of introducing an arbitrary cutoff into the strongly singular tensor potential remained as a serious difficulty common to all neutral-meson theories.

It is the purpose of the present note to investigate a simple electron-pair model for a π^0 meson in which the required large mass of the electrons arises as a relativistic effect. Although the model is semiclassical in character, it has the virtue of leading to definite mass and lifetime predictions as well as to a natural cutoff. Thus it may point the way towards the eventual formulation of a more complete description of nuclear particles and their interactions.

II. OUTLINE OF MODEL

The system that will be investigated here is essentially a straightforward extension of the Bohr-Sommerfeld model to higher velocities, except that the proton is replaced by a positron. It may therefore be regarded as a semiclassical relativistic positronium model in which the two charges revolve around a common center of mass at rest in the laboratory.

In order to simplify the problem, the effect of the intrinsic magnetic moments will be treated separately, and the discussion will be restricted to the equilibrium case of circular orbits, analogous to the original treatment of the hydrogen orbits by Bohr. By virtue of these assumptions and the equality of rest masses, the difficulties arising from the motion of the relativistic center of mass with respect to the static center of gravity are avoided. Since the system will be quantized in steady-state orbits, radiation will be assumed absent as long as the system is in its lowest allowed state as in the Bohr-Sommerfeld model.

III. RELATIVISTIC INERTIAL FORCE

The centrifugal force acting on the two charges q_1 and q_2 each having a rest mass m_0 will have the familiar form

$$F_c=m^*r_{12}\omega^2. \tag{1}$$

Here m^* is the relativistic reduced mass given by $\gamma_{12}m_0^*$ equal to $\gamma_{12}(m_0/2)$ for the special case of equal rest masses,[5] γ_{12} is the Lorentz contraction factor $[1-(v_{12}^2/c^2)]^{-\frac{1}{2}}$, r_{12} is the distance between centers, and ω is the orbital frequency of rotation.

Two points must be remembered in applying Eq. (1) in the limit of large velocities. The first is that if one wishes to express F_c in terms of the relative velocity of the two particles v_{12} rather than in terms of the angular velocity ω, one must use the relativistic rather than the Galilean law for the addition of velocities measured relative to the center of mass. Accordingly, calling these

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[1] E. Fermi and C. N. Yang, Phys. Rev. **76**, 1739 (1949).
[2] G. Wentzel, Helv. Phys. Acta **10**, 107 (1936).
[3] G. Gamow and E. Teller, Phys. Rev. **51**, 289 (1937).
[4] R. E. Marshak, Phys. Rev. **57**, 1101 (1940).

[5] For the more general case of unequal rest masses, see the mnemonic device of F. S. Crawford, Am. J. Phys. **26**, 376 (1958) and the earlier treatment of M. H. L. Pryce, Proc. Roy. Soc. (London) **A195**, 62 (1948). For the case of continuously distributed mass, see D. Bohm and J. P. Vigier, Phys. Rev. **109**, 1882 (1958).

velocities $v_{1C}=r_{1C}\omega$ and $v_{2C}=r_{2C}\omega$, respectively, one has

$$v_{12}=\frac{v_{1C}+v_{2C}}{1+[(v_{1C}v_{2C})/c^2]}=\omega(r_{1C}+r_{2C})f_r, \tag{2}$$

so that, calling $v_{1C}+v_{2C}=v_c$,

$$v_{12}=r_{12}\omega f_r=v_c f_r, \tag{3}$$

where in general

$$f_r=[1+(v_{1C}v_{2C}/c^2)]^{-1}. \tag{4}$$

In the present case of $v_{1C}=\frac{1}{2}v_c$ this reduces to

$$f_r=[1+(v_c^2/4c^2)]^{-1}. \tag{4a}$$

The correction factor f_r multiplying the relative velocity in the center-of-mass system v_c is seen to vary between the limits $f_r=1$ for the nonrelativistic case of $v_c\ll c$ and $f_r=\frac{1}{2}$ for $v_c\simeq 2c$. When rewritten in terms of v_{12}, Eq. (1) takes on the form

$$F_c=(m^*v_{12}^2/r_{12})(1/f_r)^2, \tag{5}$$

which differs from the nonrelativistic or low-velocity expression by a nearly constant factor of 4 when $v_{1C}=v_{2C}\simeq c$.

The second point to be remembered in the application of Eq. (1) is that when one is dealing with high angular velocities, the familiar precession of the perehelion frame in Sommerfeld's model will no longer be a small effect. The magnitude of the angular velocity of precession is given by

$$\Omega=[(\gamma_{12}-1)/\gamma_{12}]\omega. \tag{6}$$

It is equal to the well-known Thomas precession velocity,[6] whose value may be derived from purely kinematic considerations.[7,8] Inspection of Eq. (6) shows that as soon as γ_{12} becomes large compared with unity, instead of being negligibly small, Ω approaches ω in magnitude.

The effect of this very large precession is to divide the total angular momentum in the laboratory frame L_l into a part due to orbital motion relative to the precessing reference frame K_p, L_ω, and into a part that represents angular motion of K_p relative to the laboratory frame, L_Ω, or

$$L_l=L_\omega+L_\Omega. \tag{7}$$

The orbital angular momentum L_ω relative to K_p is given by

$$L_\omega=m^*\omega r_{12}^2, \tag{8}$$

which, in terms of v_{12} becomes

$$L_\omega=f_r^{-1}m^*v_{12}r_{12}. \tag{9}$$

[6] L. H. Thomas, Nature **117**, 514 (1926); Phil. Mag. **3**, 1 (1927).
[7] H. C. Corben and P. Stehle, *Classical Mechanics* (John Wiley & Sons, Inc., New York, 1950), Sec. 102.
[8] A. Sommerfeld, *Atombau and Spektrallinien* (Friedrich Vieweg und Sohn, Braunschweig, 1950), 7th ed., Vol. I, Appendix 12.

Using Eq. (6) for Ω, L_Ω takes on the form

$$L_\Omega=f_r^{-1}\left(\frac{\gamma_{12}-1}{\gamma_{12}}\right)m^*v_{12}r_{12}. \tag{10}$$

In terms of the total laboratory frame angular momentum L_l one obtains

$$L_\omega=[\gamma_{12}/(2\gamma_{12}-1)]L_l, \tag{11}$$

and

$$L_\Omega=[(\gamma_{12}-1)/(2\gamma_{12}-1)]L_l. \tag{12}$$

In the limit of $\gamma_{12}\gg 1$, the angular momentum is seen to divide itself equally between L_ω and L_Ω. The total centrifugal force is therefore composed of a part due to orbital motion within K_p and a part due to precession of K_p in the laboratory. In terms of L_ω, Eq. (5) gives for the former component

$$F_{c\omega}=L_\omega^2/m^*r_{12}^3, \tag{13}$$

so that the total combined inertial force may be written as

$$F_{ct}=F_{c\omega}\left(1+\frac{L_\Omega^2}{L_\omega^2}\right)=F_{c\omega}\left(1+\frac{(\gamma_{12}-1)^2}{\gamma_{12}^2}\right). \tag{14}$$

In the limit of $\beta_{12}^2\simeq 1$, the second term of Eq. (14) reduces to $(\gamma_{12}-1)/(\gamma_{12}+1)$ using the identity

$$\gamma_{12}^2\beta_{12}^2=(\gamma_{12}+1)(\gamma_{12}-1), \tag{15}$$

so that the total inertial force becomes

$$F_{ct}=\left(\frac{2\gamma_{12}}{\gamma_{12}+1}\right)\frac{L_\omega^2}{m^*r_{12}^3}. \tag{16}$$

Equation (16) shows that the effect of precession is to increase the usual centrifugal force calculated on the basis of an angular velocity ω and angular momentum L_ω by a factor that approaches 2 as $\gamma_{12}\gg 1$. When written in terms of v_{12}, using Eq. (9) to replace L_ω, one therefore obtains for the centrifugal force at high relativistic velocities the expression

$$F_{ct}=f_r^{-2}\left(\frac{2\gamma_{12}}{\gamma_{12}+1}\right)\frac{m^*v_{12}^2}{r_{12}}. \tag{17}$$

Equation (17) reduces to the classical form $(m_0v_{12}^2/r_{12})$ as γ_{12} and f_r both approach unity at low velocities. For the present case of equal rest-masses and $\gamma_{12}\gg 1$, the combined effects of the two correction factors and the reduced mass $m^*=\gamma_{12}(m_0/2)$ leads to

$$F_{ct}=4\gamma_{12}m_0v_{12}^2/r_{12}. \tag{18}$$

IV. ELECTROMAGNETIC FORCE

The inertial force as calculated above is the force that would be measured by an observer at rest relative to the proper frame of either of the two moving particles, for

instance as indicated by a spring balance.[9] To obtain the electromagnetic force that balances this inertial force, one must therefore likewise calculate the interaction as noted by an observer at rest relative to the proper frame of either particle.

This procedure has the advantage that since the charge being considered has no motion relative to this observer, the magnetic field produced by the other charge has no effect and the interaction reduces to a purely electrostatic one. The electrostatic field due to the moving charge q_2 will, however, be transformed relativistically, giving for the field produced at the position of the observer's charge q_1 the value[10,11]

$$\mathcal{E}_{12}=\frac{(1-\beta_{12}^2)}{(1-\beta_{12}^2\sin^2\varphi_{12})^{\frac{3}{2}}}\left(\frac{q_2}{r_{12}^2}\right), \qquad (19)$$

where $\beta_{12}=v_{12}/c$, and φ_{12} is the angle between v_{12} and r_{12}. This field is directed along the radius vector connecting q_1 and q_2, and leads to equal and oppositely directed forces when calculated by an observer at rest relative to q_2.

For the special case of steady-state circular orbits, the angle φ_{12} is always 90° and Eq. (19) simplifies to

$$F_{\mathrm{el}}=\gamma_{12}(q_1q_2/r_{12}^2). \qquad (20)$$

Equation (20) is seen to reduce to the ordinary static Coulomb force in the limit of $\gamma_{12}\simeq 1$. For velocities small compared to those of light, Eq. (20) may be expanded in powers of $(v_{12}/c)^2$ to give

$$F_{\mathrm{el}}=\frac{q_1q_2}{r_{12}^2}+\frac{1}{2}\frac{v_{12}^2}{c^2}\frac{q_1q_2}{r_{12}^2}+\cdots. \qquad (21)$$

Equation (21) contains in addition to the static Coulomb interaction a velocity-dependent term of the form that would ordinarily be obtained by an observer at rest in the laboratory from a consideration of the magnetic vector potential interaction arising from the motion of the charges. Thus Eq. (20) is equivalent to using a generalized velocity-dependent potential or Lorentz force law at low velocities, but it is free from the difficulties associated with the transformation of velocities and forces when these are first calculated in the laboratory frame, especially when $v_{12}\simeq c$.

V. FORCE EQUILIBRIUM CONDITION

In the steady-state circular orbits of interest here, the inertial force is at all times exactly equal to the force of attraction. Equating F_{ct} from Eq. (17) with F_{el} from Eq. (20) leads to the relation

$$(1/f_r)^2\left(\frac{2\gamma_{12}}{\gamma_{12}+1}\right)\frac{\gamma_{12}m_0{}^*v_{12}^2}{r_{12}}=\gamma_{12}\left(\frac{q_1q_2}{r_{12}^2}\right). \qquad (22)$$

For the case of f_r approximately constant, i.e., either $\beta_{12}\simeq 1$ or $\beta_{12}\simeq 0$, this expression may be solved for γ_{12} as a function of r_{12} by making use of the identity in Eq. (15) to give

$$\gamma_{12}=1/[1-(r_{m*}/r_{12})], \qquad (23)$$

where

$$r_{m*}=f_r^2q_1q_2/2m_0{}^*c^2. \qquad (24)$$

For the present case of $\gamma_{12}\gg 1$, $f_r=\frac{1}{2}$, $q_1=q_2=e$, and $m_0{}^*=(m_0/2)$, r_{m*} reduces to

$$r_m=e^2/4m_0c^2=0.70\times 10^{-13}\text{ cm}. \qquad (25)$$

Inspection of Eq. (23) shows that there exists a minimum possible value for r_{12} in an equilibrium orbit equal to r_{m*} where $\gamma_{12}\rightarrow\infty$. In the case of two equal particles of mass m_0 and charge e, Eq. (25) shows this minimum equilibrium orbit to have a diameter equal to one-quarter of the classical "shell-electron" diameter $d_0=(e^2/m_0c^2)=2.82\times 10^{-13}$ cm, independent of any assumptions as to the size of the electron's charge distribution.

This somewhat surprising result may be traced to the fact that for large velocities, γ_{12} cancels out of Eq. (22) so that in solving for r_{12}, the smallest possible value is reached for $v_{12}=c$. In the nonrelativistic case, v_{12} was allowed to go to infinity, which results in $r_{12}\rightarrow 0$ in the limit of high velocities. Thus, the existence of a minimum equilibrium orbit of finite radius may be said to follow necessarily from the existence of an upper limit to the velocity of material particles, coupled with finite values of their charge and rest mass.

It therefore appears that a straightforward application of relativistic transformation laws leads automatically to a natural "cutoff" or limitation on the approach distance of two charges in equilibrium under the action of inertial and electromagnetic forces. This result is in fact not limited to the case of Coulomb forces, so long as the forces vary more rapidly with distance than $(1/r_{12})$ and transform relativistically in the same way as do electromagnetic forces. Thus, using an arbitrary static law of force $F_{12}=a_nr^{-n}$ in Eq. (22) to replace (q_1q_2/r_{12}^2), one obtains the result

$$r_m=(f_r^2a_n/2m_0{}^*v_{12}^2)^{[1/(n-1)]}, \qquad (26)$$

which tends to a finite minimum value for all $n>1$ as $v_{12}\rightarrow c$.[12] This result may also be expressed as saying

[9] For an excellent discussion of the problem of defining the force in a relativistic system of charges, see H. Arzeliès, *La Dynamique Relativiste* (Gauthiers-Villars, Paris, 1957), Chap. X, Part A, pp. 143–148.

[10] Work cited in reference 9, p. 203.

[11] For a particularly clear discussion, see also F. K. Richtmyer and E. H. Kennard, *Introduction to Modern Physics* (McGraw-Hill Book Company, Inc., 1942), Sec. 70, p. 150.

[12] The existence of an absolute minimum approach distance between any two particles in force equilibrium arrived at here bears a striking resemblance to the occurrence of a similar limiting length in Schwarzschild's solution of Einstein's general theory of relativity for a point-particle. [See the discussion in E. T. Whittaker, *Aether and Electricity* (Thomas Nelson and Sons Ltd., London, 1953), Vol. I, p. 175 ff.] Although the magnitudes of this characteristic length differ greatly, the form of the line-element is exactly that of γ_{12} in Eq. (23) above, suggesting a possible underlying connection which will be examined further elsewhere.

that in a relativistic theory, the potential energy can never exceed the "reduced-mass energy" of the particles $m_0^{*\prime}c^2$, where

$$m_0^{*\prime}=[2\gamma_{12}/(\gamma_{12}+1)]m_0^*/f_r^2.$$

VI. ANGULAR MOMENTUM QUANTIZATION

In order to study the characteristics of the extreme relativistic system further, it is desirable to obtain expressions for the equilibrium values of the various angular momenta in terms of the force constant. This may be done conveniently by solving Eq. (16) for L_ω^2 and setting F_{ct} equal to F_{el} from Eq. (20), giving

$$L_\omega^2=\left(\frac{\gamma_{12}+1}{2\gamma_{12}}\right)m^*r_{12}\gamma_{12}e^2. \tag{27}$$

Making use of the definition of L_ω giving by Eq. (8), Eq. (27) reduces to

$$L_{\omega(\mathrm{eq})}=\left(\frac{\gamma_{12}+1}{2\gamma_{12}}\right)\left(\frac{\gamma_{12}e^2}{\beta_{12}c}\right)f_r. \tag{28}$$

Utilizing the relations between L_ω, L_Ω, and L_l given by Eqs. (7)–(12), one obtains for the equilibrium precessional angular momentum

$$L_{\Omega(\mathrm{eq})}=\tfrac{1}{2}\frac{(\gamma_{12}-1)}{(\gamma_{12}+1)}\left(\frac{\gamma_{12}e^2}{\beta_{12}c}\right)f_r, \tag{29}$$

and

$$L_{l(\mathrm{eq})}=\left(\frac{2\gamma_{12}-1}{\gamma_{12}}\right)\left(\frac{\gamma_{12}+1}{2\gamma_{12}}\right)\left(\frac{\gamma_{12}e^2}{\beta_{12}c}\right)f_r. \tag{30}$$

Examining these expressions reveals a number of interesting features. Turning first to $L_{\omega(\mathrm{eq})}$, it is seen that there exists a minimum value for equilibrium orbital momenta since the quantity $(\gamma_{12}+1)/2\beta_{12}$ passes through a minimum for $\gamma=5/3$, where $(\gamma_{12}+1)/2\beta_{12}=5/3$. Thus one arrives at the conclusion that for the model under discussion, there exists a natural lower limit to orbital angular momenta given by

$$L_{\omega(\mathrm{min})}=1.66(e^2/c)f_r. \tag{31}$$

If, for a given system, f_r varies appreciably with β_{12} in the neighborhood of this minimum, its location will be shifted, but the existence of a minimum will not be affected. For the case of two electrons, one obtains from Eq. (3) inserted into Eq. (4a) a quadratic in f_r whose solution gives

$$f_r=\tfrac{1}{2}\left[1+\left(1-\frac{v_{12}^2}{c^2}\right)^{\frac{1}{2}}\right]=\frac{\gamma_{12}+1}{2\gamma_{12}}. \tag{32}$$

The minimum $L_{\omega(\mathrm{eq})}$ now occurs for $\gamma_{12}=2$ giving for an electron pair the value

$$L_{\omega(\mathrm{min})}=1.3(e^2/c). \tag{33}$$

It is interesting to note that this is close to the smallest equilibrium angular momentum in Sommerfeld's treatment of the hydrogen atom e^2/c which is a quantity that also occurs as a "zero-point" angular momentum in all relativistic wave equations.

Turning next to Eq. (29) for $L_{\Omega(\mathrm{eq})}$, it is seen that the precessional angular momentum vanishes in the limit of $\gamma_{12}\rightarrow 1$, and reaches a limiting value $\frac{1}{2}(\gamma_{12}/\beta_{12})(e^2/c)f_r$ when $\gamma_{12}\gg 1$. Finally, the combined laboratory angular momentum $L_{l(\mathrm{eq})}$ [Eq. (30)] is seen to be very closely approximated by the simple expression

$$L_{l(\mathrm{eq})}=(\gamma_{12}/\beta_{12})(e^2/c)f_r, \tag{34}$$

both in the limits of very high and very low values of γ_{12}. Again, just as for $L_{\omega(\mathrm{eq})}$, a minimum value exists for $L_{l(\mathrm{eq})}$, indicating that in addition to the familiar low-velocity equilibrium states, there now exists a second set of orbits near $r_{12}\simeq r_m$ that meet the Bohr-deBroglie quantum condition $L_{l(\mathrm{eq})}=n\hbar$.

Inserting this value for $L_{l(\mathrm{eq})}$ in Eq. (34) and solving for β_{12}^2, one obtains a quadratic equation in β_{12}^2,

$$\beta_{12}^4-\beta_{12}^2+(\alpha f_r/n)^2=0, \tag{35}$$

where $\alpha=(e^2/\hbar c)=1/137.03$. The solutions of Eq. (35) are given by

$$\beta_{12}^2=\tfrac{1}{2}\pm\tfrac{1}{2}[1-(2\alpha f_r/n)^2]^{\frac{1}{2}}, \tag{36}$$

where the lower sign yields the familiar positronium solutions, and the upper sign gives the new high-energy states. For these low-velocity states, expanding the second term in powers of $(2\alpha f_r/n)^2$ and putting $f_r=1$ gives to lowest order the familiar Bohr condition,

$$\beta_{12}=\alpha/n, \tag{37}$$

and the kinetic energy

$$\tfrac{1}{2}m_0^*\beta_{12}^2c^2=\tfrac{1}{2}(\alpha^2/n^2)m_0^*c^2. \tag{38}$$

When m_0^* is set equal to $(m_0/2)$, Eq. (38) gives the magnitude of the binding energy for the ordinary positronium states.

The high energy solutions of Eq. (35) lead to the condition

$$\beta_{12}^2=1-(\alpha f_r/n)^2, \tag{39}$$

to first order in α^2. By virtue of the definition of $\gamma_{12}=(1-\beta_{12}^2)^{-\frac{1}{2}}$ one obtains directly

$$\gamma_{12}=n/\alpha f_r, \tag{40}$$

or, since $f_r=\frac{1}{2}$ for $\gamma_{12}\gg 1$,

$$\gamma_{12}=n(2/\alpha)=n\cdot 274. \tag{41}$$

The diameters of these orbits are obtained by solving Eq. (23) for r_{12}

$$r_{12}=r_{m^*}/(1-\gamma_{12}^{-1}), \tag{42}$$

or approximately

$$r_{12}=\left(f_r^2\frac{e^2}{2m_0^*c^2}\right)\left(1+\frac{\alpha f_r}{n}\right). \tag{43}$$

The orbits therefore all lie close to the limiting separation r_{m^*} and approach r_{m^*} as a limit with increasing

values of n. The ground state or outermost orbit in the case of an electron pair is given by

$$[r_{12}]_{n=1}=\frac{e^2}{4m_0c^2}(1+\tfrac{1}{2}\alpha), \tag{44}$$

which is essentially equal to r_m since the numerical value of $\frac{1}{2}\alpha$ is only 1/274.

Inserting the high energy values for γ_{12} and f_r into Eqs. (28), (29), and (30) shows that $L_{l(\text{eq})}=n\hbar$, and $L_{\Omega(\text{eq})}=L_{\omega(\text{eq})}=n(\hbar/2)$. Thus, in the limit of $\gamma_{12}\gg 1$, there exist states in which the orbital angular momentum within the precessing frame is quantized in half-integral values of $\hbar$.

VII. LAGRANGIAN AND HAMILTONIAN FORMULATION

In order to obtain the total energy of the relativistic pair system, it is necessary to find the Hamiltonian applicable to the extremely high-energy rotational states. Since the kinetic and potential energies no longer bear the simple relation to each other that exists in the low-velocity limit, one cannot write down the total energy of the system directly from a knowledge of the momentum and kinetic energy. The situation is further complicated by the fact that the derivative of the relativistic kinetic energy with respect to velocity no longer gives the momentum, so that it is not possible to assume that the simple classical relations between kinetic energy and centrifugal force continue to hold at very high velocities.

Since the relativistic Lagrangian for a particle moving in an electromagnetic field is known,[13] it will be taken as the starting point. For the present case of a purely static potential, the Lagrangian reduces to

$$\mathfrak{L}=-(1-\beta_{12}^2)^{\frac{1}{2}}m_0c^2-E_p+C. \tag{45}$$

Here E_p is the potential energy, which will be so chosen that for $r_{12}\rightarrow\infty$, $E_p\rightarrow 2m_0c^2$, or

$$E_p=2m_0c^2-(e^2/r_{12}). \tag{46}$$

This choice is effectively equivalent to assuming that the work done in separating the two charges to very large distances is equal to their total mass energy, or in other words, that the inertial mass is entirely due to the energy stored in the electrostatic fields. As to the constant C, by setting it equal to m_0c^2 following Sommerfeld,[14] the Lagrangian may be written in a simple form, closely analogous to the nonrelativistic case, or

$$\mathfrak{L}=\mathfrak{T}-E_p, \tag{47}$$

where

$$\mathfrak{T}=[1-(1-\beta_{12}^2)^{\frac{1}{2}}]m_0c^2=\left(\frac{\gamma_{12}-1}{\gamma_{12}}\right)m_0c^2. \tag{48}$$

[13] See work cited in reference 9, Chap. VII; also H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1950), Chap. 6, Sec. 5.

[14] Work cited in reference 8, Appendix 6, p. 667.

The quantity $\mathfrak{T}$ is often referred to as the Helmholtz kinetic potential energy.[14] It is always positive, never exceeds m_0c^2, and reduces to the nonrelativistic kinetic energy $T=\frac{1}{2}m_0v_{12}^2$ in the limit of $v_{12}\simeq c$.

With these assumptions, one obtains $(\partial\mathfrak{L}/\partial v_{12})=(\partial\mathfrak{T}/\partial v_{12})=p_{12}$, in analogy to the nonrelativistic case, and the inertial force is given by

$$F_i=(d/dt)(\partial\mathfrak{T}/\partial v_{12}). \tag{49}$$

For circular orbits, or constant absolute value of v_{12}, Eq. (49) leads to the expression for the centrifugal force $(\gamma_{12}m_0v_{12}^2/r_{12})$ of Sec. III above when the reduced mass equals m_0 and $f_r=1$, or when the source of the field is assumed to be infinitely heavy.

It is seen that the choice of the Lagrangian of Eq. (45) is consistent with the usual expression for the centrifugal force and the assumption that the rest masses of the electron and positron are purely electromagnetic in nature. The absolute value of the Hamiltonian is therefore fixed, by virtue of the definition

$$H=p\dot{q}-\mathfrak{L}. \tag{50}$$

Substituting v_{12} for $\dot{q}$ and the expression Eq. (47) for $\mathfrak{L}$ leads to the result

$$H=\mathcal{T}+E_p, \tag{51}$$

where $\mathcal{T}=(\gamma_{12}-1)m_0c^2$, the familiar expression for the relativistic kinetic energy of a particle possessing a rest mass m_0.

It is interesting to note that in the extreme relativistic case, there occur two different quantities, $\mathfrak{T}$ and $\mathcal{T}$, both of which reduce to the classical kinetic energy T in the limit of low velocities. As long as $\gamma_{12}\simeq 1$, they differ only inappreciably, but when $\gamma_{12}\gg 1$ as in the present case, the distinction becomes very significant, as can be seen by writing down their relationship in the form

$$\mathcal{T}=\gamma_{12}\mathfrak{T}. \tag{52}$$

Whereas $\mathfrak{T}$ is the "kinetic energy" which gives the correct momentum and force expressions, $\mathcal{T}$ is the "kinetic energy" which enters into the Hamiltonian or the total energy. Thus, the correct force equilibrium condition for a highly relativistic particle moving in a static Coulomb potential may be written down simply by substituting $\mathfrak{T}$ for T in the classical expression for centrifugal force $F=2T/r_{12}$. To see that this gives the same results as derived in Secs. III, IV, and V, one needs only to equate F_c with e^2/r_{12}^2, giving

$$\frac{2\mathfrak{T}}{r_{12}}=\frac{2(\gamma_{12}-1)m_0c^2}{\gamma_{12}}=\frac{e^2}{r_{12}^2}. \tag{53}$$

Solving Eq. (53) for γ_{12}, one obtains

$$\gamma_{12}=\frac{1}{1-(e^2/2m_0c^2r_{12})}, \tag{54}$$

which is precisely the result obtained earlier for the special case $f_r=1$ and $m^*=\gamma_{12}m_0$. In fact, comparing

the left-hand sides of Eqs. (53) and (22) for $f_r=1$ and $m_0^*=m_0$, one sees that one must have

$$\left(\frac{2\gamma_{12}}{\gamma_{12}+1}\right)\gamma_{12}m_0v_{12}^2=2(\gamma_{12}-1)m_0c^2=2\mathcal{T}, \quad (55)$$

which can readily be shown to be an identity by the use of Eq. (15).

It is therefore apparent that one might equally well have picked as a starting point the Hamiltonian or Lagrangian for a particle moving in an electromagnetic field, rather than the force-equilibrium condition used in Sec. V. However, this would have obscured the kinematic aspects of the situation when the particles have equal masses and the source of the field can no longer be assumed at rest in the laboratory frame. Ultimately, the basic reason for making a return to the considerations of forces necessary may be traced to the fact that whereas electromagnetic forces and energies depend only on the relative positions and velocities, centrifugal forces depend also on the distribution of masses, or on the position of the center of mass relative to the centers of the moving charges.

If one now identifies the Hamiltonian of Eq. (51) with the total energy including that of the rest masses following Sommerfeld,[14] one obtains for the case of two charges of rest mass m_0

$$H=E_T=(\gamma_{12}-1)m_0c^2-(e^2/r_{12})+2m_0c^2. \quad (56)$$

This may be written in terms of the binding energy $W_{12}=E_T-2m_0c^2$ as

$$W_{12}=(\gamma_{12}-1)m_0c^2-(e^2/r_{12}), \quad (57)$$

which is the starting point used by Sommerfeld in his treatment of the hydrogen fine structure in the limit of $v_{12}\ll c$ and $\gamma_{12}\simeq 1$. Thus one would not expect to find any appreciable deviations from the usual fine-structure for atomic levels, with the possible exception of s states, where a finite minimum approach distance may lead to slight upward shifts, as will be discussed in more detail elsewhere.

VIII. PROPERTIES OF THE HIGH-ENERGY SOLUTION

With the connection between the present treatment and the more familiar Hamiltonian and Lagrangian formulation established, it is now possible to investigate the character of the high-energy equilibrium orbits in more detail.

Neglecting for the moment the magnetic moment interaction, the total energy of the system is given by using the allowed values of γ_{12} and r_{12} from Eqs. (40) and (44) in Eq. (56), resulting in

$$E_T=\left(\frac{n}{\alpha f_r}-1\right)m_0c^2-\frac{m_0^*c^2}{f_r^2}\left(1+\frac{\alpha f_r}{n}\right)+2m_0c^2. \quad (58)$$

For the present case, $m_0^*=\frac{1}{2}m_0$ and $f_r=\frac{1}{2}$, so that Eq. (58) reduces to

$$E_T=\left(\frac{2n}{\alpha}-1\right)m_0c^2-4m_0c^2\left(1+\frac{\alpha}{2n}\right)+2m_0c^2, \quad (59)$$

or, to the extent that $\alpha/2n$ may be neglected compared to unity,

$$E_T=(2n/\alpha)m_0c^2-3m_0c^2. \quad (60)$$

For the ground state $n=1$ and with $(2/\alpha)=274$, one therefore obtains a total energy

$$E_T=271m_0c^2. \quad (61)$$

This is seen to be a very large positive energy, corresponding to a "binding" energy $W=269m_0c^2\simeq 137$ Mev, in contrast with the low-energy states, which have $W<0$. Thus, these highly relativistic states are energetically unstable,[15] and conversely, they can only be obtained when a large energy is added to the rest energy of the electron pair.

Since the mass equivalent of this energy is of the order of that observed for the π^0 mesons, one is led to the possibility that these unstable relativistic states may represent this highly unstable nuclear particle. Not only is the mass of the right order of magnitude, but also the size is sufficiently small for these particles to form part of the "core" of nucleons, whose radius is now known to be of the order of $0.3-0.4\times 10^{-13}$ cm.[16] Furthermore, there will be no net magnetic moment due to the orbital motion of two equal and opposite charges, so that if the intrinsic magnetic moments are opposed, the structure as a whole will also show no resultant magnetic moment. For spin magnetic moments opposed, the spins of oppositely charged particles are parallel, so that when the spins are oriented so as to oppose the sense of the orbital motion, the system as a whole will have zero angular momentum. The structure will therefore obey Bose statistics, and decay preferentially into two γ rays upon annihilation, again in agreement with the observed characteristics of the π^0 meson.

IX. MAGNETIC MOMENT CORRECTION TO THE TOTAL ENERGY

The system considered so far has consisted of two purely electrostatic charges possessing no intrinsic spins or magnetic moments. If the system is to be composed of an actual electron and positron, the effect of the intrinsic magnetic moments on the total energy must be investigated.

It will therefore be assumed that each charge carries a spin of absolute value $\frac{1}{2}\hbar$ and a corresponding magnetic

[15] Classically, the possibility of unstable circular orbits may be understood in terms of an effective force varying more rapidly than the inverse square law. That such a modification arises here may be seen by inserting Eq. (23) for γ_{12} in Eq. (20) and expanding in powers of (r_m^*/r_{12}) to give $F_{el}=q_1q_2/r_{12}^2+r_m^*q_1q_2/r_{12}^3+\cdots$.

[16] See the recent review article of J. L. Gammel and R. M. Thaler, in *Progress in Elementary Particle and Cosmic-Ray Physics* (North-Holland Publishing Company, Amsterdam, 1960), Vol. V, p. 99.

moment $\mathbf{\mu}_\sigma = g_\sigma \boldsymbol{\sigma}$ where g_σ is the gyromagnetic ratio associated with spin. Two types of effects will have to be examined; namely, the spin-spin and the spin-orbit contributions to the total energy of Eq. (56).

Before calculating the corrections to the potential energy in the extreme relativistic case of interest here, it will be necessary to examine the effect of the precession on the magnetic moments.

From the discussion of the energy relations in Sec. VII it is apparent that the kinetic energy associated with the precession is approximately $\gamma_{12} m_0 c^2$, or $274 m_0 c^2$ in the ground state $n=1$. The additional centrifugal force due to precession above and beyond that arising from the orbital motion within the precessing frame is therefore equivalent to increasing the inertial mass by a factor γ_{12}. The effective magnetic moment, which is inversely proportional to the mass of the particle must, therefore, be expected to decrease by the same factor γ_{12}. One therefore obtains an effective magnetic moment

$$\mu_{\mathrm{eff}} = e\hbar/2\gamma_{12} m_0 c = \tfrac{1}{2}\alpha\mu_{\mathrm{B}}, \tag{62}$$

where μ_{B} is the Bohr-magneton and $\gamma_{12} = (2/\alpha)$ for the ground state $n=1$.

The general expression for the spin-spin interaction energy is

$$E_{\sigma,\sigma} = -[3(\boldsymbol{\sigma}_1 \cdot u)(\boldsymbol{\sigma}_2 \cdot u) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2] J_{\sigma,\sigma}(r_{12}), \tag{63}$$

where $u = \mathbf{r}_{12}/|\mathbf{r}_{12}|$ and $J_{\sigma,\sigma} = g_\sigma^2/r_{12}^3$.

Inserting for g_σ the value obtained from the effective magnetic moment of Eq. (62), or

$$g_\sigma = \left(\frac{e}{(2/\alpha) m_0 c}\right), \tag{64}$$

one obtains, for the case of relativistic circular orbits and spin directions parallel, the value

$$E_{\sigma,\sigma} = (\alpha/2)^2 \mu_{\mathrm{B}}^2 / r_{12}^3 = -4 m_0 c^2, \tag{65}$$

using $r_{12} = \frac{1}{4}(e^2/m_0 c^2)$ from Eq. (44).

Similarly, for the spin-orbit interaction, the energy due to a particle of magnetic moment $\mathbf{\mu}_\sigma = g_\sigma \boldsymbol{\sigma}$ moving relative to a charge e with orbital angular momentum $\mathbf{l}$ associated with a magnetic moment $\mathbf{\mu}_l = g_l \mathbf{l}$ is

$$E_{\sigma,l} = [\boldsymbol{\sigma} \cdot \mathbf{l}] J_{\sigma,l}(r_{12}), \tag{66}$$

where $J_{\sigma,l} = g_\sigma g_l / r_{12}^3$. Again, due to the relativistic precession, one has

$$g_l = \left(\frac{e}{2(2/\alpha) m_0 c}\right). \tag{67}$$

Using $l = \hbar$, for the present case of circular orbits and both spins opposed to the direction of the orbital motion, the spin-orbit energy becomes

$$E_{\sigma,l} = \tfrac{1}{2}(\alpha/2)^2 (\mu_{\mathrm{B}}^2 / r_{12}^3). \tag{68}$$

Since each of the two particles carries a magnetic moment as well as a charge, the total spin-orbit interaction is twice this value, giving a combined magnetic potential correction of

$$E_P^{(m)} = E_{\sigma,\sigma} + 2E_{\sigma,l} = 2(\alpha/2)^2 (\mu_{\mathrm{B}}^2 / r_{12}^3). \tag{69}$$

Inserting the value of r_{12} for the ground state of the relativistic electron-positron system from Eq. (44), neglecting the small correction term involving $(\alpha/2)$, and making use of the fact that $e^2/m_0 c^2 = \hbar/[(1/\alpha) m_0 c]$, one obtains for the combined magnetic potential energy the value

$$E_P^{(m)} = -4 m_0 c^2 - 2 \cdot 2 m_0 c^2 = -8 m_0 c^2. \tag{70}$$

Applying this correction to the total energy E_T of Eq. (61) gives for the relativistic electron-pair system the energy

$$E_T = 271 m_0 c^2 - 8 m_0 c^2 = 263 m_0 c^2. \tag{71}$$

This figure is to be compared with the most recent experimental value for the total mass energy of the π^0 meson, 135.00 ± 0.05 Mev or $264.2 m_0 c^2$.[17]

X. LIFETIME OF RELATIVISTIC PAIR SYSTEM

In order to see whether the highly relativistic pair system could actually be identified with the observed π^0 meson, it is necessary to obtain an estimate of the system's lifetime against annihilation.

Since the detailed wave functions for the highly relativistic states are not known, this estimate cannot at present be obtained by the same direct calculation as the positronium lifetime. One possible way to circumvent this difficulty is to make use of the known theoretical lifetime of the low-velocity positronium states, and to arrive at the lifetime of the relativistic state on the basis of the close kinematic and dynamic analogy existing between the two systems.

From the cross section for annihilation together with the relative velocity in the center-of-mass system and the volume density of electrons, the mean lifetime of the positronium system may readily be obtained.[18] The result for the system with total angular momentum zero, i.e., singlet positronium, may be written in the form

$$\tau_0 = (1/\alpha^4)(2a_0/c) = (1/\alpha^4)(R_0/c). \tag{72}$$

Here, $2a_0 = R_0$ is the classical radius of the positronium system, α is the fine-structure constant, and c is the velocity of light, giving for τ_0 the value 1.25×10^{-10} sec.

This expression is seen to contain only geometric quantities aside from the dimensionless constant α. Now the only difference between the low-velocity and high-velocity systems lies in their scale or size. Both are quantized so that the appropriate deBroglie wavelength fits their respective orbital circumferences in the precessing perihelion frames. Therefore, if the two

[17] Walter H. Barkas and A. H. Rosenfeld, Table of mass-values in *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960).

[18] See the review articles of M. Deutsch [Progr. Nuclear Phys. **3**, 131 (1953)]; and S. DeBenedetti and H. C. Corben [Ann. Rev. Nuclear Sci. **4**, 191 (1954)].

perihelion frames are equally valid reference frames as assumed in the above derivation, then one would expect the lifetime of the high-energy state to be simply given by replacing R_0 with the smaller radius R_0' of the relativistic system.

Aside from this change, the only other factor that must be considered is that the relative velocity in the center-of-mass system, v_c, will now be given by $v_c = v_{12} f_r^{-1}$ from Eq. (3). Since in the relativistic system, $v_{12} = c$, one has $v_c = 2c$ rather than c when this correction is neglected. Inserting these values into Eq. (72) therefore leads to the following expression for the mean lifetime of the relativistic system

$$\tau_0' = (1/\alpha^4)(R_0'/2c). \tag{73}$$

Substituting $R_0' = (r_{12}/2) = \frac{1}{2}e^2/4m_0c^2$ from Eq. (25) gives the value $\tau_0' = 2.06 \times 10^{-16}$ sec, close to the latest experimental value of the π^0 lifetime, $\tau_0 = (2.0 \pm 0.4) \times 10^{-16}$ sec.[19]

The agreement of the value for the lifetime given by Eq. (73) based on the analogy to the positronium system, may be taken as further support of the hypothesis that the π^0 particle can be regarded as an electron-positron pair in a highly relativistic quantum state.

The analogy to positronium appears to be quite deep since both the π^0 meson[20] and the singlet positronium system[18] decay with the emission of two gamma rays polarized at right angles to each other. Both systems therefore have pseudoscalar parity, in addition to being characterized by a total angular momentum equal to zero.[21]

Thus the relativistic electron-pair model developed above is seen to account for all the known properties of the π^0 meson, including its total energy, decay characteristics, magnetic moment, size, spin, and parity.

XI. SUMMARY AND CONCLUSION

The above considerations appear to indicate that the relativistic two-body system may be treated in terms of a semiclassical Bohr-Sommerfeld model extended to high velocities and distances of the order of 10^{-14} cm.

This result is somewhat surprising since on the basis of quantum electrodynamics, one would have expected that large numbers of electron pairs should be created in such a highly relativistic system, leading to a complete breakdown of the classical laws of interaction. Why this does not happen is not clear, but the surprising agreement of the simple classical model with the observed characteristics of the π^0 meson suggests that a more detailed investigation of this basic question would be desirable.

The model leads to the occurrence of a natural minimum distance of approach or "cutoff," independent of any assumptions as to the physical size of the particles involved. This limit to the approach of two relativistic particles in turn leads to the possibility of a new set of quantized states. These states are characterized by very high total energies and a strong precession that accounts for half of the system's total angular momentum.[22]

Because the sizes of these new orbits and the strength of the interaction correspond to those typical of nuclear systems, the suggestion arises that electrons and positrons in relativistic states form the basic structural elements of all nuclear particles.[23,24]

Since the model of the π^0 meson involves only electromagnetic forces, one may hope that with the further understanding of the more complex particles,[25] it will be possible to arrive at an electromagnetic description of nuclear forces.

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[19] R. G. Glasser, N. Seeman, and B. Stiller, Bull. Am. Phys. Soc. **6**, 39 (1961).

[20] R. Plano, A. Prodell, N. Samios, M. Schwartz, and J. Steinberger, Phys. Rev. Letters **3**, 525 (1959).

[21] By analogy to positronium, a second type of π^0 meson should exist in which the spins are oriented anti-parallel, thereby resulting in a total angular momentum $\hbar$. Its mass may be calculated by adding the repulsive spin-spin interaction energy $4m_0c^2$ (Eq. 63) to $E_T = 271m_0c^2$ for Eq. (61), giving $E_T = 275m_0c^2$, the spin-orbit contribution being zero. In a radiative decay process, this π_1^0 particle would give rise to 3 gamma rays. Some evidence for the existence of such a π_1^0 particle decaying in this manner has in fact been reported by R. P. Ely and D. H. Frisch [Phys. Rev. Letters **3**, 565 (1959)]. Unfortunately, the evidence is not conclusive, nor is there any indication of the mass of such a particle. Further experiments, including mass-determinations, would therefore be of great value.

[22] A possible relation to the occurrence of spin angular momentum quantized in half-integral units of $\hbar$ in Dirac's relativistic wave-equation is apparent.

[23] This is supported by the existence of striking mass regularities involving integral and half-integral multiples of $(1/\alpha)m_0$ noted by a number of investigators: Y. Nambu, Progr. Theoret. Phys. (Kyoto) **7**, 595 (1952); H. Fröhlich, Nuclear Phys. **7**, 148 (1958); L. S. Levitt, Current Sci. (India) **27**, 131 (1958); K. M. Guggenheimer, Nuovo cimento **11**, 287 (1959).

[24] The related suggestion that the large mass of nucleons might be interpreted as arising from the relativistic motion of light-weight "subparticles" or "specks" was developed by W. F. G. Swann [Phys. Rev. **109**, 998 (1958)]. The author is indebted to Dr. Swann for bringing this work to his attention shortly before submission of the present paper.

[25] The extension of the present approach to include the charged π and μ mesons has been outlined elsewhere [Bull. Am. Phys. Soc. **6**, 257 (1961)], as well as the possibility that excited rotational states of these structures may account for the high-energy scattering resonances [Bull. Am. Phys. Soc. **5**, 238 (1960)]. More detailed accounts are in preparation.